SINGULAR INTEGRAL OPERATORS ON THE UNIT CIRCLE[1]

BY JOEL DAVID PINCUS

Communicated by N. Levinson, August 9, 1966

THEOREM 1. *Let U be unitary and of simple spectral multiplicity and let V be a bounded symmetric operator such that $UV-VU=e(\cdot, U^*e)$ where e is cyclic for U. Then V is unitarily equivalent to the operator L defined by*

$$Lx(\tau) = D(\tau)x(\tau) + \frac{1}{\pi i} \, \mathrm{P}\!\!\int_{\sigma(U)} \frac{k(\tau)k^*(t)}{t-\tau} x(t)dt \qquad [2]$$

where $D(\tau)$ is an essentially bounded real-valued function defined on $\sigma(U)$, the spectrum of U, and $k(\tau)$ is an essentially bounded measurable complex-valued function.

We confine ourselves without essential restriction to the case that $k(t) \neq 0$ almost everywhere on $\sigma(U)$.

Let

$$A(l, z) = \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \frac{e^{i\theta}+z}{e^{i\theta}-z} g(\nu, e^{i\theta}) \frac{d\nu}{\nu - l} d\theta$$

where

$$g(\nu, e^{i\theta}) = \frac{1}{\pi} \arg \frac{D(e^{i\theta}) - \nu - io - |k(e^{i\theta})|^2}{D(e^{i\theta}) - \nu - io + |k(e^{i\theta})|^2} \cdot$$

LEMMA 1.

$$[A(\bar{l}, \bar{z})]^{-1} = A^*\left(l, \frac{1}{z}\right) \cdot$$

LEMMA 2. *Let*

$$\phi(\nu, z) = i \exp \int_{-\pi}^{\pi} g(\nu, e^{i\theta}) \frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta$$

for $|z| < 1$. Then there exists a one-parameter family of positive singular measures, $d\sigma_\nu(\cdot)$, of finite total mass for almost all ν, and a real-valued function $\beta(\nu)$ such that

[1] This work performed under the auspices of the U. S. Atomic Energy Commission.

[2] The complex conjugate of a function T is denoted by T^*.

$$\phi(\nu, z) = i\beta(\nu) + \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma_\nu(\theta).$$

Let $\{P_\nu^{(j)}(\theta)\}_{j=1}^{m(\nu)}$ denote a complete orthonormal set in $L_2(\sigma(U), d\sigma_\nu(\cdot))$ where $m(\nu) \equiv$ dimension of $L_2(\sigma(U), d\sigma_\nu(\cdot))$. Set

$$F_j(\nu, z) = (A(\nu + io, z) \int_{-\pi}^{\pi} \frac{P_\nu^{(j)}(\theta)}{1 - ze^{-i\theta}} d\theta.$$

THEOREM 2 (*Evaluation of the spectral multiplicity of L*). *Let* $M(\nu) = \{e^{i\theta}: g(\nu, e^{i\theta}) = 1\}$. *If* $M(\nu)$ *is the union of n disjoint arcs, then* $m(\nu) = n$; *otherwise,* $m(\nu)$ *is infinite.*

THEOREM 3. *Let*

$$P_\nu(x, y) = \frac{1}{2\pi i} \lim_{\eta \downarrow 0} \left(\frac{k(e^{i\theta})}{1 - xe^{-i\theta}}, (L - \nu - i\eta)^{-1} - (L - \nu + i\eta)^{-1} \frac{k(e^{i\theta})}{1 - ye^{-i\theta}} \right)$$

for $|x| < 1$, $|y| < 1$, *where* $(f, g) = \int_{-\pi}^{\pi} f(e^{i\theta}) g^*(e^{i\theta}) d\theta$.
Then,

$$P_\nu(x, y) = \sum_{j=1}^{m(\nu)} F_j(\nu, x) F_j^*(\nu, y),$$

and

$$\int_{\sigma(L)} P_\nu(x, y) d\nu = \left(\frac{k(e^{i\theta})}{1 - xe^{-i\theta}}, \frac{k(e^{i\theta})}{1 - ye^{-i\theta}} \right).$$

