SMOOTH AND PIECEWISE LINEAR SURGERY

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Communicated by J. Milnor, June 24, 1966

We shall be concerned here with developing some new techniques of surgery on a map in both the smooth and p.l. (piecewise linear) categories. In [11] these tools are applied to the problem of deforming a homotopy equivalence between two piecewise linear manifolds until it is a piecewise linear homeomorphism, and the Hauptvermutung (which claims that topologically homeomorphic p.l. manifolds are piecewise linearly homeomorphic) is answered affirmatively for a large class of manifolds.[2]

The general problem of surgery on a map may be described as follows: Consider a finite CW pair $(X, \partial X)$ satisfying Poincaré duality in dimension n (cf. [12]), a k-dimensional bundle $E_x \to X$, and a map f: $(W, \partial W) \to (V, \partial V)$ from a compact $(n+k)$-manifold $(W, \partial W)$ to a pair of spaces $(V, \partial V)$ containing $(E_x, E_x | \partial X)$ as an open subpair. When can f be deformed (as a map of pairs) until it is transverse regular on $X \subset V$ in such a way that $(M, \partial M) = (f^{-1}(X), f^{-1}(\partial X)) \subset (W, \partial W)$ is an n-submanifold with normal bundle $E_m = f^* E_x$ and f: $(M, \partial M) \to (X, \partial X)$ is a homotopy equivalence covered by the bundle map f: $E_m \to E_x$? If this can be done, we say that f may be deformed until it is *h-regular* on X. If $f | \partial W$ is already h-regular on ∂X, when can f be deformed rel ∂W (i.e. keeping $f | \partial W$ fixed) until it is h-regular on X? For example, the pioneering work of Browder [1] and Novikov [10] dealt, in the smooth category, with the case where $W = S^{n+k}$, $V = TE_x =$ the Thom space of E_x, and $f =$ a map of degree one. Wall's generalization in [12] of the Browder-Novikov results uses $(W, \partial W) = (D^{n+k}, S^{n+k-1})$, $(V, \partial V) = (T(E_x), T(E_x | \partial X))$, and a map of degree one. Our main interest will be when f is a homotopy equivalence and the codimension k is different from two. Furthermore, the surgery results are true in both the smooth and p.l. categories, and the theorems are stated simultaneously for both cases. Thus all maps, manifolds, bundles, hypotheses, conclusions, etc. should be interpreted as all smooth or all p.l. unless the category is explicitly pinned down. For an exposition of the p.l. category, p.l. microbundles, and p.l. transverse regularity see [13]. Recall that by the Kister-Mazur Theorem every p.l. microbundle contains a unique *p.l. bundle*—a

[1] Supported in part by National Science Foundation Grant GP-5804.

[2] These results are the content of the author's Ph.D. Thesis directed by W. Browder at Princeton University.

fibre bundle with fibre R^k and group the set of p.l. homeomorphisms of R^k onto itself keeping the origin fixed.

The procedure in the proof of the surgery theorems is to first make f transverse regular on X so that

$$(*) \qquad \begin{array}{l}(M, \partial M) = (f^{-1}(X), f^{-1}(\partial X)) \subset (W, \partial W) \text{ is an} \\ n\text{-submanifold with normal bundle } E_m = f^*E_x.\end{array}$$

Then we add handles up through the middle dimension to M and ∂M inside $W\times[0, 1]$ and $\partial W\times[0, 1]$ and deform f, preserving (*) at each step, so as to make M and ∂M 1-connected and kill the kernels of the homology maps $f_*: H_*(M)\to H_*(X)$ and $f_*: H_*(\partial M)\to H_*(\partial X)$. Whitehead's Theorem [5, p. 113] then implies that f is a homotopy equivalence. Of course, if $f|\partial W$ is already h-regular on ∂X, handles are added only to M and $f|\partial W$ is kept fixed by the deformation. To summarize the main results, when the codimension k is equal to one and E_x is trivial, f can always be made h-regular under reasonable circumstances; however, when $k\geqq 3$ and $f|\partial W$ is to be kept fixed there is in general an obstruction, which can nonetheless be avoided when $\partial X\neq 0$ and $f|\partial W$ is allowed to vary.

