SPECTRAL OPERATORS ON LOCALLY CONVEX SPACES

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1. Let C be the complex plane, $S(C)$ the tribe of all Borel parts of C, $B^\infty(C)$ the algebra of bounded complex-valued Borel measurable functions defined on C and $M^1(C)$ the set of bounded complex Radon measures on C. Let E be a locally convex space[1] which is separated, quasi-complete and barrelled. A family $\mathfrak{F}=(m_{x,x'})_{x\in E, x'\in E'}$ of measures belonging to $M^1(C)$ is called a *spectral family* on C if there exists a representation $f\rightarrow U_{\mathfrak{F},f}$ of the algebra $B^\infty(C)$ into the algebra[1] $L(E, E)$ mapping 1 onto I and satisfying the equations $\int_C f dm_{x,x'} = \langle U_{\mathfrak{F},f}x, x'\rangle$ for all $f\in B^\infty(C)$, $x\in E$, $x'\in E'$. By $P_{\mathfrak{F}}$ we denote the *spectral measure* defined on $S(C)$ by the equations $P_{\mathfrak{F}}(\sigma)=U_{\mathfrak{F},\phi_\sigma}$ (ϕ_σ is the characteristic function of σ). A linear mapping T of (the vector space) $D_T\subset E$ into E *commutes* with $\mathfrak{F}$ if $TU_{\mathfrak{F},f}\supset U_{\mathfrak{F},f}T$ for all $f\in B^\infty(C)$.

Let T be a linear mapping of $D_T\subset E$ into E. We say that $\lambda\in\hat{C}$ (=the one point compactification of C) belongs to the *resolvent* set $r(T)$ of T if there is a neighborhood V of λ such that: (i) $zI-T$ is a one-to-one mapping of D_T onto E and $(zI-T)^{-1}\in L(E, E)$ for each $z\in V-\{\infty\}$; (ii) $\{(zI-T)^{-1} | z\in V-\{\infty\}\}$ is a bounded part of $L(E, E)$. The set $\operatorname{sp}(T)=\hat{C}-r(T)$ is the *spectrum* of T. If $\operatorname{sp}(T)\not\ni\infty$ we say that T is *regular.*

By an *admissible* set we mean a directed (for $\subset$) set of closed parts of C whose union is C, having a countable cofinal part and containing with $A\subset C$ every closed part of A. We denote below by $\mathfrak{C}_0$ and $\mathfrak{C}_1$ the admissible set of all compact parts of C and all closed parts of C, respectively. Let $\mathfrak{C}$ be an admissible set and T a closed linear mapping of $D_T\subset E$ into E. We say that T is a $\mathfrak{C}$-*spectral operator* if there is a spectral family $\mathfrak{F}$ on C such that:

(D_{I}) T commutes with $\mathfrak{F}$;

(D_{II}) $TU_{\mathfrak{F},f}\in L(E, E)$ for each $f\in B^\infty(C)$ whose support is compact and belongs to $\mathfrak{C}$;

(D_{III}) $\operatorname{sp}(T_\sigma)\subset\sigma^-$ for every[2] $\sigma\in\mathfrak{C}$.

[1] E barrelled means that every weakly bounded part of the dual space E' is equicontinuous; E quasi-complete means that every bounded closed part of E is complete. $L(E, E)$ is the algebra of all linear continuous mappings of E into E endowed with the topology of uniform convergence on the bounded parts of E.

[2] For a set $A\subset C$ we denote by A^- the closure of A in $\hat{C}$.

(For $\sigma \in S(C)$ we denote by T_σ the mapping $x \to Tx$ of $D_T \cap E_\sigma$ into E_σ, where $E_\sigma = P_{\mathfrak{F}}(\sigma)(E)$.)

THEOREM 1. *Let* $\mathcal{C}$ *be an admissible set and* T *a closed linear mapping of* $D_T \subset E$ *into* E. *Then there is at most one spectral family on* C *satisfying* (D_{I}), (D_{II}) *and* (D_{III}).

For a $\mathcal{C}$-spectral operator T we shall denote by $\mathfrak{F}_T$ the unique spectral family on C satisfying (D_{I}), (D_{II}) and (D_{III}).

THEOREM 2. *Let* T *be a* $\mathcal{C}$-*spectral operator. Then every* $A \in L(E, E)$ *commuting with* T *commutes with* $\mathfrak{F}_T$.