The proof of this theorem follows from a residue calculation and the algebraic relations

$$i\left(\frac{A(\xi + io, x)}{A(\xi + io, y)} - \frac{A(\xi - io, x)}{A(\xi - io, y)} \right) \equiv q(\xi, x, y),$$

$$\frac{1}{2} \frac{1}{1 - x\bar{\omega}} q\left(\xi, x \frac{1}{\bar{\omega}}\right) = \sum_{1}^{m(\xi)} F_j(\xi, x) F_j^*(\xi, \omega)$$

using Lemmas 1 and 2.

This last theorem can be written in the form of an eigenfunction expansion. Thus, set

$$x_j(\xi, \theta) = \frac{1}{k(e^{i\theta})} \lim_{\eta \downarrow 0} [F_j(\xi, (1 + \eta)e^{i\theta}) - F_j(\xi, (1 - \eta)e^{i\theta})]$$

and

$$S_j f(\xi) \equiv \int_{-\pi}^{\pi} f(\theta) x_j^*(\xi, \theta) d\theta,$$

whenever $f(\theta)$ belongs to the domain of the absolutely continuous part of L, the integral existing in the mean square sense, with

$$Sf(\xi) \equiv \{S_1 f(\xi), \cdots, S_{m(\xi)} f(\xi)\}.$$

THEOREM 4. *Let*

$$\{g_1(\xi), \cdots, g_{m(\xi)}(\xi)\} = g(\xi)$$

be a vector in the direct integral Hilbert space $\mathcal{H}^*$ formed with respect to Lebesgue measure and the multiplicity function $m(\xi)$. Let

$$Tg(\theta) = \int_{\sigma(V)} \sum_{1}^{m(\nu)} g_j(\nu) x_j(\nu, \theta) d\nu.$$

Then $ST = 1$ and $TS = 1$, and

$$\int_{\sigma(V)} \sum_{1}^{m(\nu)} |g_j(\nu)|^2 d\nu = \int_{\sigma(U)} |f(e^{i\theta})|^2 d\theta.$$

Furthermore $SLf(\xi) = \xi Sf(\xi)$.

The last two theorems imply that L, and hence V, has an absolutely continuous spectral measure if the spectrum of U is not the entire circle. If the spectrum of U should be the whole unit circle, then the absolutely continuous part of V is diagonalized exactly according to the results presented above. However, in this case when $D(e^{i\theta}) \pm |k(e^{i\theta})|^2 = \xi^{\pm}$ are constant, infinitely degenerate eigenmanifolds corresponding to the eigenvalues $\xi^{\pm}$ can appear. Let us see why this is so. If $y(\xi, \tau)$ is an eigenvector of L, so that $Ly(\xi, \tau) = \xi y(\xi, \tau)$ we have

$$[D(\tau) - \xi] y(\xi, f) + \frac{1}{\pi i} \oint_{-\pi}^{\pi} \frac{k(\tau) k^*(t)}{t - \tau} y(\xi, t) dt.$$

From this, we may conclude that

$$[D(\tau) - \xi + |k(\tau)|^2] \phi(\xi, \tau^+) = [D(\tau) - \xi - |k(\tau)|^2] \phi(\xi, \tau^-)$$

where

$$\phi(\xi, z) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{k^*(t)}{t - z} y(\xi, t)dt.$$

Thus, if $D(\tau) - \xi - |k(\tau)|^2 = 0$, for example, then $\phi(\xi, \tau^+) = 0$. But, by the Plemelj formula this means that

$$\begin{aligned}\phi(\xi, \tau^+) &= \tfrac{1}{2}k^*(\tau)y(\xi, \tau) + \frac{1}{2\pi i} \;\mathrm{P}\!\!\int_{|t|=1} \frac{k^*(t)y(\xi, t)}{t - \tau} dt \\ &= \tfrac{1}{2}(I + H_U)k^*y,\end{aligned}$$

where

$$H_U x(\tau) = \frac{1}{\pi i}\int_{\sigma(U)} \frac{x(t)}{t - \tau} dt.$$

A relatively simple argument now shows that H_U has purely absolutely continuous spectrum if $\sigma(U)$ is not the whole circle—and thus $k^*y = 0$ or $y = 0$ in this case; but H_U has an infinitely degenerate eigenmanifold associated with the eigenvalues -1 and 1 if $\sigma(U)$ is the whole circle.

