ON THE NUMBER OF POSITIVE INTEGERS LESS THAN x AND FREE OF PRIME DIVISORS GREATER THAN x^c

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Dr. Chowla recently raised the following question regarding the number of positive integers less than x and free of prime divisors greater than x^c, which number is here denoted by $f(x, c)$: For every fixed positive c, is $\liminf_{x\to\infty} f(x, c)/x > 0$?

This paper, while incidentally answering this question in the affirmative, proves more, in fact the best[1] possible result in this direction, namely:

THEOREM A. *A function $\phi(c)$ defined for all $c>0$ exists such that*

(1) *$\phi(c)>0$ and is continuous for $c>0$;*

(2) *for any fixed c*

$$f(x, c) = x\phi(c) + O(x/\log x)$$

where the "O" is uniform for c greater than or equal to any given positive number.

Notation. The following symbols are used for the entities mentioned against them:

p, p_r: any prime.

$S(x, p)$: the set of integers less than x each divisible by p and free of prime divisors greater than p.

$T(x, p)$: the set of integers less than x each free of prime divisors greater than p.

$N[K]$: the number of members of K, where K denotes any finite set of integers.

$F(t)$: $\sum_{p\leqq t} (1/p)$ where p runs through primes.

Preliminary lemmas.

LEMMA I. *For $c \geqq 1$, the theorem is true, and*

$$f(x, c) = x\phi(c) + O(1).$$

PROOF. This is obvious. In fact, for these values of c, $\phi(c)=1$, and

$$|f(x, c) - x\phi(c)| \leqq 1.$$

LEMMA II. *If $p_1 \neq p_2$, the sets $S(x, p_1)$ and $S(x, p_2)$ are distinct, and*

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[1] The "O" of the theorem cannot be improved upon, as will be seen in the sequel.

$N[S(x, p)] = N[T(x/p, p)]$.

PROOF. This is obvious, from the unique factorization theorem.

LEMMA III. *The number of primes less than or equal to* x *is* $O(x/\log x)$ *and* $F(x) = \log\log x + b + O(1/\log x)$ *where* b *is constant.*

This is well known and is an "elementary theorem" in the theory of primes.

LEMMA IV. *If* $0 < c_1 \leqq 1$, *and Theorem* A *is true for* $c \geqq c_1$, *then it is true for* $c \geqq c_1/(1+c_1)$.

PROOF. By hypothesis $\phi(c)$ is defined for $c \geqq c_1$, and

$$\begin{aligned} &\phi(c) > 0 \text{ and is continuous for } c \geqq c_1, \\ (3)\qquad &f(x, c) = x\phi(c) + O(x/\log x) \text{ uniformly for } c_1 \leqq c \leqq 1, \\ &f(x, c) = x\phi(c) + O(1) \text{ for } c \geqq 1, \qquad \text{by Lemma 1.} \end{aligned}$$

Also, obviously,

(4) $\phi(c)$ is bounded and monotonic increasing, though possibly not strictly so, for $c \geqq c_1$.

Let now

$$(5)\qquad c_2 = c/(1+c_1) \text{ and } c_2 \leqq d \leqq c_1 \text{ (obviously } c_2 < c_1).$$

Now

$$\begin{aligned} f(x, c_1) - f(x, d) &= \sum_{x^d < p \leqq x^{c_1}} N[S(x, p)] &&\text{(by Lemma II)} \\ &= \sum_{x^d < p \leqq x^{c_1}} N\left[T\left(\frac{x}{p}, p\right)\right] &&\text{(by Lemma II)} \\ &= \sum_{x^d < p \leqq x^{c_1}} f\left(\frac{x}{p}, \frac{\log p}{\log x - \log p}\right) \\ &= \sum_{x^d < p \leqq x^{c_1}} \left\{\frac{x}{p}\, \phi\left(\frac{\log p}{\log x - \log p}\right)\right\} \\ &\quad + \sum_{p \geqq x^{1/2}} O(1) + \sum_{p < x^{1/2}} O\left(\frac{x}{p \log (x/p)}\right) \end{aligned}$$

by (3) and (5), since in this range

$$(6)\qquad \frac{\log p}{\log x - \log p} > \frac{d}{1-d} \geqq \frac{c_2}{1-c_2} = c_1.$$