THEOREM 1.1. *Suppose that* $(V, \partial V)=(Y\cup Z, Y_0\cup Z_0)$ *with* $(X, \partial X)=(Y\cap Z, Y_0\cap Z_0)$, $E_x\to X$ *is the trivial line bundle*, $\pi_1(X)=\pi_1(V)=0$; *and if* n *is even, either* $\pi_2(Y, X)=0$ *or* $\pi_2(Z, X)=0$. *Assume* $n\geqq 5$ *and* $H_n(Y, Y_0)=H_n(Z, Z_0)=0$. *If* $f:(W, \partial W)\to(V, \partial V)$ *is a homotopy equivalence having* $f|\partial W$ *already h-regular on* ∂X, *then* f *may be deformed* rel ∂W *until it is h-regular on* X.

REMARK. This result with the added condition that for all n both relative π_2's vanish was first proved in [2] by W. Browder. Our approach is similar to [2]. D. Sullivan has recently eliminated all the π_2 restrictions in the hypothesis using methods of surgery in the stable range.

THEOREM 1.2. *Suppose* $n\geqq 5$, $k\geqq 3$, V *and* X *are* 1-*connected*, $H_{n+k-1}(V-X, \partial V-\partial X)=0$, *and* f *is a homotopy equivalence with* $f:\partial W\to\partial V$ *already h-regular on* ∂X. *Then there is an obstruction*

$$\begin{aligned} c^n(f) &\in 0, && n \text{ odd} \\ &\in Z, && n\equiv 0 \bmod 4 \\ &\in Z_2, && n\equiv 2 \bmod 4 \end{aligned}$$

depending only on the homotopy class rel ∂W *of* f, *which vanishes iff* f *may be deformed* rel ∂W *until it is h-regular on* X.

For $n \equiv 0$, 2 mod 4 c^n may be described geometrically: Make f the transverse regular rel ∂W on X and let $M = f^{-1}(X)$. Then for any ring Λ the kernel $K_*(M; \Lambda)$ of the homology map $f: H_*(M; \Lambda) \rightarrow H_*(X; \Lambda)$ and the cokernel $K^*(M; \Lambda)$ of the cohomology map $f^*: H^*(X; \Lambda) \rightarrow H^*(M; \Lambda)$ satisfy Poincaré duality. If $n = 4l$, there is a nonsingular bilinear pairing $\sigma: K_{2l}(M; Q) \otimes K_{2l}(M; Q) \rightarrow Q$, and $c^{4l}(f) =$ signature of σ divided by 8. If $n = 4l+2$, $K_{2l+1}(M; Z)$ can be made free abelian by surgery and there is a map $\Phi: K_{2l+1}(M; Z) \rightarrow Z_2$ satisfying $\Phi(x+y) = \Phi(x) + \Phi(y) + x \cdot y$ mod 2. $c^{4l+2}(f) =$ Arf invariant of Φ (cf. [14]).

As a corollary to the proof that c^n has such a geometric interpretation there is

ADDENDUM 1.3. *$f: (W, \partial W) \rightarrow (V, \partial V)$ can be made h-regular* rel ∂W *on X* iff *$f \times id: (W, \partial W) \times (D^k, \partial D^k) \rightarrow (V, \partial V) \times (D^k, \partial D^k)$ can be made h-regular* rel $\partial W \times D^k \cup W \times \partial D^k$ *on X for $k \geq 3$.*

REMARK 1. It is possible that $\partial X = 0$, in which case $X \subset V - \partial V$.

REMARK 2. J. Levine [8] first did surgery below the metastable range on framed, smooth manifolds inside products of spheres. W. Browder in [2] has a very comprehensive theory of smooth surgery in low codimensions.

The obstruction c^n is a "cobordism invariant" in the sense of

PROPOSITION 1.4. *Suppose $n \geq 6$, $k \geq 3$; $V, \partial V, X, \partial X$ are* 1-*connected*; *$H_{n+k-1}(V - X, \partial V - \partial X) = 0$; and f is a homotopy equivalence. Then the obstruction $c^{n-1}(f|\partial W)$ to making $f: \partial W \rightarrow \partial V$ h-regular on ∂X is zero.*

Another situation where $c^n(f)$ vanishes is when $n \equiv 0$ mod 4, V is an $(n+k)$-manifold, and f is a stable tangential equivalence.

