ON STRUCTURAL STABILITY[1]

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The purpose of this note is to sketch a proof of

THEOREM A. *A C^2 diffeomorphism (on a compact, boundaryless manifold) which satisfies Axiom* A *and the strong transversality condition is structurally stable.*

This is (one direction of) a conjecture of Smale [**3**]. The case where the nonwandering set is finite is the main theorem of [**4**]. For background, see [**2**] and [**3**]. Details will be given in a subsequent publication.

1. **An infinitesimal condition.** Throughout, M denotes a smooth, compact, boundaryless manifold and $f: M \to M$ a diffeomorphism. A *chart* on M is a pair (α, U) where U is an open subset of M and α maps a neighborhood of $\overline{U}$ diffeomorphically onto an open subset of Euclidean space $\boldsymbol{R}^m$.

Let $\mathfrak{X}^0(M)$ denote the Banachable space of all continuous vector fields on M. Let $f^\#: \mathfrak{X}^0(M) \to \mathfrak{X}^0(M)$ be the continuous linear operator defined by $f^\# \eta = Tf^{-1} \circ \eta \circ f$ for $\eta \in \mathfrak{X}^0(M)$.

Fix a Riemannian metric on M and let d denote the corresponding metric on M; i.e., for $x, y \in M$, $d(x, y)$ is the infimum of the lengths of all curves from x to y. We define a new metric d_f by

$$d_f(x, y) = \sup_n d(f^n(x), f^n(y))$$

where the supremum is over all integers n. Let $\mathfrak{X}_f(M)$ denote the set of all $\eta \in \mathfrak{X}^0(M)$ with the property that for every chart (α, U) on M there exists $K > 0$ such that

$$| \eta_\alpha(x) - \eta_\alpha(y) | \leqq K d_f(x, y)$$

for all $x, y \in U$. Here $\eta_\alpha: U \to \boldsymbol{R}^m$ is defined by $T\alpha \circ \eta(x) = (\alpha(x), \eta_\alpha(x))$ for $x \in U$. By standard techniques $\mathfrak{X}_f(M)$ can be made into a Banachable space. The inclusion $\mathfrak{X}_f(M) \to \mathfrak{X}^0(M)$ is continuous and for any finite cover of M by charts (α, U) the K's above can be chosen small if η is sufficiently close to 0 in $\mathfrak{X}_f(M)$.

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As f is C^2, $f^{\#}$ restricts to a continuous linear operator on $\mathfrak{X}_f(M)$. Recall that a continuous linear operator is split surjective if and only if it has a continuous linear right inverse. Let 1 denote the identity operator.

THEOREM B. *If* $1-f^{\#}:\mathfrak{X}_f(M)\to\mathfrak{X}_f(M)$ *is split surjective, then* f *is structurally stable.*

SKETCH OF PROOF. We must show that if g is a diffeomorphism which is sufficiently close to f in the C^1 topology, then there is a homeomorphism $\phi:M\to M$ such that $g\circ\phi=\phi\circ f$. Following Moser [1] we write $\phi=\exp(\eta)$ and $g=f\circ\exp(\xi)$ where η and ξ are vector fields (η is C^0 and ξ is C^1). Here $\exp:TM\to M$ is an exponential map. The equation $g\circ\phi=\phi\circ f$ becomes $\exp(\xi)\circ\exp(\eta)=f^{-1}\circ\exp(\eta)\circ f$ which may be written in the form

$$(1-f^{\#})(\eta) = R(\eta)$$

where $R:\mathfrak{X}^0(M)\to\mathfrak{X}^0(M)$ is a C^1 map which is C^1 close to zero for small $\eta\in\mathfrak{X}^0(M)$ when the C^1 norm $\|\xi\|_1$ of ξ is small. To find a solution for this last equation it suffices to solve $\eta=J\circ R(\eta)$ where J is a continuous linear right inverse to $1-f^{\#}$ on $\mathfrak{X}_f(M)$. It is not difficult to show that for arbitrarily small closed neighborhoods B of 0 in $\mathfrak{X}_f(M)$, $J\circ R$ maps B to B and is a contraction in the C^0 norm provided $\|\xi\|_1$ is sufficiently small (i.e., g is C^1 close to f). Hence this last equation may be solved (in a preassigned neighborhood of 0 in $\mathfrak{X}_f(M)$) via the Banach contraction principle.

