HEREDITARILY RETRACEABLE ISOLS[1]

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In [3], it was shown that the recursive isomorphism type of a co-r.e. retraceable set may contain nonretraceable sets. The question naturally arises whether *every* nonrecursive, co-r.e. retraceable set gives rise, under some recursive permutation, to a nonretraceable set. It is an obvious corollary to the first theorem announced in this note that the answer is "no."

If an infinite retraceable set has no nonretraceable regressive subsets, we term it *hereditarily retraceable*; a *hereditarily retraceable isomorphism type* is then a recursive isomorphism type consisting exclusively of such sets. Likewise, a *hereditarily retraceable isol* is a recursive equivalence type each member of which is an immune retraceable set having no nonretraceable regressive subsets. Such, then, are the objects referred to in the title of the note; the existence of a continuum of them follows from Theorem 1 below. Our terminology is, in all other respects, that of [1], [2].

THEOREM 1. *If α is an infinite retraceable set, then α has an infinite retraceable subset β with the following property:*

$(\forall f)$*(f a one-one partial recursive function$\Rightarrow$every regressive subset of $f(\beta)$ is retraceable.)*

Moreover, if α has recursively enumerable complement then we can satisfy the additional requirement that β have recursively enumerable complement.

The proof, which will appear in detail elsewhere, is accomplished by means of a simple priority scheme which (a) applies to any retracing function, and (b) is designed to exploit the following lemma (the truth of the lemma is obvious):

LEMMA 0. *If an infinite set β of natural numbers is regressed by the partial recursive function f, then either β is retraceable or there are infinitely many numbers $b \in \beta$ such that $f(b) > b$.*

REMARK 1.5. With regard to the analogy

$$\frac{\text{recursive}}{\text{r.e.}} = \frac{\text{retraceable}}{\text{regressive}}$$

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suggested in [2], we observe that the existence of hereditarily retraceable sets is a negative result of a rather crippling sort; for the theorem stating that every infinite recursive set has a nonrecursive r.e. subset is surely as basic a result as can be found in the theory of r.e. sets.

We conclude the note with a theorem which is, in part, an application of Theorem 1, and which concerns the question of a regressive extension of [1, Proposition P4]: Proposition P4 of [1] asserts that the composition of the principal functions (i.e., the order-of-magnitude enumeration functions) of two infinite retraceable sets enumerates a retraceable set. In Theorem 2 we shall consider not only composition of *principal* functions of regressive sets, but also (and doubtless more naturally for the class of regressive sets) *composition of regressive functions*. (The terms *regressive function* and *retraceable function* signify as in [2]; note that if α is an infinite retraceable set then the notions *principal function of* α and *retraceable function with range* α coincide.)

THEOREM 2. (i) *If f is an everywhere-defined regressive function and g is an everywhere-defined retraceable function, then fg (defined by $(fg)(x) = f(g(x))$) is a regressive function. However:*

(ii) *There exist retraceable functions h with the following property: $(\forall g)(\exists \alpha)(\forall f)$ [g an everywhere-defined retraceable function$\Rightarrow$(α is a regressive set such that if f is a regressive function with range α then hgf is not regressive)];*

(iii) *There exist retraceable functions h with the following property: $(\forall g)(\exists f)$ [g an everywhere-defined retraceable function$\Rightarrow$(f is the principal function of a regressive set and hgf is not regressive)];*

(iv) *There exist functions f and g such that g is the principal function of a retraceable set, f is the principal function of a regressive set, and fg does not enumerate a regressive set.*

Part (i) of Theorem 2 is trivial to verify, and part (iv) is established by an easy *ad hoc* construction which we shall not describe here. Parts (ii) and (iii) are obtained from Theorem 1 in the following manner. Let γ be an infinite retraceable set, and let g be the retraceable function enumerating γ (i.e., g is the principal function of γ). We shall indicate the proof for (ii); the argument on behalf of (iii) is virtually identical with that for (ii). Applying Theorem 1, let β be a hereditarily retraceable set (any one will do); let h be the retraceable function whose range is β. Let D be the degree of unsolvability of the range of the composite function hg. Now, if we look carefully at the proof of Theorem T5 of [1], and take into account the fact ([2]) that recursively separable, recursively equivalent retraceable sets have regressive union, we see (with the help of a trivial countability argument) that the following assertion is true: there exists a regressive set α such that α is not retraceable in D (i.e., α is not retraced by a function partial recursive in D).[2] Let f be a regressive function with range α. Suppose hgf were regressive. Then, since the range of h is hereditarily retraceable, hgf would have to enumerate a retraceable set. But then, by using partial recursive functions r and t such that r retraces range(hg) and t retraces range(hgf), we could obtain a function p, partial recursive in D, such that p retraces α: contradiction.

Thus, each of the two obvious candidates for a direct regressive extension of [1, Proposition P4] turns out to be a rank falsehood. On the other hand it is easy, on the basis of [1, Proposition P4] and [2, Proposition 5], to define a binary operation of "functional composition" on the class of regressive *isols*. (Note that by [2, Proposition 3] it is optional whether we regard a regressive isol as a recursive equivalence type whose members are regressive *sets*, or as one whose members are regressive *functions*.)

REFERENCES

1. J. C. E. Dekker and J. Myhill, *Retraceable sets*, Canad. J. Math. **10** (1958), 357–373.

2. J. C. E. Dekker, *Infinite series of isols*, Proc. Sympos. Pure Math. Vol. **5**, Amer. Math. Soc., Providence, R. I., 1962, pp. 77–96.

3. T. G. McLaughlin, *Retraceable sets and recursive permutations*, Proc. Amer. Math. Soc. **17** (1966), 427–429.

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[2] It is a straightforward matter to define such a function from g's to α's without any use of the axiom of choice.