SOME CLASSICAL THEOREMS ON OPEN RIEMANN SURFACES[1]

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There are many approaches to the study of open Riemann surfaces. I shall mention only three of these. First, one can ask how to generalize the classical theory of compact Riemann surfaces; that is, the theory of algebraic curves over the complex numbers. My talk will be concerned with this approach. Secondly, one can ask how much of the classical theory of meromorphic functions in the plane or unit disk carries over to more general domains. I shall not be concerned with this problem today although perhaps this is a more reasonable approach than the first, since open surfaces do not really seem a proper object for algebraic investigation. (This, however, will not prevent me from speaking on the subject.) Thirdly, one can deal with the problem of classification of surfaces. This topic is, I think, almost unavoidable in any discussion of open surfaces since it is difficult to make general statements which do not trivialize for some important class of surfaces. This will be particularly true for theorems with algebraic origins, although there are notable exceptions. Theorems concerning periods of differentials will make little sense in the context, say, of the unit disk. Consequently, I shall have to discuss to some extent the classification problem in order that you understand the types of surfaces where one can reasonably hope for analogues of theorems from classical algebraic geometry.

The classical theorems I want to discuss are the following: Abel's theorem, the Riemann-Roch theorem, and the theorem of Torelli. Let me remind you of the classical theorems in a form that seems most easily generalized. The classical theory may be said to start with the observation that the only functions meromorphic on the Riemann sphere are the rational functions. On the Riemann sphere we may prescribe the zeros and poles of a rational function subject only to the restriction that the numbers of zeros and poles be the same if we adopt the usual conventions when counting multiple values.

If one considers the field of meromorphic functions on a compact Riemann surface, then, algebraically, this field is a finite (algebraic)

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extension of the field of rational functions. Each nonconstant meromorphic function on a compact surface takes each value the same number of times, and, in fact, represents the surface as a finite sheeted branched covering of the Riemann sphere. However, when we ask about the possibility of prescribing zeros and poles on a nonsimply connected compact surface, we discover that we have lost some freedom. Already, in the case of elliptic functions, which are meromorphic functions on a torus, we know that the difference between the zeros and poles in a fundamental parallelogram must be some period of the elliptic function. This first case of Abel's theorem indicates that to understand the problem of prescribing zeros and poles, one must consider objects other than the functions; one must consider differentials, or if you like what amounts to the same thing, one must consider multi-valued functions. Abel's theorem concerns three types of mathematical objects: meromorphic functions, abelian differentials, and divisors. Let me talk a little bit about differentials and divisors.

By an abelian (or meromorphic) differential one means a differential that is locally the differential of a meromorphic function. (I am really talking about abelian differentials of the first and second kind.) The integrals of such abelian differentials can be thought of as multi-valued functions. While it makes no sense to talk about the value of a differential at a point of a Riemann surface, it does make sense to talk about the zeros and poles of an abelian differential. If an abelian differential has no poles, it is said to be of the first kind, or regular. On a compact Riemann surface, there are no regular single-valued nonconstant analytic functions; however, if the genus is greater than zero, there are everywhere regular multi-valued functions whose differentials, the single-valued mathematical objects, form the vector space of abelian differentials of the first kind. This vector space has dimension g, the genus of the surface involved. Denote this space by Γ_a Let me remark here, that a very convenient way of forming meromorphic functions is to take the quotient of two linearly independent abelian differentials, since this will be a single-valued function.

By a divisor on a surface we mean nothing more than a zero chain; that is, a finite set of points with an integer associated with each point. For a meromorphic function its zeros and poles form a divisor where the associated integer is positive or negative depending on whether the point is a zero or a pole, and the value of the integer is the multiplicity. For an arbitrary divisor, the sum of the associated integers is said to be its degree. Clearly, the degree of the divisor of a meromorphic function is zero, since the numbers of zeros and poles are equal.

