BOOK REVIEWS

Ergodic Properties of Algebraic Fields, by Yu. V. Linnik. Translated by Michael S. Keane. Band 45 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, Springer-Verlag, 1968. $12.10

As one can see from the title, this book covers rather unusual material. Linnik has tried in it to connect two rather different branches of mathematics: Algebraic Number Theory and Ergodic Theory. It is a *tour de force* which he carries off quite well.

Previous works connecting ergodic theory with number theory have generally concentrated on proving results in metric number theory by using the ergodic theorem. One finds a suitable ergodic transformation (usually on the interval $[0, 1]$), constructs an invariant measure on it, looks at a reasonable function, and discovers via the ergodic theorem that almost every number has some property. For instance, one might let $T(x) = \{10x\}$ (where $\{a\}$ = fractional part of a). Then T is ergodic on $[0, 1]$, and Lebesgue measure is invariant under T. Let f be the characteristic function of $[.1, .2)$, and plug f and T into the ergodic theorem; out comes the statement that for almost every real number, a tenth of the digits in its decimal expansion are 1's.

Linnik is concerned with a different and subtler connection. Perhaps the best way to explain it is by means of an example (which, indeed, is the first example treated in the book). Let m be an integer $\equiv 1$ or $2 \pmod 4$ or $\equiv 3 \pmod 8$ and, consider the Diophantine equation $x^2+y^2+z^2=m$. This has primitive integral solutions (i.e., solutions where x, y, z have no common factor). What Linnik is after is an ergodic theorem concerning these primitive solutions. There are two immediate difficulties which arise. First of all, the ergodic theorem involves an ergodic transformation, and thus far there is no transformation at all on the solutions. A much more serious problem is that the number of primitive solutions is finite. It is not at all clear what can be meant by an ergodic transformation on a finite set.

The first problem is settled by constructing the transformation; the second needs more discussion. Linnik's eventual goal is to show that the primitive solutions are "uniformly distributed" on the sphere. It is not too hard to make sense of this notion. The primitive solutions are points on the 2-sphere $S_m = x^2+y^2+z^2=m$. Suppose that there are r_m such solutions. Let Λ be a sector from the origin cutting out the solid angle ω, and suppose that there are λ_m primitive solutions in the set $S_m \cap \Lambda$. Then the primitive points will be uniformly distributed if

$$\lim_{m\to\infty} \frac{\lambda_m}{r_m} = \frac{\omega}{4\pi}\cdot$$

(Here, m runs through only the congruence classes given above.) The analogous problem for the n-sphere ($n \geqq 4$) has been investigated; it turns out that the primitive points are indeed uniformly distributed in these cases.

Now the nature of the ergodic theorem to be proved becomes clearer. We let T be the transformation, and let f_m be the characteristic function of $S_m \cap \Lambda$. Then the desired theorem is something of the following sort

Let $X = (x, y, z)$ be a primitive solution. Then for almost all X,

$$\frac{f_m(X) + f_m(TX) + \cdots + f_m(T^{n-1}X)}{n} = \frac{\omega}{4\pi} + e(n, m),$$

where n is chosen to be sufficiently large (compared with m), and $e(n, m) \to 0$, as $m, n \to \infty$. (Here, "almost all" means that the number of X for which the theorem fails is $o(r_m)$.)

The theorem Linnik actually proves is similar to this one, but has some technical differences. One picks a prime $q \geqq 3$ and an integer $k \geqq 1$, and restricts attention to those m for which the Kronecker symbol

$$\left(\frac{-m}{q}\right)$$

is 1. The numbers q and k are used to define T. This definition is rather complicated; I might mention (in the style of Dr. Watson, whose tantalizing allusions to unreported cases of Sherlock Holmes often consisted of the remark that they involved matters of the greatest importance to some crowned heads of Europe) that it depends on some results on integral quarternions and binary quadratic forms. Finally, Linnik does not consider the whole sphere, but rather the integral points in a certain spherical sector Ω of spherical angle $\pi/6$. (This restriction is the result of certain normalizations, and does not affect the nature of the results.) The sector Λ is required to lie inside Ω, and the conclusion now reads: For almost all x,

$$\frac{f_m(X) + \cdots + f_m(T^{n-1}X)}{n} = \frac{6\omega}{\pi} + e(n, m, \Lambda, q, k),$$

where Λ, q, and k are fixed, n is sufficiently large compared with m, and $e(n, m, \Lambda, q, k) \to 0$ as $m, n \to \infty$. This theorem implies the uniform distribution result for the 3-sphere, but with the restriction that

$$\left(\frac{-m}{q}\right) = 1.$$

As Linnik notes, this last condition is somewhat unnatural; some work of Malyshev shows that it could be removed if the generalized Riemann hypothesis were true.

One can attempt to carry out a similar program with other families of algebraic surfaces; Linnik treats the case of one- and two-sheeted hyperboloids. The behavior of the flows in these cases is closely tied up with the arithmetic of real and imaginary quadratic fields, respectively. In fact, Linnik is able to prove a result about the coefficients of certain Dirichlet series which is also a consequence of the Riemann hypothesis for the L-series associated with imaginary quadratic fields. The connection between the surfaces and the fields is that the natural matrix transformations which preserve the surfaces give representations of the integers of the fields. Given the fields, therefore, one could study appropriate matrix representations of the integers and attempt to prove theorems about the corresponding surfaces. Linnik discusses this aspect of the problem briefly. The book closes with a discussion of some further unsolved problems.

The book also contains one other chapter, unrelated to the rest. In it Linnik proves some theorems which show that the behavior of certain sums (earlier discussed by Davenport and Erdös) can be used for a model of Brownian motion. In other words, it is also possible to link algebraic number theory and the theory of stochastic processes.

The book is not easy to read. That state of affairs is not Linnik's fault, but the material's. Much of the book is concerned with difficult estimates of error terms, and at times the reader may get the feeling that he is watching a man unravel a plate of spaghetti. Linnik makes things as clear as possible under the circumstances; his style is good, and the Russian habit of repeating at the beginning of each chapter the key results proved earlier is a big help. Despite his efforts, though, the book is somewhat daunting.

LAWRENCE CORWIN