Applying Theorem 1.2 to the special case $(W, \partial W) = (V, \partial V) = (D^n \times D^k, \partial(D^n \times D^k))$ and $(X, \partial X) = (D^n \times 0, S^{n-1} \times 0)$ gives a computation of the homotopy groups $\pi_n(G_k, \mathrm{SPL}_k)$ for $n \leq k-2$. Recall that G_k is the semisimplicial complex of degree one maps of $R^k - 0$ to itself and SPL_k is the subcomplex determined by the p.l. homeomorphisms of R^k onto itself keeping the origin fixed. G_k may be equivalently described as the complex of degree one maps of (D^k, S^{k-1}) to itself keeping 0 fixed. Let $\mathrm{SP}\Lambda_k \subset G_k$ be the subcomplex of degree one p.l. homeomorphisms of D^k onto itself. Then the stability theorems of Haefliger-Wall in [4] imply that the map $\pi_n(G_k, \mathrm{SP}\Lambda_k) \rightarrow \pi_n(G_k, \mathrm{SPL}_k)$ induced by radial extension is an isomorphism for $n \leq k-2$. Now consider the set of degree one maps f of the triple $D = (D^n \times D^k; S^{n-1} \times D^k, D^n \times S^{k-1})$ to itself so that $f: S^{n-1} \times D^k \rightarrow S^{n-1} \times D^k$ is a p.l. homeomorphism onto and $f|S^{n-1} \times 0$ is the identity. f is not necessarily fibre preserving. Two such maps f_0 and f_1 are said to be *equivalent* provided there is a map $\bar{f}$ of $(D\times[0, 1]; D\times 0, D\times 1)$ to itself for which $\bar{f}|D\times j=f_j(j=0, 1)$, $\bar{f}|S^{n-1}\times 0\times[0, 1]=\text{identity}$, and $\bar{f}$ is a p.l. homeomorphism of $S^{n-1}\times D^k\times[0, 1]$ onto itself. Denote the resulting set of equivalence classes by $H^{n,k}$, which becomes a group under composition. Now if the class $[f]\in H^{n,k}$ $(n\geqq 5, k\geqq 3)$ is represented by a map $f:D\to D$, then $f|\partial D^{n+k}$ is h-regular on $S^{n-1}\times 0$ and the obstruction $c^n(f)$ to making f h-regular rel ∂D^{n+k} on D^n is defined by Theorem 1.2.

THEOREM 1.5. *For $n\geqq 5$, $k\geqq 3$ the correspondence $f\to c^n(f)$ defines a homomorphism*

$$\begin{aligned} c^n: H^{n,k} &\to 0, && n \text{ odd} \\ &\to Z, && n\equiv 0 \bmod 4 \\ &\to Z_2, && n\equiv 2 \bmod 4. \end{aligned}$$

In fact c^n is an isomorphism (assuming $k>n/2+1$ whenever $n=6$ or 14).

If $n\leqq k-2$ it follows from [4] that

LEMMA 1.6. *The homomorphism $\pi_n(G_k, \mathrm{SPA}_k)\to H^{n,k}$ is an isomorphism.*

COROLLARY 1.7. *For $n\leqq k-2$*

$$\begin{aligned} \pi_n(G_k, \mathrm{SPL}_k) = \pi_n(G_k, \mathrm{SPA}_k) &= 0, && n \text{ odd } or\ n=0 \\ &= Z, && n\equiv 0 \bmod 4 \ and\ n\geqq 4 \\ &= Z_2, && n\equiv 2 \bmod 4. \end{aligned}$$

REMARK. When $n\geqq 5$, 1.7 follows immediately from 1.5 and 1.6. For $0\leqq n\leqq 4$ special arguments must be used. For example, if $n=4$, let $f:D\to D$ represent the class $[f]\in\pi_4(G_k, \mathrm{SPA}_k)$ and deform f rel ∂D^{4+k} until it is transverse regular on $D^4\times 0$. Then $[f]\to$index $f^{-1}(D^4\times 0)/16$ gives the desired isomorphism.

The next theorem was inspired by [12, Theorem 2], which concerns the h-regularity problem for maps $f:(D^{n+k}, \partial D^{n+k})\to(T(E_x), T(E_x|\partial X))$ of degree one.

