MODULAR REPRESENTATION ALGEBRAS[1]

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Let G be a cyclic p-group, K a field of characteristic p, and KG the group algebra of G over K. The *representation ring* $a(KG)$ is generated by symbols $[M]$, one for each isomorphism class $\{M\}$ of finitely generated left KG-modules, with relations

$$[M] + [M'] = [M \oplus M'], \quad [M][N] = [M \otimes_K N].$$

The *representation algebra* $A(KG)$ is defined as $C \otimes_Z a(KG)$, where Z is the ring of rational integers, C the complex field. The aim of this note is to give a simple proof of the following theorem of Green [**1**].

THEOREM. *The representation algebra $A(KG)$ is semisimple.*

Since G is a cyclic p-group, the algebra $A(KG)$ is finite dimensional (and commutative), having C-basis $\{v_1, \cdots, v_q\}$, where $q = [G:1]$, and where $v_r = [V_r]$. Here, V_r denotes the unique indecomposable KG-module of dimension r. We set $A_0 = R \otimes_Z a(KG)$, where R is the real field. Then $A(KG) = C \otimes_R A_0$, and it suffices to prove that A_0 is semisimple, or equivalently, that A_0 has no nonzero elements of square zero.

By the *components* of a module we mean the indecomposable summands in a direct sum decomposition of the module.

LEMMA 1 (ROTH [**4**], RALLEY [**3**]). *The number of components of $V_r \otimes V_s$ is precisely* $\min(r, s)$.

PROOF. Let H_r be the $r \times r$ matrix with 1's above the main diagonal and zeros elsewhere, let E_r be the $r \times r$ identity matrix, and let λ be an indeterminate over K. Then the number of components of $V_r \otimes V_s$ is the same as the number of invariant factors of $(\lambda E_r + H_r)^s$ different from 1. This easily yields the desired result.

Let us write

$$v_r v_s = \sum_{t=1}^{q} a_{rst} v_t, \qquad 1 \leqq r, s \leqq q.$$

Then the coefficients $\{a_{rst}\}$ are nonnegative integers, and Lemma 1 asserts that

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$$\sum_{t=1}^{q} a_{rst} = \min (r, s), \qquad 1 \leqq r, s \leqq q.$$

LEMMA 2. *The quadratic form*

$$\sum_{r,s=1}^{q} \min (r, s) X_r X_s$$

is positive definite.

PROOF. One verifies that the given form coincides with $(X_1 + \cdots + X_q)^2 + (X_2 + \cdots + X_q)^2 + \cdots + X_q^2$.

We now show that if $u \in A_0$ satisfies $u^2 = 0$, then necessarily $u = 0$. Write $u = \sum_{r=1}^{q} \alpha_r v_r$, $\alpha_r \in R$. Then

$$0 = u^2 = \sum_{r,s} \alpha_r \alpha_s v_r v_s = \sum_{r,s,t} \alpha_r \alpha_s a_{rst} v_t,$$

whence

$$\sum_{r,s} \alpha_r \alpha_s a_{rst} = 0, \qquad 1 \leqq t \leqq q.$$

Summing on t, we obtain

$$\sum_{r,s} \min (r, s) \alpha_r \alpha_s = 0,$$

so by Lemma 2, $\alpha_r = 0$ for $1 \leqq r \leqq q$. This completes the proof.

The above technique has also been used by Hannula [2].

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