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A RENEWAL THEOREM FOR DISTRIBUTIONS ON R^1 WITHOUT EXPECTATION[1]

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ABSTRACT. Let $U\{I\}$ be the expected number of visits to an interval I of a random walk associated with a distribution on R^1 in the domain of attraction of a stable law with exponent $\frac{1}{2}<\alpha\leqq 1$. Theorem A gives asymptotic expressions for $U\{I\pm t\}$ as $t\to\infty$. Such expressions are not valid when $0<\alpha\leqq\frac{1}{2}$ without additional hypotheses on F. These are indicated in Theorem B.

1. Theorem 1 of [3] extends to distributions on all of R^1 as follows: (Notation as in [3] or [4, Chapter XI].) Let F be a probability distribution on $(-\infty, \infty)$ and for any measurable set I put

$$U\{I\} = \sum_{n=0}^{\infty} F^{n*}\{I\}$$

finite or not. As in [3] we assume F is nonarithmetic. (See note (iv) in §2 below.)

THEOREM A. *Suppose*

$$1 - F(t) + F(-t) = t^{-\alpha}L(t), \qquad t > 0, \tag{1}$$

and

$$\lim_{t\to\infty} \frac{F(-t)}{1-F(t)} = \frac{q}{p} \tag{2}$$

where $0<\alpha\leqq 1$, $p+q=1$ *and* L *is slowly varying at* ∞. *Then when* $\frac{1}{2}<\alpha<1$,

$$\lim_{t\to\infty} t^{1-\alpha}L(t)(U\{I+t\} + U\{I-t\}) = \frac{\sin \pi\alpha}{\pi(p^2 + 2pq\cos\pi\alpha + q^2)}|I| \tag{3}$$

and

$$\lim_{t\to\infty} \frac{U\{I-t\}}{U\{I+t\}} = \frac{q}{p} \tag{4}$$

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for every bounded interval I of length $|I|$. If $\alpha=1$, $p\neq q$ and $\int_{-\infty}^{\infty}|x|F\{dx\}=\infty$, then (4) *remains valid but* (3) *becomes*

$$\lim_{t\to\infty} m(t)(U\{I+t\}+U\{I-t\})=(p-q)^{-2}|I| \tag{5}$$

where

$$m(t)=\int_0^t (1-F(x)+F(-x))\,dx \sim \int_{-t}^{t}|x|F\{dx\}, \qquad t\to\infty.$$

We postpone to §3 the case $0<\alpha\leqq\frac{1}{2}$.

2. **Discussion.** (i) Conditions (1) and (2) together are, of course, the necessary and sufficient conditions for F to be in the domain of attraction of a stable law with exponent α; see [**4**, p. 544].

(ii) Note that $p=1$, $q=0$ in (2) includes the extreme possibility, previously considered in [**3**], that $F(t)=0$ for all $t<0$.

(iii) The restriction $p\neq q$ in the case $\alpha=1$ as well as $m(\infty)=\infty$ is essential. A random walk induced by an F for which $p=q$, $\alpha=1$ and $m(\infty)=\infty$ can be persistent (whether or not it is will depend on more detailed properties of L). If persistent $U\{I\}$, the expected number of visits to I, is infinite, so (4) and (5) are vacuous. If $m(\infty)<\infty$, F has a finite absolute mean and then the classical renewal theorem applies, see [**5**, p. 368]; a finite mean can occur only when $\alpha\geqq 1$.

(iv) The restriction to nonarithmetic distributions is not essential; when F is arithmetic, Theorem A is true as stated, provided in (3), (4) and (5) one uses half-open intervals with length a multiple of the span of F. J. A. Williamson [**8**] has proved results similar to ours for discrete distributions in R^d, $d\geqq 1$. See also [**6**]. However, these authors did not consider $\alpha=1$, so Theorem A gives new information in this case.

3. **The case** $0<\alpha\leqq\frac{1}{2}$. When $0<\alpha\leqq\frac{1}{2}$ in (1), Theorem A is not true without further restrictions on F. See [**8**, §5] for counterexamples with discrete F. The following theorem gives an indication of the sort of restrictions needed.