Before stating Abel's theorem let me talk a little more about compact surfaces. Topologically, a compact surface of genus g is a sphere with g handles.[3] One can form a basis for the first homology group by taking a pair of curves associated with each handle. If we label the handles $1, 2, \cdots, g$, then we have cycles A_k and B_k for the kth handle. They intersect each other but no other curves in the homology basis. A basis of this kind, of which there are many, will be called a canonical homology basis. An abelian differential (of the first or second kind) will have well-defined periods around any cycle.

For a torus, the genus is one. If we consider a torus as a period parallelogram of an elliptic function with opposite sides identified, the one abelian differential of the first kind is dz. The periods are precisely the integrals of dz over the sides of the period parallelogram. These sides form an A and a B cycle.

Abel's theorem answers the following question. On a compact Riemann surface, when is a given divisor of degree zero the zeros and poles of a meromorphic function; that is, the divisor of a meromorphic function? The answer is roughly as follows. Since the given divisor D has degree zero, we can join the points with positive coefficients to those with negative coefficients with curves to form a one-chain γ. The boundary, of this one-chain, $\partial\gamma$, is D in the sense of singular homology. Abel's theorem says that given γ so that $\partial\gamma = D$, then there must be a one-*cycle*, c, that is, a one-chain whose boundary is zero, so that to integrate any abelian differential of the first kind α over γ gives the same result as integrating α over c. Thus, over the one-chain $\gamma - c$, any abelian differential of the first kind must have zero period. Let me state this precisely now.

ABEL'S THEOREM. *Let W be a compact Riemann surface. Let D be a divisor of degree zero. D is the divisor of a meromorphic function if and only if there is a one-chain γ so that $\partial\gamma = D$ and $\int_\gamma \alpha = 0$ for all abelian differentials of the first kind, α.*

Since the second condition is vacuous in case W is the Riemann sphere, the theorem reduces to the fact that any divisor of degree zero is the divisor of a rational function.

If one is perverse enough to refuse to form meromorphic functions by taking quotients of differentials, one can still use Abel's theorem and the fact that the dimension of Γ_a is finite to prove the existence or meromorphic functions on compact surfaces. This remark is silly when made in the classical context, but it does have some relevance to the open case when the genus is infinite.

[3] I assume the surface is oriented.

I shall not explain the Riemann-Roch theorem in detai . Its statement is more complicated. Let me say that it involves abelian differentials (even of the third kind), meromorphic functions, divisors, and the finite genus, g, of the surface. It is definitely not silly to remark that the theorem provides existence theorems for meromorphic functions with prescribed singularities. These existence theorems do, however, depend on the genus being a finite number. Since the usual statement of the Riemann-Roch theorem involves explicitly the genus of the surface, one might wonder how one can hope to generalize this to a surface of infinite genus. However, you will recall that in some proofs of the theorem, the final formula is a consequence of the equality of dimension of two spaces of differentials. Since the genus does not enter explicitly into the definition of these vector spaces, one can hope to find a generalization at this point.

Now we wish to generalize Abel's theorem to open Riemann surfaces. My remarks will have most pertinence to the situation of infinite genus. A sphere with a countable infinity of handles, which become very small, or the surface of an infinite ladder furnish examples of such surfaces.

Another instructive example is the following. Let $\{a_n\}$ be a sequence of distinct complex numbers converging to the origin. Join the pairs a_{2n-1}, a_{2n} by disjoint slits. Now cut the Riemann sphere along each of these slits. Join two copies of such a multiple slit sphere by cross-identifying across corresponding slits in the usual way. In this way we obtain a two-sheeted branched covering of the Riemann sphere minus the origin, with branch points of order two above each of the a_n's. Had we chosen but $2g+2$ such points, a_n, the resulting branched covering of the whole Riemann sphere would have been a compact hyperelliptic surface of genus g. Since the branch points converge to the origin in the case of an infinity of the a's, there are no manifold points on the surface corresponding to the origin. The resulting surface is of infinite genus and is called a transcendental hyperelliptic surface.