Suppose that $(P, \partial P)$, (Q, S), (R, S) are finite CW pairs with $\partial P=Q\cup R$ and $Q\cap R=S$; and that $(P; Q, R)$ is a triple satisfying Poincaré duality in dimension n (i.e. $H^i(P, R)=H_{n-i}(P, Q)$) while (Q, S) and (R, S) are Poincaré pairs in dimension $n-1$. Possibly $S=0$ or one of Q or $R=0$; but not both $Q=R=0$! As usual let $E_p\to P$ be a k-bundle which is neighborhood of $(P, \partial P)\subset(V, \partial V)$ and let $f:(W, \partial W)\to(V, \partial V)$ be a map. Then $\partial f=f:\partial W\to\partial V$ is said to be *h-regular with respect to* (R, S) provided there is an $(n-1)$-manifold $C\subset\partial W$ (with $\partial C=D$), a regular neighborhood U of C mod D in ∂W (cf. [7]) and a normal bundle ∂E_c of $(C, D)\subset(U, \partial U)$ so that

(i) $(C, D)=\partial f^{-1}(R, S)\cap U$ and $\partial f:(C, D)\to(R, S)$ is a homotopy equivalence,

(ii) $\partial f:\partial E_c\to\partial E_r=E_p|R$ is a bundle map, and

(iii) $\partial f(\partial W-\text{int }U)\subset\partial V-\text{int }R$.

If $R=0$, then these conditions are already satisfied.

THEOREM 1.8. *Suppose f is a homotopy equivalence; V, ∂V, P, and $Q\neq 0$ are simply connected; $H_{n+k-1}(V-P, \partial V-\partial P)=0$; and ∂f is h-regular with respect to (R, S). Let $n\geqq 6$ and $k\geqq 3$ unless $n=6$ or 14, in which case $k>n/2+1$. Then there is a deformation*

$$f_t\colon (W, \partial W, \partial W - \text{int } U) \to (V, \partial V, \partial V - \text{int } R)$$

of f which is kept fixed on U so that f_1 is h-regular on P. In particular, if $(A; B, C, D)=f_1^{-1}(P; Q, R, S)$, then $f_1:(A; B, C, D)\to(P; Q, R, S)$ is a homotopy equivalence.

REMARK 1. A useful case to bear in mind is where $R=0$. Then a first application of the theorem produces an n-manifold $(A, \partial A)\subset(W, \partial W)$ together with a homotopy equivalence $f_a:(A, \partial A)\to(P, \partial P)$. A second application with (abusing notation momentarily) $(W, \partial W)=(W, \partial W)\times(I, \partial I)$, $(V, \partial V)=(V, \partial V)\times(I, \partial I)$, $P=P\times I$, $Q=\partial P\times I$, and $R=P\times 0\cup P\times 1$ implies uniqueness up to h-cobordism of such $(A, \partial A)$ and f_a. Here $I=[0, 1]$.

REMARK 2. If $(V, \partial V)=(P, \partial P)\times(D^k, \partial D^k)$, then Theorem 1.8 may be regarded as the analogue in the homotopy category of the Cairns-Hirsch Smoothing Theorem of [6].

Up to now we have been concerned with making homotopy equivalences h-regular; this is "surgery without Thom complexes." Historically, in the Browder-Novikov theory of surgery, h-regularity was first studied in the stable range ($k\gg n$) for maps $f:S^{n+k}\to T(E_x)$ of degree one. This is "surgery with Thom complexes." For example, it was shown in [1] that, for a 1-connected Poincaré duality space X^n, if

(**) $n\geqq 5$ is odd or if $n=4l\geqq 8$ and Index $X^n=L$-genus of the dual Pontryagin classes of E_x,

then f can be made h-regular. Next, Levine in [9] and Wall in [12] observed the same to be true in the metastable range $k>n/2+1$. However, for $k\leqq n/2+1$ there are other obstructions besides Index $X^n=L$-genus. We have

PROPOSITION 1.9. *Suppose* (**) *is satisfied and* $n=4r$, $k=2r+1$. *Then there is an integer* $l(f)$, *depending only on the homotopy class of* f, *which vanishes iff* f *may be deformed until it is h-regular on* X^n.

REMARKS. The obstruction $l(f)$ is a generalization of Haefliger's linking number invariant of [3]. If E_x is the trivial $(2r+1)$-bundle over $X^{4r}=S^{2r}\times S^{2r}$, then any integer may be realized by such an $l(f)$. Finally, in contrast to Proposition 1.9, W. Browder in [2] has shown that all the possible obstructions to h-regularity in low codimension vanish under suspension: if $f:S^{n+k}\to TE_x$ has degree one, $k\geqq 2$, and (**) holds, then $Sf:S^{n+k+1}\to T(E_x\oplus\epsilon^1)$ can be deformed to h-regular position on X^n.

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