THEOREM B. *Suppose F satisfies* (1) *and* (2) *and is absolutely continuous with bounded density $f(t)=F'(t)$. Suppose further that either*

(i) *$f(t)=O(L(|t|)/|t|^{\alpha+1})$ for all t; or*

(ii) *$f(t)$ ultimately decreases as $|t|$ increases; or*

(iii) *$f(t)$ is absolutely continuous on $|t|\geqq b$ with $f'(t)=O(L(|t|)/|t|^{\alpha+2})$.*

Then the conclusion of Theorem A *follows under* (i) *for $\frac{1}{4}<\alpha\leqq 1$, and for all $0<\alpha\leqq 1$ under* (ii) *or* (iii).

Discrete versions of (i) and (ii) are known. See [**8**, §3] and [**6**, p. 232]; see also [**2**] where an even stronger monotonicity condition is imposed.

The proof of Theorem B is messy. It together with applications of A and B to convolution type integrals and extensions to higher dimensions will appear elsewhere.

4. **Proof of Theorem A.** The methods of [**3**, §§3–6] can be straightforwardly adapted to construct a proof of Theorem A. Here is a sketch. Put $\phi(\theta)=\int_{-\infty}^{\infty} e^{ix\theta}F\{dx\}$ and for any t write $L(t)=L(|t|)$, $m(t)=m(|t|)$. Note that, when $\alpha=1$, m is slowly varying and $L(t)=o(m(t))$, $t\to\infty$; cf. [**4**, p. 272].

LEMMA 1. *For* $0<\alpha<1$,

$$\frac{1}{1-\phi(\theta)}\sim\frac{\cos(\pi\alpha/2)\pm i(p-q)\sin\pi\alpha/2}{\Gamma(1-\alpha)(p^2+2pq\cos\pi\alpha+q^2)}\,\frac{|\theta|^{-\alpha}}{L(1/\theta)}$$

as $\theta\to 0^{\pm}$. *If* $\alpha=1$ *but* $p\neq q$ *and* $m(\infty)=\infty$, *then*

$$\frac{1}{1-\phi(\theta)}\sim\frac{\pi}{2}(p-q)^{-2}\left|\frac{d}{d\theta}\left(\frac{1}{m(1/\theta)}\right)\right|+i(p-q)^{-1}\frac{1}{\theta m(1/\theta)}$$

as $|\theta|\to 0$. (*The real and imaginary parts on the left are to be considered as having the corresponding asymptotic form on the right.*)

REMARK. Except possibly for the form given here when $\alpha=1$, asymptotic expressions for $1-\phi$ equivalent to those in Lemma 1 are well known and occur often in the literature. With the obvious modifications, the method used in proving Lemma 2 of [**3**] can be used here as well.

From now on when $\alpha=1$ we assume $p\neq q$ and $m(\infty)=\infty$.

COROLLARY. *Let* g *be any bounded function continuous at* $\theta=0$, *and put* $w_t(\theta)=\operatorname{Re}(e^{-it\theta}/(1-\phi(\theta)))$. *If* $0<\alpha<1$, *then*

$$\lim_{t\to\pm\infty}|t|^{1-\alpha}L(t)\int_{|\theta|\le B/|t|}g(\theta)w_t(\theta)\,d\theta$$

$$=\frac{2g(0)}{K\Gamma(1-\alpha)}\int_0^B\frac{b\cos y\pm d\sin y}{y^\alpha}\,dy$$

where $b=\cos\pi\alpha/2$, $d=(p-q)\sin\pi\alpha/2$ *and* $K=p^2+2pq\cos\pi\alpha+q^2$. *When* $\alpha=1$ *we have*

$$\lim_{t\to\pm\infty} m(t) \int_{|\theta|\leq B/|t|} g(\theta)w_t(\theta)\, d\theta = \frac{2g(0)}{(p-q)^2}\left(\frac{\pi}{2} \pm (p-q) \int_0^B \frac{\sin y}{y}\, dy\right).$$

This Corollary follows from Lemma 1 and properties of regularly varying functions. See Lemmas 3 and 4 of [3].

LEMMA 2. *Let g be any function with compact support which satisfies $|g(\theta+h)-g(\theta)|=O(h)$ uniformly in θ. Write $\rho(t)=|t|^{1-\alpha}L(t)$ when $\alpha<1$ and $\rho(t)=m(t)$ when $\alpha=1$. Then, for $\frac{1}{2}<\alpha\leq 1$ and $B>1$,*

$$\limsup_{|t|\to\infty} \left| \rho(t) \int_{|\theta|\geq B/|t|} g(\theta)w_t(\theta)\, d\theta \right| = O\left(\frac{1}{B^{2\alpha-1}}\right).$$

(See [3, (5.11)] or [6, §3.5].)