In dealing with one-cycles on an open surface, it is necessary to distinguish between dividing and nondividing cycles. A simple closed curve on a surface is said to be a dividing cycle if, when we cut the surface along the curve, the surface is, indeed, divided into two pieces. Otherwise, a simple closed curve will be called a nondividing cycle.

At first glance, the project of generalizing any theorem concerning meromorphic functions to open surfaces might seem futile because of the work of Behnke and Stein [**3**]. They showed that on an open Riemann surface one can have an everywhere regular analytic function with any discrete set of zeros. However, results have been obtained by putting restrictions on the functions and differentials under consideration. I will describe some of these restrictions.

The regular analytic differentials on open surfaces to be considered will be Dirichlet bounded [2, Chapter V]; that is, if α is an everywhere regular analytic differential (one-form), then α will be said to be Dirichlet bounded if the Dirichlet integral $\int\int_W \alpha \wedge {}^*\bar{\alpha}$ is finite. If locally $\alpha = df = f'dz$, then the Dirichlet integral is $2\int\int_W |f'|^2 dxdy$. The term "regular" is unnecessary in describing Dirichlet bounded analytic differentials, since poles always force the Dirichlet integral to be infinite. The space of Dirichlet bounded analytic differentials is a separable Hilbert space which we shall denote by Γ_a. A meromorphic differential, in this context, is usually required to have a Dirichlet bounded integral outside of some compact set. Thus, it will have a finite number of poles. Also, we shall have occasion to distinguish those differentials whose periods over dividing cycles are zero. Such differentials are called semiexact and we shall denote the semiexact differentials in Γ_a by Γ_{ase}.

So much for the differentials. What restrictions must be placed on meromorphic functions to obtain an Abel's theorem? Here the requirements are more technical and I will not describe them all. The class of suitable meromorphic functions is called *quasi-rational* by Ahlfors [2, p. 315]. A meromorphic function is said to be quasi-rational if the meromorphic differential $d \log f = (f'/f)\, dz$ is Dirichlet bounded and exact outside some compact set. This last requirement has as a consequence that the number of zeros and the number of poles of a quasi-rational function must be equal and finite. There are additional requirements which I shall omit. Since the conditions on $d \log f$ are linear, the quasi-rational functions are closed under multiplication.

With these definitions we can state Ahlfor's generalization of Abel's theorem. A divisor D of degree zero is the divisor of a quasi-rational function if and only if there exists a one-cycle γ whose boundary is D and such that $\int_\gamma \alpha = 0$ for all Dirichlet bounded semiexact abelian differentials, α. (A very similar result was proved independently by Kusunoki [9].)

In a somewhat similar spirit Royden [16] has produced a generalization of the Riemann-Roch theorem for a wide class of open surfaces. Rodin [15], using Sario's principal functions, was able to give Royden's theorem its most general statement for arbitrary surfaces. Thus, Abel's theorem and the Riemann-Roch theorem seem to, and in fact *do*, have very general analogues on open surfaces. The theorems are true generalizations, for they reduce to the classical theorems if the surface in question is compact.

There is, however, a catch. There is no guarantee that the class of quasi-rational functions for a given surface contains anything more than the constants. The same situation holds for the meromorphic functions in the Royden-Rodin version of the Riemann-Roch theorem. Apropos of my previous remark, it is definitely not true that the quotient of two Dirichlet bounded abelian differentials is a quasi-rational function. In fact, that such quotients in general have any reasonable properties as meromorphic functions is yet to be proved, if it is true. Moreover, since the genus is in general infinite, we cannot use these theorems to prove the existence of quasi-rational functions. If the class of quasi-rational functions trivializes, the theorems are not devoid of content, however. In Ahlfors' generalization of Abel's theorem, it simply means that no divisor of degree zero has the stated property, a statement not without interest. Still, I think it would be disappointing if the class of quasi-rational functions always was trivial except in the classical case of compact surfaces where all meromorphic functions are quasi-rational.

So an important question persists. Do there exist surfaces of infinite genus which admit nonconstant quasi-rational functions? Interestingly enough, this question seems to have been answered by Maurice Heins before it was asked.

