A SHORT PROOF OF A THEOREM OF BARR-BECK

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Let $\mathbf{C}$ be a category. Let $(P, \mathfrak{M})$ be a projective structure [K−W] where P are the set of $\mathfrak{M}$-projectives and the set of P-proper morphisms. Then the following are true.

I. $(P, \mathfrak{M})$ is determined by a cotriple iff there is a coreflexive subcategory $\mathbf{C}' \subset \mathbf{C}$ with the properties:

(1) $|\mathbf{C}'| \subset P$,

(2) the coreflexions are in $\mathfrak{M}$.

II. If $S \dashv T$, where $S: \mathbf{C} \to \mathbf{D}$ and $T: \mathbf{D} \to \mathbf{C}$, and if $(P, \mathfrak{M})$ is a projective structure in $\mathbf{C}$ determined by a cotriple G, then the projective structure $(rSP, T^{-1}\mathfrak{M})$ is determined by the cotriple SGT, where rSP is the collection of retracts of SP. Moreover, if $(P, \mathfrak{M})$ is induced by a cotriple G, then $(rSP, T^{-1}\mathfrak{M})$ is induced by SGT.

The proofs of these two statements are omitted here. As a corollary of the above statements, we have the following.

III (Barr-Beck). The triple cohomology of groups coincides with the Eilenberg MacLane cohomology.

IV (Barr-Beck). The triple cohomology of associative algebras coincides with the Hochschild cohomology.

For detailed statements of the above, see [$\mathrm{B}-\mathrm{B}_1$].

We now prove III. Let $(\mathbf{G}, \pi)$ be the category of groups over the group π. Let M be a π-module. Then there is an adjoint pair

$$(\mathbf{G}, \pi) \underset{T}{\overset{S}{\rightleftarrows}} \pi\text{-Mod}$$

where $S(W) = Z\pi \otimes_W IW$ with $IW = \ker(Z(W) \to Z)$ and $T(M) = M \times_\varphi \pi$, the semidirect product of M and π with respect to the π-module structure $\varphi: \pi \to \mathrm{Aut}(M)$ (cf. [$\mathrm{B}-\mathrm{B}_2$], where $S(W)$ is denoted by $\mathrm{Diff}_\pi(W)$). Now the free group cotriple on the category $\mathbf{G}$ of groups gives a cotriple on $(\mathbf{G}, \pi)$. Let $(P, \mathfrak{M})$ be the corresponding projective structure. Then $(rSP, T^{-1}\mathfrak{M})$ is a projective structure in π-Mod. To show $(rSP, T^{-1}\mathfrak{M})$ is induced by the free functor cotriple on π-Mod, it suffices to show that SP contains all free π-modules. Since P are retracts of free groups and IF are free

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F-modules (cf. [M, p. 123]), $S(F)$ are indeed free π-modules. Now by adjointness, we have

$$[GW, M \times_{\varphi} \pi] \cong [SGW, M]$$

where G is the free cotriple on **G**. We can conclude that SGW is homotopic to the bar resolution of $S(W)$ by direct computation or by invoking the following:

V. Let A be a preadditive category. Let **G** be a cotriple on A where the functor G is additive. Then the cotriple complex of every $B \in A$ is a projective resolution.

The statement IV can be proved similarly with the pair of functors,

$$(K\text{-alg}, \wedge) \underset{T}{\overset{S}{\rightleftarrows}} \wedge^e\text{-Mod}$$

with $S(\Gamma) = J\Gamma \otimes_{\Gamma^e} \wedge^e$, where $J\Gamma = \ker(\wedge \otimes_K \wedge^{\mathrm{opp}} \to \wedge)$, and $T(M) = \wedge^e * M$ [B]. The proof goes after we observe that $J\Gamma$ is a free Γ^e-module if Γ is free K-algebra, [C−E, p. 181].

Added in Proof. Another way to prove III and IV is to observe that if T preserves and reflects epimorphisms, then the projective structure $({}_{\gamma}SP, T^{-1}\mathfrak{M})$ is an absolute projective structure if $(P, \mathfrak{M})$ is.

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