EIGENFUNCTION EXPANSIONS AND SCATTERING THEORY FOR PERTURBED ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS[1]

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1. A number of papers discussing the spectral decomposition and eigenfunction expansion for partial differential operators appeared in the last few years. Browder [1], [2], [3], [4], Gårding [5] and Mautner [12] proved the existence of an abstract eigenfunction expansion for elliptic partial differential operators. In 1953 A. Ya. Povzner [13] considered the detailed spectral decomposition of $-\Delta+q(x)$. This was completed by T. Ikebe [6] who used the theory of wave operators as developed by Kato [8] and Kuroda [10], [11].

In this note we investigate an eigenfunction expansion for the operator $P(D)+q(x)$ where $P(D)$ is a linear homogeneous elliptic partial differential operator with constant coefficients. Detailed proofs of the results will appear elsewhere.

2. The Euclidean n-space will be denoted by R_n or M_n with elements $x=(x_1, \cdots, x_n)$ or $k=(k_1, \cdots, k_n)$ respectively. $\int f(x)dx$ denotes integration with respect to Lebesgue measure. We set

$$-D_j = \frac{\partial}{i\partial x_j} \qquad \text{for } 1 \leqq j \leqq n.$$

Let $P(x)$ be a homogeneous elliptic polynomial, i.e. $P(x) \geqq c|x|^{2p}$ where $2p$ is the order of $P(x)$. Then $P(D)=P(D_1, \cdots, D_n)$ is a linear homogeneous elliptic partial differential operator. All through this note we assume that $4p>n$. It is well known that $P(D)$ can be extended to a selfadjoint operator $\tilde{P}(D)$ in $L_2(R_n)$. Let $q(x) \in C_{2[n/2]}$ with $q(x)=O(|x|^{-n-h})$ for some $h>0$. Then by Theorem 1 of [11], $\tilde{P}(D)+q(x)$ is a selfadjoint operator in $L_2(R_n)$. Let $\{E_t\}$ and $\{P_t\}$, $-\infty<t<+\infty$, be the resolutions of the identity for $\tilde{P}(D)$ and $\tilde{P}(D)+q(x)$ respectively. Define

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$$U_t = \exp(itH)\exp(-itH_0),$$
$$W_\pm = s - \lim U_t, \qquad t \to \pm\infty$$

whenever the strong limits exist. $W_\pm$ are the wave operators.

For the sake of simplicity we will use the notation $H_0 = \tilde{P}(D)$ and $H = H_0 + Q$ where Q is the operator of multiplication by $q(x)$ in $L_2(R_n)$.

THEOREM 1. (i) *$H = H_0 + Q$ has pure point spectrum in $(-\infty, 0)$. Furthermore the negative eigenvalues are of finite multiplicity with zero as the only possible limit point.*

(ii) *For $f(x) \in P_0 L_2(R_n)$ we have the expansion*

$$f(x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=1}^{N} f_j u_j(x) \quad \textit{with} \quad \sum_{j=1}^{\infty} |f_j|^2 < +\infty$$

where $\{u_j, j = 1, 2, \cdots\}$ are the eigenfunctions corresponding to the eigenvalues $h_j, j = 1, 2, \cdots$ in $(-\infty, 0)$ counted according to their multiplicity and

$$f_j = \int_{R_n} f(x)[u_j(x)]^- dx, \qquad j = 1, 2, \cdots.$$

(iii) *For $f(x) \in D(H) \cap P_0 L_2(R_n)$, H has the diagonal representation*

$$Hf(x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j=1}^{N} h_j f_j u_j(x).$$

DEFINITION 1. Let A be a selfadjoint operator in $L_2(R_n)$. For $f(x) \in L_2(R_n)$, $z \in \Omega$ and $\operatorname{Im} z > 0$, where $\Omega = \{z \mid 0 < a < \operatorname{Re} z < b < +\infty, 0 \leq \operatorname{Im} z \leq c\}$ for some a, b, and c, let

$$\{(A - z)^{-1} f\}(x) = \int_{R_n} K(x - y; z) f(y) dy$$

where $K(x; z)$ is defined for all $z \in \Omega$. We say that A has property K, if

$$K(x; z) = \frac{V_z(x_1, \cdots, x_n)}{|x|^m}$$

where $m = (n-1)/2$, $V_z(x_1, \cdots, x_n)$ is continuous in $\Omega \times R_n$ and bounded in R_n, uniformly in $z \in \Omega$.

DEFINITION 2. $s > 0$ is a singular point for the operator $H = H_0 + Q$ if

$$\text{(s)} \qquad -f(x) = \int_{R_n} K(x - y; s) q(y) f(y) dy$$

for some non-null $f \in B$, where B is the Banach space of all continuous functions vanishing at infinity with the sup norm.