To understand what Heins did, we must first look a little closer at the problem of classifying Riemann surfaces. The scheme of classification proposed by Ahlfors and Sario was to classify together surfaces where some distinguished class of functions trivializes. Let HD, AD, HB, and AB stand respectively for the following classes of functions: Dirichlet bounded harmonic functions: Dirichlet bounded analytic functions; bounded harmonic functions; and bounded analytic functions. Let O_{HD}, O_{AD}, O_{HB}, and O_{AD} stand for the classes of surfaces where HD, AD, HB, and AD respectively reduce to the constant functions. Notice that the compact surfaces are in all of these classes. Every time a class of functions is discovered, a new class of surfaces arises. Classification theory has many objects for lts study. The statement $O_{HB} \subset O_{AB}$ merely means that if a surface admits no nonconstant bounded harmonic functions, then it admits no nonconstant bounded analytic functions; an easy observation since the real part of a bounded analytic function will be a bounded harmonic function. Similarly, $O_{HD} \subset O_{AD}$. Another important class of surfaces are called parabolic and denoted O_g. Here g stands for a Green's function; thus, parabolic surfaces are defined by the fact that there are no Green's functions on them. For the purposes of classification theory, a Green's function is a positive harmonic function on a surface punctured at one point where the function has a logarithmic pole. By the minimum principle for harmonic functions one sees easily that compact surfaces are also parabolic. Among the classes of surfaces so far introduced, in many ways the parabolic surfaces are closest to the compact surfaces even though they can well have infinite genus. In particular, all differentials in Γ_a are semiexact; that is, $\Gamma_a = \Gamma_{ase}$ for parabolic surfaces.

Between the various classes, the following inclusion relations are known [2, Chapter IV]:

$$O_G \subsetneqq O_{HB} \subsetneqq O_{HD} \subsetneqq O_{AD},$$

$$O_G \subsetneqq O_{HB} \subsetneqq O_{AB} \subsetneqq O_{AD}.$$

Roughly speaking, classification theory deals with the order relations between the O classes and, in particular, whether the inclusions are strict or not.

The significance of a surface being in O_{AD}, for our present problem, is this: on such surfaces any differential in Γ_a is determined by its periods. For if α and α' have the same periods, $\alpha - \alpha'$ is exact. Thus the function which is the integral of $\alpha - \alpha'$ is an AD function and so is a constant. Thus $\alpha = \alpha'$.

What Heins did was to study a class of meromorphic functions on parabolic surface, a class he called functions of bounded valence [7]. (Actually he studied functions on more general surfaces of class O_{AB}, but I shall not discuss this.) In general, a meromorphic function, f, is said to be of bounded valence if there is some integer N so that f assumes any value at most N times counting multiplicities. Thus, such a function represents the underlying surface as a finite sheeted covering of part of all of the Riemann sphere. Let BV denote this class of functions on a surface. The constants are considered to be in BV. On a general surface, BV is not a field. However, if the surface in question is parabolic, then Heins showed that BV is, in fact, a field.

The importance of BV functions is this. If the surface in question is parabolic, then all quasi-rational meromorphic functions are BV functions. The additional restrictions in the definition of quasi-rational, that I did not describe, are always met on parabolic surfaces, so in that case the definition as presented is complete. Also, the class of functions considered by Royden are functions of bounded valence on parabolic surfaces.

I shall now state the central theorem of Heins which shows why Abel's theorem and the Riemann-Roch theorem hold on parabolic surfaces. In fact, this theorem exhibits the mechanism by which these theorems are almost reduced to the classical theorems on compact surfaces.

