A FAMILY OF SIMPLE GROUPS ASSOCIATED WITH THE SIMPLE LIE ALGEBRA OF TYPE (F_4)

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In this note we obtain a family of simple groups, which also seem to be new, by applying the method we used in [3] to the Chevalley groups of type (F_4). The orders of the finite groups in the family are

$$q^{12}(q-1)^2(q+1)(q^2+1)(q^3+1)(q^6+1),$$

where $q=2^{2n+1}$, $n=1, 2, 3, \cdots$.

Let $\mathfrak{g}$ be the simple Lie algebra of type (F_4) over the complex number field, and Σ the root system of $\mathfrak{g}$. Let the Coxeter-Dynkin diagram of Σ be

$$\begin{array}{ccccccc} 2 & & 2 & & 1 & & 1 \\ \circ & \text{———} & \circ & \text{———} & \circ & \text{———} & \circ . \\ a_1 & & a_2 & & a_3 & & a_4 \end{array}$$

Let P be the additive group generated by Σ, and $\phi: P \to P$ a homomorphism defined by (see [2, Exposé 24, p. 4]),

$$\phi(a_1) = 2a_4, \qquad \phi(a_2) = 2a_3, \qquad \phi(a_3) = a_2, \qquad \phi(a_4) = a_1.$$

Then for any $r \in \Sigma$ we have $\phi(r) = \lambda(r)\bar{r}$, where $\lambda(r)$ is the length of the root r and where $r \to \bar{r}$ is a permutation of order 2 of Σ.

Let K be a field of characteristic 2 which admits an automorphism $t \to t^\theta$ such that $2\theta^2 = 1$. Define the algebra $\mathfrak{g}_K$ over K and the automorphisms $x_r(t)$, where $r \in \Sigma$, $t \in K$, of $\mathfrak{g}_K$ as in [1], and let G be the group generated by all the $x_r(t)$. Then we have:

(1) The group G admits an automorphism $x \to x^\sigma$ such that

$$x_r(t)^\sigma = x_{\bar{r}}(t^{\lambda(\bar{r})\theta})$$

for all $r \in \Sigma$, $t \in K$.

(2) The group G^1 of all elements x in G such that $x = x^\sigma$ is simple if K has more than two elements.

In order to describe the group G^1 more closely, let $\mathfrak{U}$ be the subgroup of G generated by all the $x_r(t)$ with $r > 0$, and set $\mathfrak{U}^1 = \mathfrak{U} \cap G^1$. For $r \in \Sigma$, $r > 0$, $\lambda(r) = 1$, set

$$\alpha(t) = \begin{cases} x_r(t^\theta)x_{\bar{r}}(t) & \text{if } r + \bar{r} \notin \Sigma. \\ x_r(t^\theta)x_{\bar{r}}(t)x_{r+\bar{r}}(t^{\theta+1}) & \text{if } r + \bar{r} \in \Sigma. \end{cases}$$

[1] This work was done while the author held a Research Associateship of the Office of Naval Research, U. S. Navy.

Then $\alpha(t) \in \mathfrak{U}^1$, and from the 24 positive roots we obtain 12 such elements: $\alpha_1(t), \alpha_2(t), \cdots, \alpha_{12}(t)$. We have:

(3) Every element $x \in \mathfrak{U}^1$ is written uniquely as

$$x = \alpha_1(t_1)\alpha_2(t_2) \cdots \alpha_{12}(t_{12}),$$

where $t_i \in K$.

For any homomorphism $\chi: P \rightarrow K^*$ define $h(\chi) \in G$ as in [1]. Also see [1] for the meaning of the symbol $\omega(w)$, where $w \in W$, the Weyl group of Σ. We have:

(4) $h(\chi) \in G^1$ if and only if $\chi(a_4) = \chi(a_1)^\theta$, $\chi(a_3) = \chi(a_2)^\theta$.

(5) The group W^1 of all $w \in W$ such that $[w(r)]^- = w(\bar{r})$ for all $r \in \Sigma$ is of order 16, and for each $w \in W^1$ we can take $\omega(w)$ in G^1.

(6) Every element x in G^1 is written uniquely as $x = uh(\chi)\omega(w)u'$, where: $u \in \mathfrak{U}^1$; $h(\chi) \in G^1$; $w \in W^1$ (we take $\omega(w)$ in G^1); u' is a product of $\alpha(t)$ for which $w(r) < 0$.

If K is a finite field of $q = 2^{2n+1}$ elements, where $n \geqq 1$, then we can set $t^\theta = t^m$, where $m = 2^n$. The order of G^1 can be computed from the above, since for each $w \in W^1$ we can find easily all $r \in \Sigma$ such that $r > 0$, $w(r) < 0$.

References

1. C. Chevalley, *Sur certains groupes simples*, Tôkoku Math. J. (2) vol. 7 (1955) pp. 14–66.

2. *Séminaire C. Chevalley, Classification des groupes de Lie algébriques*, vol. 2, Paris, 1956–1958.

3. R. Ree, *A family of simple groups associated with the simple Lie algebra of type* (G_2), Bull. Amer. Math. Soc. vol. 66 (1960) pp. 508–510.

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