Let now $\mathcal{C}$ be an admissible set of parts of C and $\mathfrak{F} = (m_{x,x'})_{x \in E, x' \in E'}$ a spectral family on C. Consider the following property concerning $\mathfrak{F}$: $P\mathcal{C}$). Given $x \in E$, $x' \in E'$ there is $\sigma(x, x') \in \mathcal{C}$ such that the supports of the measures $m_{Qx,x'}$ are contained in $\sigma(x, x')$ for all $Q \in L(E, E)$ commuting with $\mathfrak{F}$.

THEOREM 3. *Let* T *be a* $\mathcal{C}$-*spectral operator and suppose that* $\mathfrak{F}_T$ *has property* $P\mathcal{C}$). *Then* $\mathrm{sp}(T_\sigma) \subset \sigma^-$ *for all* $\sigma \in \mathcal{C}_1$.

THEOREM 4. *Let* T *be a* $\mathcal{C}$-*spectral operator. Then*[3]: (4.1) $S(\mathfrak{F}_T) \subset \mathrm{sp}(T)$. (4.2) *If* $\mathfrak{F}_T$ *has property* $P\mathcal{C}$) *then* $S(\mathfrak{F}_T)^- = \mathrm{sp}(T)$.

2. We say that an operator $S \in L(E, E)$ is *scalar* if there is a spectral family $\mathfrak{F} = (m_{x,x'})_{x \in E, x' \in E'}$ on C of measures with compact support such that $\int_C z\,dm_{x,x'} = \langle Sx, x' \rangle$ for all $x \in E$, $x' \in E'$; we write in this case $S = U_{\mathfrak{F},z}$. An operator $Q \in L(E, E)$ is *quasi-nilpotent* if $\lim_{n \to \infty} |\langle Q^n x, x' \rangle|^{1/n} = 0$ for all $x \in E$, $x' \in E'$.

THEOREM 5. (5.1) *Let* $T \in L(E, E)$ *be a* $\mathcal{C}_0$-*spectral operator and suppose that* $\mathfrak{F}_T$ *has property* $P\mathcal{C}_0$). *Then* $T = U_{\mathfrak{F}_T,z} + Q$, *where* Q *is quasi-nilpotent, and* T, $U_{\mathfrak{F}_T,z}$, Q *commute. Further if* $T = S + R$ *where* S *is scalar,* R *quasi-nilpotent and where* T, S, R *commute, then* $S = U_{\mathfrak{F}_T,z}$ *and* $R = Q$. (5.2) *Let* $\mathfrak{F}$ *be a spectral family on* C *of measures with compact support and* Q *a quasi-nilpotent operator commuting with* $\mathfrak{F}$. *Then* $T = U_{\mathfrak{F},z} + Q$ *is a* $\mathcal{C}_0$-*spectral operator and* $\mathfrak{F} = \mathfrak{F}_T$.

3. In what follows we denote by Φ an arbitrary directed family of closed barrelled subspaces of E having the properties: (i) the set $E_0 = \bigcup_{F \in \Phi} F$ is dense in E; (ii) a linear mapping T of E_0 into E_0 verifying the relations $T(F) \subset F$ for all $F \in \Phi$ is continuous if T_F(T_F is the mapping $x \to Tx$ of F into F) is continuous for all $F \in \Phi$; (iii) given

[3] We denote by $S(\mathfrak{F}_T)$ the closure in C of the union of the supports of the measures belonging to $\mathfrak{F}_T$.

$x \in E$ and $x' \in E'$ there is $x_0 \in E_0$ verifying the equations $\langle Tx, x'\rangle = \langle Tx_0, x'\rangle$ for each $T \in L(E, E)$ such that $T(F) \subset F$ for all $F \in \Phi$. Given Φ let $L_\Phi(E, E)$ be the set of all $T \in L(E, E)$ such that: (j) $T(F) \subset F$ for all $F \in \Phi$; (jj) T_F is regular for all $F \in \Phi$ and $\mathrm{sp}(T_{F'}) \subset \mathrm{sp}(T_{F''}) \subset \mathrm{sp}(T)$ if $F', F'' \in \Phi$, $F' \subset F''$. For $T \in L_\Phi(E, E)$ we write $A(T) = \bigcup_{F \in \Phi} \mathrm{sp}(T_F)$.