THEOREM (HEINS). *Let W be a parabolic Riemann surface admitting nonconstant functions of bounded valence. Assume g is infinite. Then there exists a compact surface W_0 and an analytic mapping π from W into W_0 so that for each f in BV on W, there is an f_0, meromorphic on W_0 so that $f=f_0 \circ \pi$. The set of points, S, in W_0 not covered maximally by π has capacity zero. (π is necessarily a finite sheeted covering of, say, n sheets.)*

Thus, BV on W is isomorphic to the field of meromorphic functions on the compact surface W_0. Moreover, the divisor of any BV function, f, on W is the divisor of the corresponding f_0 lifted via π. In fact, Abel's theorem, and the Riemann-Roch theorem on W are nothing more than the corresponding theorem on W_0 lifted via π. Let me show how this takes place for Abel's theorem.

It is not difficult to show that a function f_0 on W_0 lifts via π to a quasi-rational function, f, on W if and only if the zeros and poles of f_0 do not intersect S, the set of W_0 not covered maximally. Thus π tells us how to lift functions and divisors from W_0 to W. Moreover, since π is finite sheeted, we can lift an $\alpha_0 \in \Gamma_a(W_0)$ to an $\alpha \in \Gamma_a(W)$ via π. This does not quite complete the proof of one half of Abel's theorem on W, since $\Gamma_a(W)$ consists of more than the differentials lifted from W_0. I shall now complete one-half of Abel's theorem on W.

Suppose f is a quasi-rational function on W being the lift of f_0 via π. Let γ_0 be a path on $W_0 - S$ so that $\partial\gamma_0$ is the divisor of f_0. If γ is γ_0 lifted via π, that is, $\gamma = \pi^{-1}(\gamma_0)$, then $\partial\gamma$ is the divisor of f. Suppose we choose γ_0 so that all differentials in $\Gamma_a(W_0)$ vanish over γ_0. Here we use the classical formulation of Abel's theorem. Take $\alpha \in \Gamma_a(W)$. We want to show that $\int_\gamma \alpha = 0$. Let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ where γ_i is the part of γ lying on the ith sheet. Similarly, let α be α_i on the ith sheet. (This is a little imprecise, but it can be done precisely in each parametric disk, which is all that is necessary.) Then $\int_\gamma \alpha = \sum_{i=1}^n \int_{\gamma_i} \alpha_i$. Now we can define a Dirichlet bounded differential α_0 on $W_0 - S$ by adding together the determinations of α on the various sheets. Then $\int_\gamma \alpha = \sum \int_{\gamma_i} \alpha_i = \int_{\gamma_0} \alpha_0$ by the definition of γ. But S, being a set of capacity zero, is a removable singularity for α_0; thus α_0 extends to be a differential in $\Gamma_a(W_0)$. By the classical theorem $\int_{\gamma_0} \alpha_0 = 0$, and so $\int_\gamma \alpha = 0$. Thus one-half of Abel's theorem is proven.

I cannot prove the other half of Abel's theorem by this method. One still needs, apparently, the existence theorems on open surfaces. However, Heins' theorem does tell *why* Abel's theorem holds on parabolic surfaces, even if it does not provide a complete proof. Similar considerations hold for the Riemann-Roch theorem on W.

That there are parabolic surfaces admitting BV functions is easy to show, and I will illustrate this shortly with the transcendental hyperelliptic surface. Heins has also shown that there are parabolic surfaces admitting no functions of bounded valence [6]. Consequently, let me refine, provisionally, the classification of parabolic surfaces by introducing the class, $\tilde{O}_g$, the class of parabolic surfaces admitting nonconstant BV functions.

Now let me illustrate the theorem of Heins by the transcendental hyperelliptic surface. Call W the two sheeted covering of W_0, the Riemann sphere. S, $(\subset W_0)$ consists only of the origin. Let π be the natural projection of W into W_0. Using a type of argument first introduced by P. J. Myrberg [13], one can show that any BV function on W is a rational function on W_0 lifted via π to W.