From Lemma 1 and the recurrence criterion [7, p. 34] it follows that $U\{I\}<\infty$ for bounded I. Define

$$\mu_t\{I\} = \rho(t)(U\{I+t\} + U\{-I+t\}).$$

LEMMA 3. *For every $a>0$ and all λ,*

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\mu_t\{dx\} = 2\rho(t)\int_{-\infty}^{\infty} g_a(\theta+\lambda)w_t(\theta)\, d\theta \tag{6}$$

where $g_a(\theta)=(1/a)(1-|\theta|/a)$ for $|\theta|\leq a$, $g_a(\theta)=0$ elsewhere, and $\gamma_a(x)=2(1-\cos ax)/a^2x^2=\int_{-\infty}^{\infty} e^{ix\theta}g_a(\theta)\, d\theta$.

NOTE. Lemma 3 may be proved as in [3, §4]. See also [1, p. 221] and [5]. The proof is quite easy when $\alpha<1$ since in this case $|1-\phi(\theta)|^{-1}$ is locally integrable about $\theta=0$. Note also that in both Lemmas 2 and 3 one needs to know that $|1-\phi(\theta)|$ vanishes only at $\theta=0$. But this is true if and only if F is nonarithmetic. This is not a critical problem however, and a proof in the arithmetic case may be made using the methods given here. (In fact the proof is slightly less messy when F is arithmetic since a direct formula for the renewal measure weights is available; the auxiliary functions g_a, γ_a do not appear.) See [3, §2(ii)] or [6] or [8].

Here is the proof of Theorem A. Write the integral on the right-hand side of (6) as the sum of the integral over $|\theta|\leq B/|t|$ plus the integral over $|\theta|>B/|t|$. Let $t\to+\infty$ (or $-\infty$) and apply the Corollary and Lemma 2. Next, let $B\to\infty$, evaluate the improper integrals which arise and substitute $(2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-i\lambda x)\gamma_a(x)\, dx$ for $g_a(\lambda)$. Then,

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\mu_t\{dx\} = 2pC\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\, dx \tag{7}$$

where C is the constant occurring on the right in (3) or (5). (If $t \to -\infty$, the p on the right in (7) is replaced by $q=1-p$.) As (7) is true for all $a>0$ and real λ, it follows from [3, Lemma 8], or [1, p. 218] that

$$\mu_t\{I\} \to 2pC|I| \quad \text{and} \quad \mu_{-t}\{I\} \to 2qC|I|$$

as $t \to \infty$ for every interval I. From this and the definition of μ_t we get the conclusion of Theorem A whenever I or $\bar{I}$ is symmetric about the origin. The conclusion for arbitrary I follows by putting $I = I_0 + \delta$ where $\bar{I}_0$ is symmetric and observing that $\rho(t \pm \delta) \sim \rho(t)$ as $t \to \infty$.

References

1. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968. MR **37** #4841.

2. N. G. de Bruijn and P. Erdös, *On a recursion formula and on some Tauberian theorems*, J. Res. Nat. Bur. Standards **50** (1953), 161–164. MR **14**, 973.

3. K. B. Erickson, *Strong renewal theorems with infinite mean*, Trans. Amer. Math. Soc. **151** (1970), 263–291.

4. W. Feller, *Introduction to probability theory and its applications*. Vol. II, Wiley, New York, 1966. MR **39** #3535.

5. W. Feller and S. Orey, *A renewal theorem*, J. Math. Mech. **10** (1961), 619–624. MR **24** #A581.

6. A. Garsia and J. Lamperti, *A discrete renewal theorem with infinite mean*, Comment. Math. Helv. **37** (1962/63), 221–234. MR **26** #5630.

7. Donald S. Ornstein, *Random walk*. I, Trans. Amer. Math. Soc. **138** (1969), 1–43.

8. J. A. Williamson, *Random walks and Riesz kernels*, Pacific J. Math. **25** (1968), 393–415. MR **37** #2328.

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