Hence

$$\text{(6a)}\quad f(x, c_1) - f(x, d) = x\int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) + O\left(\frac{x}{\log x}\right)$$

by Lemma III, where the "O" is uniform with respect to d in virtue of (3) and (6), and the Riemann-Stieltjes integral on the right exists in virtue of (4) and the continuity of $\phi(c)$ for $c \geqq c_1$. Using integration by parts for the integral and using (3) and (4) and Lemma III, we obtain (see Note 1a)

$$f(x, c_1) - f(x, d) = x\int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right)\frac{dt}{t \log t} + O\left(\frac{x}{\log x}\right)$$

(uniformly for $c_2 \leqq d \leqq c_1$)

$$\text{(7)}\qquad = x\int_{d}^{c_1} \phi\left(\frac{u}{1-u}\right)\frac{du}{u} + O\left(\frac{x}{\log x}\right)$$

(uniformly for $c_2 \leqq d \leqq c_1$).

This, together with (3), proves the existence of $\phi(c)$ for $c \geqq c_1/(1+c_1)$, though it is not yet clear whether $\phi(c) > 0$ in $c_1/(1+c_1) \leqq c \leqq c_1$, and is continuous therein.

Now, let

$$\text{(8)}\qquad \frac{c_1}{1+c_1} \leqq d_1 < d_2 \leqq c_1.$$

Then (7) gives

$$\frac{f(x, d_2) - f(x, d_1)}{x} = \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right)\frac{du}{u} + O\left(\frac{1}{\log x}\right)$$

$$\to \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right) du \qquad \text{as } x \to \infty.$$

Hence

$$\text{(9)}\qquad \phi(d_2) - \phi(d_1) = \int_{d_1}^{d_2} \phi\left(\frac{u}{1-u}\right)\frac{du}{u} > 0,$$

by (3), since in the range $d_1 < u < d_2$, we have $u/(1-u) > c_1$.

This shows that $\phi(d_2) \neq 0$ for any d_2 of the kind specified in (8); for, if $\phi(d_2)$ were zero, then obviously $\phi(d_1)$ would be zero and their difference also would be so, contrary to (9). Also, obviously $\phi(d_2) \geqq 0$. Hence

$$\text{(10)}\qquad \phi(c) > 0 \qquad \text{for } c > c_1/(1 + c_1).$$

Also by (9) and the hypothesis

(11) $\phi(c)$ is continuous for $c \geqq c_1/(1+c_1)$.

Using now the results (7), (10), and (11) and repeating the above argument with $c_1/(1+c_1)$ in place of c_1, and noting that the positiveness of $\phi(c_1)$ is not needed in the above argument, it follows that $\phi(c_1/(1+c_1))>0$.

This completes the proof of the lemma.

PROOF OF THE THEOREM. Lemma I holds for $c\geqq 1$ and Lemma IV for $0<c<1$, if we note that the hypotheses of the lemma are satisfied for $c_1=1$, and hence by the result of the lemma for $c_1=1/2$ and hence by induction for $c_1=1/n$ (n positive integral), and that $1/n\to 0$ as $n\to\infty$.

COROLLARY. *For* $0<c_1<c_2\leqq 1$,

$$\phi(c_2) - \phi(c_1) = \int_{c_1}^{c_2} \phi\left(\frac{u}{1-u}\right)\frac{du}{u}$$

(*see also Note* 1b).

PROOF. A finite set of numbers, $d_0, d_1, d_2, \cdots, d_n$, obviously exists such that

$$c_1 = d_0 < d_1 < d_2 < \cdots < d_n = c_2;$$

and

$$d_r \geqq \frac{d_{(r+1)}}{1 + d_{(r+1)}} \qquad (r = 0, 1, 2, \cdots, n-1).$$

To each of these intervals (d_r, d_{r+1}) apply the result (9) and add; the corollary follows at once.

REMARKS. The "O" of the theorem cannot be improved upon. This can be seen as follows:

Obviously $f(x, 1)=x+O(1)$ and

$$f(x, 1) - f\left(x, \frac{1}{2}\right) = \sum_{x^{1/2}<p\leqq x} \left[\frac{x}{p}\right] = x \sum_{x^{1/2}<p\leqq x}\left(\frac{1}{p}\right) - G(x)$$

where $G(x)=\sum_{x^{1/2}<p\leqq x}\{x/p\}$ and $\{y\}$ denotes fractional part of y.

Hence $f(x, 1/2)=x(1-\log 2)+o(x/\log x)+G(x)$ by the prime number theory (see Note 1c). But $G(x)>kx/\log x$ where k is a fixed positive number, as can be easily seen from the prime number theory.