If f is a BV function, it is no essential restriction to assume its poles are on a compact set of W, say, on the set that projects onto the exterior of the unit disk. Thus, f is bounded off a compact set in W, since all values of f which are large in absolute value must occur near the poles of f. Let f_1 and f_2 be the values of f on the two sheets of W. Then $g=(f_1-f_2)^2$ is well defined on W, and takes the same values at corresponding points of the two sheets. Consequently, g is the lift from W_0 of a function g_0 which is seen to be bounded in the unit disk punctured at the origin. Moreover, by its definition g_0 is zero at all the a's, the branch points. Consequently, g_0 is identically zero and so f_1 is identically equal to f_2. Thus, f takes the same values at corresponding points of the two sheeted covering, and so is the lift of some function meromorphic on the Riemann sphere, that is, a rational function.

Since the reason for the validity of the generalization of Abel's theorem is so explicit in the case of parabolic surfaces, one might hope that for surfaces in a wider class, such an illumination might be available. Except for the generalization explicit in Heins' work, the situation remains essentially unknown. Some glimmers of a negative sort occur in the work of Kuromochi [8] and [4, p. 128]. In particular, if W is a surface where HD is finite dimensional, then the presence of nontrivial quasi-rational functions implies the surface is parabolic. Thus, in that strange class of surfaces strictly between the parabolic surfaces and O_{HD}, there are no nonconstant quasi-rational functions.

Also, one knows that if the surface is the interior of a bordered surface, then again the quasi-rational functions are constants. But for surfaces between these two extremes, the situation is unclear.

In a certain sense, the known generalizations of the classical theorems so far discussed may be a little disappointing, being such direct reflections of the classical situation. I would like to discuss, therefore, a theorem whose generalization to open surfaces, if and when it ever comes, will definitely not be disappointing: the theorem of Torelli. Let me return to the compact case to state the theorem.

Remember that on a compact surface of genus g, we have a canonical homology basis consisting of g pairs of cycles, the A cycles and the B cycles. Since the space Γ_a has dimension g, and the integrals over the separate A cycles are independent linear functionals on Γ_a, we can find a basis $\phi_1, \phi_2, \cdots, \phi_g$ of Γ_a so that $\int_{A_i}\phi_j=\delta_{ij}$. The $g\times g$ matrix (B_{ij}) where $B_{ij}=\int_{B_i}\phi_j$ is called the Riemann matrix of the surface associated with the particular canonical homology basis. This Riemann matrix is clearly determined by the conformal type of the surface. The theorem of Torelli is the converse of this statement. If two surfaces have identical Riemann matrices for some choice of canonical homology bases, then the surfaces are conformally equivalent.

For parabolic surfaces of infinite genus, it is now possible to state, and meditate upon, the infinite dimensional analogue of Torelli's theorem. This was shown possible by Ahlfors in 1947. Notice that we cannot expect the theorem to hold without restrictions on the surfaces, since the plane and the disk have the same Riemann matrix, namely, the empty matrix. Consequently, from now on all surfaces will be assumed to be parabolic.

First, we must find a basis for the homology. Since Dirichlet bounded differentials have vanishing periods over dividing cycles on parabolic surfaces, it suffices to consider a basis modulo dividing cycles. Ahlfors [1], [2, Chapter I] shows that there does exist a canonical homology basis of A and B cycles, infinite in number, but having otherwise the same properties as in the compact case. That there are Dirichlet bounded abelian differentials $\phi_1, \phi_2, \cdots$ so that $\int_{A_i}\phi_j=\delta_{ij}$, is also due to Ahlfors [1]. However, for the infinite dimensional case, it is not clear that such ϕ_i are uniquely determined by their A periods. However, if W is parabolic, Ahlfors [1] again has shown that a canonical homology basis does exist, with respect to which differentials in Γ_a are determined by their A periods. Thus, there is a well-defined infinite Riemann matrix of B periods for some canonical homology bases. But we are not out of the woods yet. If

W is a compact surface and W' is W minus a set of capacity zero, then W and W' have the same B matrices, since the set of capacity zero does not affect $\Gamma_a(W)$ at all. So I would add the additional assumption of no planar ends to eliminate this difficulty. So Torelli's theorem might be expected to go like this:

Suppose W and W' are homeomorphic parabolic Riemann surfaces of infinite genus without planar ends. Suppose for corresponding canonical homology bases, Dirichlet bounded abelian differentials are determined by their A periods. Suppose, finally, that the Riemann matrices are identical. Then W and W' are conformally equivalent.