An application to the theory of self-adjoint Toeplitz matrices. Let $k(e^{i\theta})$ be positive and integrable on $(-\pi, \pi)$. Then if P is the orthogonal projector from $L_2(-\pi, \pi)$ to the Hardy space $\mathcal{H}^2$, we can represent the Toeplitz operator in the form $Tf = Pkf, f \in \mathcal{H}^2$ [1].

If $f \in \mathcal{H}^2$, then $kf \in L_2$ and, in the sense of mean convergence

$$k(e^{i\theta})f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

where

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$$

but

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{2\pi i}\int_{|t|=1} \frac{k(t)f(t)}{t - z} dt, \quad |z| < 1.$$

Thus

$$\lim_{r \uparrow 1} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} = Pk(e^{i\theta})f(e^{i\theta}).$$

But the Plemelj formula can be used to evaluate this limit, and we obtain

$$Tf = \frac{1}{2} k(e^{i\theta})f(e^{i\theta}) + \frac{1}{2\pi i} \mathrm{P}\int_{|\tau|=1} \frac{k(\tau)f(\tau)}{\tau - \exp(i\theta)} d\tau.$$

Let T as written on the right-hand side above be considered as a self-adjoint operator on $L_2(k) = L_2(-\pi, \pi; k(e^{i\theta})d\theta)$. Let us denote the closure in this space of finite linear combinations of the form $\sum_{n=0}^{N} b_n e^{in\theta}$ by $\mathcal{H}_k^2$.

Then, if $x \in (\mathcal{H}_k^2)^\perp$,

$$(Tx, y)_{L_2(k)} = (x, Ty)_{L_2(k)} = 0, \qquad \forall y \in L_2(k).$$

Thus $(\mathcal{H}_k^2)^\perp$ is the null manifold of T.

The reader will see that the spectral analysis of T restricted to $\mathcal{H}_k^2$ is carried out immediately by an easy application of the results of the preceding section. For a different approach see [2], [3].

Final Remark. The proof of our main result is carried out by means of a reduction to a new general theory of singular Riemann-Hilbert boundary value problems:

$$\phi^+(\xi, \lambda) = G(\xi, \lambda)\phi^-(\xi, \lambda)$$

where

$$G(\xi, \lambda) = 1 + \int_{-\infty}^{\infty} \frac{dM_\lambda(\nu)}{\nu - \xi - io},$$

and $dM_\lambda(\cdot)$ is a one-parameter purely singular positive measure [4]. This reduction also makes it possible to give a much more transparent deduction of the author's previous results about singular integral equations on the line [5]. Furthermore, it leads to a spectral theory for self adjoint coupled systems of singular integral equations [6].

References

1. P. R. Halmos, *A glimpse into Hilbert space*, Vol. 1, Lectures on Modern Mathematics, (T. Saaty, ed.), Wiley, New York, 1963.

2. R. S. Ismagilov, *The spectrum of Toeplitz matrices*, Dokl. Akad. Nauk SSSR **150** (1963), 769–772 = Soviet Math. Dokl. **4** (1963), 462–465.

3. Marvin Rosenblum, *A concrete spectral theory for self-adjoint Toeplitz operators*, Amer. J. Math. **87** (1965), 709–718.

4. J. D. Pincus, *A singular Riemann-Hilbert problem*, 1965 Summer Institute on Spectral Theory and Statistical Mechanics, Brookhaven National Laboratory, Upton, N. Y.

5. ———, *Commutators, generalized eigenfunction expansions and singular integral operators*, Trans. Amer. Math. Soc. **121** (1966), 358–377.

6. ———, *Commutators and systems of singular integral equations*, (to appear).

Brookhaven National Laboratory