It remains to show that $\phi=\exp(\eta)$ is a homeomorphism. It is homotopic to the identity and hence of degree one and hence onto; we need only show it is one-one. Suppose $\phi(x)=\phi(y)$. Then $\phi(f^n(x))=g^n(\phi(x))=g^n(\phi(y))=\phi(f^n(y))$ so we may assume that

$$d_f(x, y) \leqq 2d(x, y).$$

Given a finite cover of M by charts (α, U) there exists $a>0$ such that

$$ad(x, y) \leqq d(\exp(\dot{x}), \exp(\dot{y})) + |v-w|$$

for each chart (α, U) of the cover, all $x, y\in U$, all sufficiently small $\dot{x}\in T_xM$ and $\dot{y}\in T_yM$ and where $T\alpha\dot{x}=(\alpha(x), v)$ and $T\alpha\dot{y}=(\alpha(y), w)$. Taking $\dot{x}=\eta(x)$ and $\dot{y}=\eta(y)$ we obtain

$$ad(x, y) \leqq d(\phi(x), \phi(y)) + |\eta_\alpha(x)-\eta_\alpha(y)|.$$

However, $|\eta_\alpha(x)-\eta_\alpha(y)|\leqq Kd_f(x, y)\leqq 2Kd(x, y)$ where K can be made small. Hence

$$(a - 2K)d(x, y) \leqq d(\phi(x), \phi(y)) = 0$$

which implies that $x = y$. End of argument.

f is called *expansive* iff there exists $\epsilon > 0$ such that $d_f(x, y) \geqq \epsilon$ for all $x, y \in M$ with $x \neq y$. If f is expansive, then $\mathfrak{X}_f(M) = \mathfrak{X}^0(M)$ (as Banachable spaces). Now an Anosov diffeomorphism is expansive and f is Anosov if and only $f^\#$ is a hyperbolic operator on $\mathfrak{X}^0(M)$. Thus the structural stability of Anosov diffeomorphisms (see [1]) follows from Theorem B.

2. **Locally Anosov diffeomorphisms.** For $Z \subseteq M$ let

$$Z^f = \bigcup_n f^n(Z), \qquad Z^{f+} = \bigcup_{n \geqq 0} f^n(Z), \qquad Z^{f-} = \bigcup_{n \leqq 0} f^n(Z).$$

Thus $Z^f = Z^{f+} \cup Z^{f-}$ is the smallest f-invariant set containing Z. We shall say that Z is *f-unrevisited* iff the conditions $x \in Z$ and $f^n(x) \in Z$ imply that $f^q(z) \in Z$ for q between 0 and n.

DEFINITION. f is *locally Anosov* iff there exist open sets $Z_1, \cdots, Z_k \subseteq M$; continuous subbundles E_i^s and E_i^u of $TM | Z_i^f$ $(i = 1, \cdots, k)$; a Riemannian metric $\| \ \|$ on M; and a real number λ with $0 < \lambda < 1$ such that for $i, j = 1, \cdots, k$ and $\sigma = s, u$:

(1) $\bar{Z}_i$ is f-unrevisited;
(2) $M = Z_1^f \cup \cdots \cup Z_k^f$;
(3) $\bar{Z}_i \cap \bar{Z}_j = \emptyset$ for $i \neq j$;
(4) $TM | Z_i^f = E_i^s \oplus E_i^u$;
(5) $Tf(E_{ix}^\sigma) = E_{if(x)}^\sigma$ for $x \in Z_i^f$;
(6) $\| Tf\dot{x} \| \leqq \lambda \| \dot{x} \|$ for $\dot{x} \in E_i^s | Z_i$, $\| Tf^{-1}\dot{y} \| \leqq \lambda \| \dot{y} \|$ for $\dot{y} \in E_i^u | Z_i$;
(7) $E_{ix}^s \subseteq E_{jx}^s$ and $E_{jx}^u \subseteq E_{ix}^u$ for $x \in Z_i^{f+} \cap Z_j^{f-}$;
(8) E_i^σ is d_f-Lipschitz, i.e., it is given locally as the span of vector fields in $\mathfrak{X}_f(M)$.

Note that an Anosov diffeomorphism is locally Anosov. (Take $k = 1$ and $Z_1 = M$ and choose λ and $\| \ \|$ as in [1].)

THEOREM C. *If f is C^2 and locally Anosov, then $1 - f^\#: \mathfrak{X}_f(M) \to \mathfrak{X}_f(M)$ is split surjective.*

SKETCH OF PROOF. Choose a partition of unity $\theta_1, \cdots, \theta_k$ subordinate to the cover $Z_1^f, \cdots, Z_k^f$. For $\eta \in \mathfrak{X}_f(M)$, $\operatorname{supp}(\theta_i\eta) \subseteq Z_i^f$ and we may resolve $\theta_i\eta$ into components along E_i^s and E_i^u. Thus $\theta_i\eta = \eta_{is} + \eta_{iu}$ where $\eta_{i\sigma}(x) \in E_{ix}^\sigma$ for $x \in Z_i^f$ and $\sigma = s, u$ and $\operatorname{supp}(\eta_{i\sigma}) \subseteq \operatorname{supp}(\theta_i) \subseteq Z_i^f$. Clearly, the linear operator $\eta \rightsquigarrow \eta_{i\sigma}$ is a continuous linear operator on $\mathfrak{X}_f(M)$. We define $J_{i\sigma}(\eta)$ by the "Neumann series":

$$J_{iu}(\eta) = \sum_{n=0}^{\infty} (f^\#)^n(\eta_{iu}), \qquad J_{is}(\eta) = -\sum_{n=1}^{\infty} (f^\#)^{-n}(\eta_{is}).$$

It can be shown that these series converge uniformly to an element of $\mathfrak{X}_f(M)$ (compare with [1]) and that each $J_{i\sigma}$ is a continuous linear operator on $\mathfrak{X}_f(M)$. Clearly

$$(1 - f^\#)J_{i\sigma}(\eta) = \eta_{i\sigma}$$

so we may define $J: \mathfrak{X}_f(M) \to \mathfrak{X}_f(M)$ by

$$J = \sum_{\sigma = s,u} \sum_{i=1}^{k} J_{i\sigma}.$$

Then

$$(1 - f^\#)J(\eta) = \sum_\sigma \sum_i \eta_{i\sigma} = \eta.$$

End of argument.

3. Proof of Theorem A. Let f be a C^2 diffeomorphism which satisfies Axiom A and the strong transversality condition. Theorem A is proved by showing that f is locally Anosov.

Let Ω be the nonwandering set of f and E^s, E^u the invariant splitting of $TM|\Omega$ guaranteed by Axiom A. Write $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ as in the spectral decomposition theorem (see [2]). We say that $\Omega_j \leqq \Omega_i$ iff $Z_j^{f-} \cap Z_i^{f+} \neq \varnothing$ for all neighborhoods Z_i of Ω_i and Z_j of Ω_j. This is a partial ordering (see [2]).

To verify that f is locally Anosov one must extend $E^\sigma|\Omega_i$ $(\sigma = s, u; i = 1, \cdots, k)$ to E_i^σ defined on Z_i^f where Z_i is a neighborhood of Ω_i. This is done via induction on the above partial ordering and the construction is similar to the construction of the "tubular families" of [4].

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