Since a proof of Torelli's theorem seems inaccessible now, one must compromise in order to stay in business. Consequently, as a line of approach to the theory of open surfaces, one can impose conditions on the infinite Riemann matrix and ask what follows. In the same spirit one can ask, more generally, what follows from conditions imposed on Γ_a. I shall close with a discussion of some problems that arise in this program.

First, let us look at differentials in $\Gamma_a(W)$ (g is infinite) which behave as if they were on a compact surface. By this I mean, they are exact outside some compact set. Let Γ_a' be the set of such differentials. Clearly, such differentials can arise, for consider the situation of Heins' theorem. Suppose π maps W into W_0 as before. Suppose the genus of W_0 is greater than zero so that we may lift a nonzero differential α_0 from $\Gamma_a(W_0)$ to $\Gamma_a(W)$. Since the set S where π does not cover maximally is small, we can find curves $\{A_k, B_k\}_{k=1}^{g}$ in a canonical homology basis for W_0 which do not touch S. These curves lift to a compact set in W, and it is easy to see that outside this set on W any α lifted from $\Gamma_a(W_0)$ is exact.

Naturally, one asks if this is the general situation for $\Gamma_a'(W)$, where W is parabolic. It almost is. The corresponding result is as follows:

If the dimension of $\Gamma_a'(W)$ is greater than one, then $W \in \tilde{O}_g$. Moreover, $\Gamma_a'(W)$ is the lift via π of Heins' theorem of $\Gamma_a(W_0)$, where W_0 is the compact surface of Heins' theorem. Consequently, $\Gamma_a'(W)$ is finite dimensional, having dimension equal to the genus of W_0.

The hypothesis is necessary. One can find parabolic surfaces where Γ_a' is one dimensional and the surface is not of class $\tilde{O}_g$.

The proof is an easy consequence of a difficult theorem of Heins [6], generalized by Royden [17], concerning rings of bounded analytic functions on parabolic ends. The theorem of Heins-Royden allows us to conclude that if α and α' are in $\Gamma_a'(W)$, then the meromorphic function α/α' is, indeed, a function of bounded valence. To prove this last assertion we proceed as follows. There is an open, relatively compact set Ω in W so that in each component of $W-\Omega$, α and α' are exact. Let $\overline{E}$ be such a component; $\overline{E}$ is a bordered surface, often called a parabolic end. Let a and a' be Dirichlet bounded analytic functions on $\overline{E}$ whose differentials are α and α' respectively. By the Dirichlet principle for parabolic ends it follows that a and a' are bounded analytic functions on $\overline{E}$. Now Royden's theorem for parabolic ends, which Heins proved in the case of one ideal boundary point, is as follows: *Let $\overline{E}$ be a parabolic end admitting nonconstant bounded analytic functions. Then there exists a compact surface E_0 and an analytic map, τ, of finite valence from $\overline{E}$ into E_0 so that if a is a bounded analytic function on $\overline{E}$, then there exists an a_0, analytic on $\tau(\overline{E})$, so that $a=a_0 \circ \tau$. Moreover, $\tau(\overline{E})$ is almost compact, the difference being an AB removable set.* (Thus, a_0 is analytic on the closure of $\tau(\overline{E})$ in E_0, and is a function of bounded valence.) If a_0 and a_0' are the functions on $\tau(\overline{E})$ corresponding (by Royden's theorem) to a and a' on $\overline{E}$, then da_0/da_0' is a meromorphic function on $\tau(\overline{E})$ of finite valence, by the parenthetical remark above. Since α/α' is the lift of this function via τ, we see that α/α' is of finite valence on $\overline{E}$. Now $W-\Omega$ has at most a finite number of components $\overline{E}$, so α/α' is of bounded valence on all of W.