THEOREM 6. *If $T \in L_\Phi(E, E)$ then* $\mathrm{sp}(T) = A(T)^-$.

THEOREM 7. *If $T \in L_\Phi(E, E)$ then there exists a unique continuous representation $\tilde{f} \to \tilde{f}(T)$ of*[4] *$H(A(T))$ into $L(E, E)$ having the properties*: (7.1) $\tilde{1}(T) = I$; (7.2) $\tilde{z}(T) = T$. *Further $\tilde{f}(T) \in L_\Phi(E, E)$ and* $\mathrm{sp}(\tilde{f}(T)) = f(A(T))^-$ *(f is an element in the equivalence class $\tilde{f}$).*

Let $T \in L(E, E)$ be a $\mathcal{C}_0$-spectral operator. Suppose that $\mathfrak{F}_T = (m_{x,x'})_{x \in E, x' \in E'}$ has property $P\mathcal{C}_0$) and let $\Phi = (E_\sigma)_{\sigma \in \mathcal{C}_0}$. Then Φ has the properties (i), (ii), (iii) and $T \in L_\Phi(E, E)$. Moreover:

THEOREM 8. *The operator $\tilde{f}(T)$ is $\mathcal{C}_1$-spectral for each $\tilde{f} \in H(A(T))$ and*

$$(1) \qquad \langle f(T)x, x'\rangle = \sum_{j=0}^{\infty} \frac{1}{j!} \int_C f^{(j)} dm_{Q^j x, x'}, \qquad \text{for } x \in E,\ x' \in E',$$

where Q is the quasi-nilpotent part of T. The series (1) *converges absolutely and uniformly for given $x \in E$ and $x' \in A$ (A is an arbitrary equicontinuous part of E').*

4. Let $\mathcal{C}$ be an admissible set, $(\sigma(n))$ an increasing sequence of compact parts belonging to $\mathcal{C}$ whose union is C, $T: D_T \to E$ a $\mathcal{C}$-spectral operator, $\mathfrak{F}_T = (m_{x,x'})_{x \in E, x' \in E'}$ and $E_\infty = \bigcup E_{\sigma(n)}$. Let T_∞ be the restriction of T to $E_\infty \subset D_T$ and $\mathfrak{F}_T^\infty = (m_{x,x'}^\infty)_{x \in E_\infty, x' \in E'_\infty}$. Here E_∞ is endowed with the topology, inductive limit of the topologies of the subspaces $E_{\sigma(n)}$ of E, and, for $x \in E_{\sigma(n)} \subset E_\infty$ and $x' \in E'_\infty$, $m_{x,x'}^\infty = m_{x,y'}$ if $y' \in E'$ is such that $x'_{E_{\sigma(n)}} = y'_{E_{\sigma(n)}}$.

THEOREM 9. (9.1) *T_∞ is a $\mathcal{C}_1$-spectral operator, $\mathfrak{F}_{T_\infty} = \mathfrak{F}_T^\infty$ and $\mathfrak{F}_{T_\infty}$ has property $P\Sigma$) (Σ is the smallest admissible set containing $(\sigma(n))$).* (9.2) *T is the closure of T_∞.* (9.3) $\mathrm{sp}(T_\infty) = S(\mathfrak{F}_T)^-$.

Further $A \in L(E, E)$ commutes with T if and only if $A(E_\infty) \subset E_\infty$ and A_{E_∞} commutes with T_∞. Also T is "scalar" if and only if T_∞ is scalar; if T is "scalar" and f is such that $\phi_{\sigma(n)} f \in B^\infty(C)$ for all n then $f(T)_\infty = f(T_\infty)$.

[4] For the definition of $H(A)$, $A \subset C$ (endowed with the "van Hove topology"), see for instance [5] (where A is supposed compact) and [4, pp. 255–256].

5. The subject matter of this note has been suggested by [2] and by [1; 3]. The Theorems 1, 2, 5 and 8 are essentially generalizations of the corresponding results in [2; 3]. The results of paragraph 4 show some of the relations between the unbounded spectral operators defined in [1] and the (everywhere defined continuous) spectral operators defined above. The definition of the spectrum and of the quasi-nilpotent operator were suggested by definitions given in [5; 6], respectively.

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