THEOREM 2. *Let $H_0 = \tilde{P}(D)$ have property K. Let (a, b), $0 < a < b < +\infty$, have no singular points. Then:*

(i) *H has no eigenvalues in (a, b). Moreover, the spectrum of H in (a, b) is absolutely continuous.*

(ii) *For $f \in (P_b - P_a)L_2(R_n)$, we can define the generalized Fourier transform $f^{\wedge}(k) \in L_2(M_n)$ of f as follows*

$$f^{\wedge}(k) = (2\pi)^{-n/2} \operatorname{l.i.m.} \int_{R_n} [F(x, k)]^{-} f(x) dx$$

where $F(x, k)$ is uniformly continuous and bounded on $R_n \times M_n$. $f^{\wedge}(k) = 0$ if $P(k) \notin (a, b)$. Furthermore $f^{\wedge}(k) \in L_2(M_n)$ with $\|f^{\wedge}\| = \|f\|$.

(iii) *For $f \in (P_b - P_a)L_2(R_n)$ we have*

$$f(x) = (2\pi)^{-n/2} \int_{a < P(k) < b} F(x, k) f^{\wedge}(k) dk.$$

(iv) *For $f \in (P_b - P_a)L_2(R_n)$ we have the diagonal representation for H*

$$Hf(x) = (2\pi)^{-n/2} \int_{a < P(k) < b} P(k) F(x, k) f^{\wedge}(k) dk.$$

(v) *For every $k \in M_n$ with $a < P(k) < b$*

$$P(D)F(x, k) + q(x)F(x, k) = P(k)F(x, k).$$

Part (v) of Theorem 2 shows that the spectral decomposition we obtained is actually an eigenfunction expansion. This is clear if we note that $F(x, k)$ is a weak solution of the elliptic partial differential equation

$$P(D)u(x) + q(x)u(x) - P(k)u(x) = 0$$

(cf. F. John [7]).

The following result is suggested by Theorem 2.

THEOREM 3. *H acting in $(P_b - P_a)L_2(R_n)$ is unitarily equivalent to H_0 acting in $(E_b - E_a)L_2(R_n)$. To be precise we have*

$$H = [W_-(E_b - E_a)] H_0 [W_-(E_b - E_a)]^*$$

*where * denotes the inverse of a partially isometric operator.*

Next we show if $q(x)$ is sufficiently restricted $\tilde{P}(D) + Q$ has no singular points in $(0, +\infty)$.

THEOREM 4. *Let $\tilde{P}(D)$ have property K. Let $q(x)$ satisfy the inequality*

$$| q(x) | \leqq \begin{cases} 0 & \text{for } |x| < R_0, \\ C_0 |x|^{-n-1} & \text{for } |x| > R_0, \end{cases}$$

where $C_0C' < 1$ if $|V_z(x_1, \cdots, x_n)| < C'$ and $R_0 > \sup(2\omega_n, 2)$ with ω_n designating the surface area of the unit sphere in R_n. Then $H = H_0 + Q$ has $(0, +\infty)$ as its absolutely continuous spectrum.

This is proved by assuming that (s) has a solution $f \in B$. Then we estimate the integral and obtain a uniform bound on $f(x)$, which when repeated shows that $f(x) \equiv 0$.

To investigate the singular points in the rest of this note we restrict the discussion to $P(D) = (D_1^2 + \cdots + D_n^2)^p = (-\Delta)^p$ in R_{2m+1} for $m = 1, 2, \cdots$ and $4p > 2m+1$. Using the fact that $K(x; z) \in L_2(R_{2m+1})$ if $\operatorname{Im} z > 0$ we obtain

THEOREM 5. *For $z \in \Omega$ with $\operatorname{Im} z > 0$ we have:*
(i) *For $r > 0$ and $d = 1, 2, \cdots, p-1$*

$$(-\Delta)_x^d K(x - y; r^{2p}) = -\frac{(-i)^{m+1}\pi}{p(2\pi)^{m+1}} r^{2d} \frac{r^{m+1-2p} e^{ir|x-y|}}{|x - y|^m} + O(|x|^{-m-1}),$$

(ii) $(-\tilde{\Delta})^p$ *has property K.*

LEMMA. *Let $f \in B$ be a solution of*

$$\text{(r)} \qquad f(x) = -\int_{R_{2m+1}} K(x - y; r^{2p}) q(y) f(y) dy.$$

Then

$$f(x) = \frac{\pi(-i)^{m+1} r^{m+1-2p}}{p(2\pi)^{m+1}} \frac{e^{ir|x|}}{|x|^m} \int_{R_{2m+1}} e^{ir\langle x/|x|, y\rangle} q(y) f(y) dy + O(|x|^{-m-1/2-h/2}) + O(|x|^{-m-1}).$$

Moreover

$$\frac{d}{d|x|}(-\Delta)^q f(x) = (ir)^{2q+1} f(x) + O(|x|^{-m-h/2}) + O(|x|^{-1-m}).$$

Using this lemma and Green's theorem we obtain

THEOREM 6. *Let $f \in B$ be a solution of the homogeneous integral equation* (r). *Then $f(x)$ is an L_2 eigenfunction of $H = (-\Delta)^p + q(x)$.*

The above lemma generalizes the well-known "radiation" condition of Sommerfeld. Now Theorems 1 and 2 give us the complete description of $H = (-\tilde{\Delta})^p + q(x)$ acting in $(I - P_0)L_2(R_{2m+1})$.

THEOREM 7. *On the positive real axis, H has absolutely continuous spectrum and eigenvalues. The point spectrum is a closed, denumerable, nowhere dense set.*

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