Thus, the hypothesis that dim $\Gamma_a'(W) \geqq 2$ implies that $W \in \tilde{O}_g$. Let W_0 be the compact surface of the Heins theorem and π the map from W into W_0. Take $\alpha \in \Gamma_a'(W)$. Let α_0' be a meromorphic differential on W_0 whose poles do not lie on S, the set in W_0 not covered maximally by π. (Since the genus of W_0 may be zero, at this point, we have to assume the possibility of poles for α_0'.) Nevertheless, if α' is the lift to W of α_0' via π, we see that outside a compact set of W, α' is Dirichlet bounded and exact. The argument of the preceding paragraph again implies that α/α' $(=f)$ is a BV function and so the lift of some function from W_0. Since $\alpha=\alpha' f$ and α' and f are both lifts from W_0, we see that α is also. This completes the proof of the theorem.

Once again, making quasi-algebraic assumptions drives us back to the class $\tilde{O}_g$. Another observation which reflects the closeness of $\tilde{O}_g$ to the compact surfaces concerns the group of conformal self-maps possible for surfaces of this class. The classical fact is that if the genus of a compact surface is greater than one, then the group of conformal self-maps is a finite group, quite often the group with one element, in fact (but not always). It is fairly easy to see that the same result holds for any parabolic surface where the dimension (always finite) of Γ_a' is positive. The argument goes as follows: any conformal self-map of W maps Γ_a' into itself in a natural way. If α is in

$\Gamma_a'(W)$, $\alpha \neq 0$, there is a cycle c so that $\int_c \alpha \neq 0$. If $\{g_1, g_2, \cdots\}$ is an infinite set of distinct conformal self-maps of W, then an infinite number of the curves $g_i(c)$ go outside any compact set to where all elements of $\Gamma_a'(W)$ are exact. But some member of $\Gamma_a'(W)$ has a nonzero period over each one of these $g_i(c)$. We have reached the desired contradiction.

It follows that the only surfaces in $\tilde{O}_g$ which have infinite groups of conformal self-maps must have the genus of the Heins compact surface, W_0, equal zero, that is, BV on such a surface must be isomorphic to the rational functions. (A result of this type was first proved by Ozawa [**14**]. Also, Mizumoto [**11**], [**12**] has studied this problem.) Of course, in general, one expects groups of conformal self-maps of an open surface to be infinite.

I shall conclude with a last example of the type of theorem that would presumably be unnecessary if we knew Torelli's theorem for open parabolic surfaces. It is known that the Riemann matrix B for a closed surface can never have the following form:

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \quad [10].$$

In the case of an open surface, since B is an infinite matrix, this fact can be generalized in several ways.

First we know that B cannot have the form:

$$\begin{pmatrix} B_1 & & & \\ & B_2 & 0 & \\ & 0 & B_3 & \\ & & & \ddots \end{pmatrix}$$

where the B_i are finite square matrices. For each row is the B periods of a ϕ which has a single nonzero A period. Thus, each row represents a basis element of Γ_a'. But since the dimension of Γ_a' is finite, the form indicated is impossible.

A second stronger statement would be that the form

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

is impossible where B_1 is a finite matrix and B_2 is an infinite matrix. I know the answer if B_1 is a 1×1 matrix. Here a proof of Gerstenhaber [**5**] in the compact case generalizes. If B_1 is bigger, we know that W is of class $\tilde{O}_g$, but more I do not know. One would think that all the information in the Heins theorem might be of help, and perhaps it is. Further possible decompositions of B are conceivable, and lead to interesting questions, and unanswered questions, as far as I know.

The approach to the theory of open surfaces I have discussed seems to be fruitful. One can range over the whole theory of algebraic curves and pick a theorem to generalize. But perhaps there is a moral to be drawn from this, also. As I remarked earlier, open surfaces of infinite genus are not truly algebraic entities. The fruitful generalizations seem to lead to a rather small class of surfaces, $\tilde{O}_g$, where the relation to the classical theory is as direct as is conceivable. Still, the subject never lacks interesting questions, and this, I think, makes the effort worthwhile.

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