NOTE 1. (a) The deduction of (7) from (6a) is based on the following:

We observe that the Riemann-Stieltjes integral

$$\int_{x^d}^{x^{c_1}} \phi\left(\frac{\log t}{\log x - \log t}\right) dF(t) = \left[F(t)\phi\left(\frac{\log t}{\log x - \log t}\right)\right]_{x^d}^{x^{c_1}} - \int_{x^d}^{x^{c_1}} F(t)d\phi$$

$$= \left[(\log\log t + b)\phi\left(\frac{\log t}{\log x - \log t}\right)\right]_{x^d}^{x^{c_1}} - \int_{x^d}^{x^{c_1}} (\log\log t + b)d\phi + R$$

which, by the substitution $t = x^u$ performed after integration by parts of the integral on the right, equals

$$\int_d^{c_1} \phi\left(\frac{u}{1-u}\right)\frac{du}{u} + R$$

where, by Lemma III,

$$R = O\left(\frac{1}{d \log x}\right) - O\left(\frac{1}{d\log x}\right) - \int_{x^d}^{x^{c_1}} O\left(\frac{1}{\log t}\right) d\phi$$

where, on account of (4), the integral also is $O(d^{-1}\log^{-1} x)$.

(b) The integral equation for $\phi(c)$ may be used for successive computation of the function. Starting with $\phi(c) = 1$ for $c \geqq 1$, one observes that $c/(1-c) \geqq 1$ for $1/2 \leqq c < 1$ so that

$$\phi(c) = 1 - \int_c^1 \frac{du}{u} = 1 + \log c \quad \text{for } 1/2 \leqq c \leqq 1.$$

This result can be used to compute $\phi(c)$ on the interval (1/3, 1/2) and so on; and an easy induction shows that

$$\phi(c) = 1 + \sum_{r=1}^{\infty} (-1)^r \psi_r(c), \qquad c > 0,$$

where $\psi_1(c) = \int_c^1 du/u$ for $c \leqq 1$, and $\psi_1(c) = 0$ for $c > 1$, and

$$\psi_r(c) = \int_c^1 \psi_{r-1}\left(\frac{u}{1-u}\right)\frac{du}{u}, \qquad \text{for } c > 0.$$

One notes that $\psi_r(c) = 0$ for $c \geqq 1/r$ (so that the infinite series is actually a finite sum) and also that the functions $\psi_r(c)$, $r \geqq 2$, are not elementary functions.

(c) From the known result $\pi(x)=x/\log x+x/\log^2 x+o(x/\log^2 x)$ follows

$$\sum_{x^{1/2}<p\leqq x}\frac{1}{p}=\int_{x^{1/2}}^{x}\frac{1}{t}\,d\pi(t)=\frac{\pi(x)}{x}-\frac{\pi(x^{1/2})}{x^{1/2}}+\int_{x^{1/2}}^{x}\frac{\pi(t)}{t^2}\,dt$$
$$=-\frac{1}{\log x}+o\left(\frac{1}{\log x}\right)$$
$$+\int_{x^{1/2}}^{x}\left[\frac{t}{\log t}+\frac{t}{\log^2 t}+o\left(\frac{t}{\log^2 t}\right)\right]\frac{dt}{t^2}$$
$$=\log 2+o\left(\frac{1}{\log x}\right).$$

NOTE 2. From the corollary to the theorem follows, for $0<c_1<c_2\leqq c_1/(1-c_1)$ and $c_2\leqq 1$,

$$\phi(c_2)-\phi(c_1)\geqq\phi(c_2)\log c_2/c_1>\phi(c_2)(1-c_1/c_2)$$

from which follows $\phi(c_2)/c_2>\phi(c_1)/c_1$, whence, arguing as in the proof of the corollary, one sees that $\phi(c)/c$ is a strictly monotonic increasing function in $0<c\leqq 1$.

$\phi(c)$ has other interesting properties, which will be published shortly. One such is that for every fixed n, $\phi(c)/c^n\to 0$ as $c\to +0$. This result, together with the now obvious result

$$\lim_{x\to\infty}\frac{f(x,c)}{x}=\phi(c)>0,\qquad c \text{ fixed and positive},$$

was communicated to Dr. Chowla in August of 1947. I understand from his reply that Vijayaraghavan already was in possession of a proof of his (Chowla's) conjecture that

$$\liminf_{x\to\infty}\frac{f(x,c)}{x}>0,\qquad \text{for } c \text{ positive and fixed.}$$

In conclusion, I wish to thank the referees for their suggestions which have led to the clarification and additions of content contained in Note 1.

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