SYMMETRY IN MEASURE ALGEBRAS[1]

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It is well known that the measure algebra of a locally compact group G is not symmetric, i.e. the set of Gelfand transforms is not closed under complex conjugation. However, if these transforms are restricted to the character group Γ, they are symmetric. In his paper [1] Rudin asks: Is there a set larger than the closure of Γ on which the transforms are symmetric?

If G is the real line the answer is yes.

Let G be the real line; we consider the algebra $M(G)$ of all regular Borel measures with convolution as multiplication. The maximal ideal space $\mathfrak{M}$ of $M(G)$ is compact and Γ (also the real line) is an open subset of $\mathfrak{M}$. Let S be the largest subset of $\mathfrak{M}$ on which the Gelfand transforms are closed under conjugation.

Let Q be an independent, compact, perfect set (of Lebesgue measure 0) which supports a positive measure σ whose Fourier-Stieltjes transform vanishes at infinity (see [2]). Without loss of generality we may suppose that the

$$\sup\{ \mid \hat{\sigma}(\gamma) \mid : \gamma \in \Gamma\} = \sup\left\{\left|\int_{-\infty}^{\infty} e^{iyx} d\sigma(x)\right| : y \in G\right\} = 1.$$

Now let $U = \{h \in \mathfrak{M}: |\hat{\sigma}(h)| < 1/4\}$. Since σ vanishes at infinity the set $A = \Gamma - U$ is compact.

Pick an absolutely continuous measure λ so that $\hat{\lambda} \equiv 1$ on A.

We are now in position to define a member of S which is not in the closure of Γ. We define a function to be identically -1 on all of Q but one point x, and there its value is $+1$. Since Q is independent we can extend this function to a homomorphism χ_σ on G to the circle group; since Q is perfect χ_σ is not continuous but, clearly, χ_σ is σ-measurable. Now let $H = \{\mu \in M(G): \chi_\sigma \text{ is } \mu\text{-measurable}\}$ and let $I = \{\phi \in M(G): \phi \perp \mu \text{ for every } \mu \in H\}$. Šreider [3] has shown that H is an algebra, I is an ideal and $M(G) = H + I$ (direct sum). Now pick $\chi_0 \in A$ such that $|\hat{\sigma}(\chi_0)| > 3/4$; and define $h_0(\mu) = \hat{\mu}(h_0) = \int \chi_0(x) d\mu_H(x)$, where μ_H is the projection of μ on H. It can be shown that $h_0 \in S$ (since H is self-adjoint) and that $\lambda \in I$ (since χ_σ is not continuous); thus if we let W be the neighborhood of h_0 determined by σ, λ, and $1/4$, then $W \cap \Gamma = \phi$ and the result is proved.

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REMARK. Obviously the more general theorem is true: *Let G be a locally compact abelian group. If there is a singular measure μ on G whose (Gelfand) transform vanishes on the boundary of the character group Γ and there exists a noncontinuous character on G which is μ-measurable, then $S\neq\bar{\Gamma}$.*

REFERENCES

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2. ———, *Fourier-Stieltjes transforms of measures on independent sets*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 199–202.

3. Yu. A. Šreider, *The structure of maximal ideals in rings of measures with convolution*, Amer. Math. Soc. Translations no. 81, Providence, 1953.

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