let x be fixed, $-1<x<1$. We obtain for the roots of the polynomial (19) in z the condition

$$\frac{1+xz}{(1+2xz+z^2)^{1/2}} = x_\nu \tag{20}$$

where x_ν denotes a root of P_n. Or

$$z = \frac{x(x_\nu^2 - 1) \pm x_\nu((1-x_\nu^2)(1-x^2))^{1/2}}{x^2 - x_\nu^2}, \tag{21}$$

thus the roots in z are all real. Using the trivial inequality (16) the assertion follows.

STANFORD UNIVERSITY

NOTE ON THE EIGENVALUES OF THE STURM-LIOUVILLE DIFFERENTIAL EQUATION

GERALD FREILICH

In discussing eigenvalues and eigenfunctions of the Sturm-Liouville differential equation

$$L(u) + \lambda\rho u = 0, \qquad L(u) = (pu')' - qu,$$

with

$$\left.\begin{aligned} p(x) &\geqq m > 0 \\ q(x) &\geqq 0 \\ \beta \geqq \rho(x) &\geqq \alpha > 0 \end{aligned}\right\} \quad \text{for } a \leqq x \leqq b, \text{ and for some } \alpha, \beta, \text{ and } m,$$

and the boundary conditions

$$u(a) = c_1 u(b), \qquad u'(a) = c_2 u'(b), \qquad c_1 c_2 p(a) = p(b),$$

we find that we can represent our eigenfunctions as unit normals in the directions of the principal axes of an ellipsoid in function space. We define our function space F as the set of all functions $v(x)$, $a\leqq x\leqq b$, which satisfy the boundary conditions of the Sturm-Liouville equation. The origin of our space will be the function $u(x)=0$. We can now metrize F by defining our inner product (u, v) for

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$u \in F$, $v \in F$, as

$$(u, v) = \int_a^b \rho uv dx.$$

Also $|u| = ((u, u))^{1/2}$, and u is orthogonal to v if and only if $(u, v) = 0$. Let me now define

$$D(u, v) = \int_a^b (pu'v' + quv)dx.$$

Notice $D(u, u) = \int_a^b (p(u')^2 + qu^2)dx \geqq 0$.

In terms of D, I shall define an infinite-dimensional ellipsoid in our function space. Take a unit vector in F, that is, $(u, u) = 1$. On this, lay off a length $r = 1/(D(u, u))^{1/2}$. The set of points of F thus determined for all unit vectors of F constitutes the ellipsoid. It can be shown that the unit normals in the directions of the principal axes constitute a complete orthonormal set of eigenfunctions. Furthermore, if we arrange this set such that u_1 is the unit vector in the direction of the longest principal axis, u_2 the unit vector in the direction of the next longest principal axis, and so on, then $\lambda_1 = D(u_1, u_1)$, $\lambda_2 = D(u_2, u_2), \cdots$ constitute an increasing set of eigenvalues, λ_i being the eigenvalue of u_i.

To show that $\lambda_n \approx n^2$ as $n \to \infty$, we must define the nth principal axis $u_n(x)$ independently of the previous $n-1$ principal axes $u_i(x)$, $i = 1, 2, \cdots, n-1$. This is customarily done in the following manner:

Take a set of $n-1$ linearly independent vectors of F, $v_1, v_2, \cdots, v_{n-1}$, through the origin and consider all unit vectors $u \perp v_i$, $i = 1, 2, 3, \cdots, n-1$. Hence $(u, u) = 1$, $(u, v_i) = 0$. Let $\min D(u, u) = f(v_i)$. Then $\max f(v_i) = \lambda_n$.

The way I define λ_n runs as follows:

Let $v_1, v_2, \cdots, v_n$ be a set of n linearly independent vectors of F. Consider

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

Then we normalize u getting $u/|u|$. It is easy to see that $D(u/|u|, u/|u|) = D(u, u)/(u, u)$. Now let $g(v_i) = \max D(u, u)/(u, u)$ for all u as defined above. Then

$$\min g(v_i) = \lambda_n.$$

To prove this, let $v_i = u_i$, for $i = 1, 2, \cdots, n$. Then

$$(u, u) = \left(\sum c_i u_i, \sum c_j u_j\right) = \sum c_i c_j (u_i, u_j) = \sum c_i^2,$$

$$\frac{D(u,\,u)}{(u,\,u)}=\frac{D(\sum c_iu_i,\,\sum c_ju_j)}{\sum c_i^2}=\frac{\sum c_i^2\lambda_i}{\sum c_i^2}\leqq\frac{\sum c_i^2\lambda_n}{\sum c_i^2}=\lambda_n,$$

since $\lambda_i \leqq \lambda_n$ for $i=1, 2, \cdots, n-1$. Hence $g(u_i)\leqq\lambda_n$. However, if $c_i=0$, for $i=1, 2, \cdots, n-1$, $c_n=1$, then $u=u_n$, and hence $D(u, u)/(u, u)=\lambda_n$, $g(u_i)=\lambda_n$. To complete the proof I must show that for all v_i, $g(v_i)\geqq\lambda_n$.

Let $u=c_1v_1+c_2v_2+\cdots+c_nv_n$ be such that $u\perp u_i$ for $i=1, \cdots, n-1$, that is, $(u, u_i)=0=c_1(u_i, v_1)+\cdots+c_n(u_iv_n)$. This is always possible since I have $n-1$ equations in n unknowns. Now by the definition of u_n, for all vectors $u\perp u_i$, $i=1, 2, \cdots, n-1$,

$$\lambda_n = D(u_n,\, u_n) \leqq \frac{D(u,\,u)}{(u,\,u)}\,.$$

Hence $g(v_i)=\max D(u, u)/(u, u)\geqq\lambda_n$.

With this definition, we can now develop inequalities for the eigenvalues. Consider

$$(p_1u')' - q_1u + \lambda\rho_1u = 0, \tag{1}$$

$$(p_2u')' - q_2u + \mu\rho_2u = 0 \tag{2}$$

with

$$p_1(x) \leqq p_2(x), \qquad q_1(x) \leqq q_2(x), \qquad \rho_1(x) \geqq \rho_2(x) \qquad \text{for } a \leqq x \leqq b,$$

and with both equations satisfying similar boundary conditions. Denoting the eigenvalues of (1) by

$$\lambda_1,\, \lambda_2,\, \lambda_3,\, \cdots$$

and those of (2) by

$$\mu_1,\, \mu_2,\, \mu_3,\, \cdots,$$

we shall prove that $\lambda_n\leqq\mu_n$ for $n=1, 2, \cdots$.

Let $v_1, v_2, \cdots, v_n$ be a fixed set of n independent vectors. Let $u=c_1v_1+\cdots+c_nv_n$. For (1) the normalized u is $u/|u|_1$ and $D_1(u/|u|_1, u/|u|_1)=D_1(u, u)/(u, u)_1$, $g_1(v_i)=\max D_1(u, u)/(u, u)_1$. For (2) the normalized u is $u/|u|_2$, and $g_2(v_i)=\max D_2(u, u)/(u, u)_2$. Since $D_1(u, u)=\int(p_1(u')^2+q_1u^2)dx$, $D_2(u, u)=\int(p_2(u')^2+q_2u^2)dx$, it follows that $D_2(u, u)\geqq D_1(u, u)$. Since $(u, u)_1=\int\rho_1u^2dx$, $(u, u)_2=\int\rho_2u^2dx$, it follows that $(u, u)_1\geqq(u, u)_2$. Hence $D_2(u, u)/(u, u)_2\geqq D_1(u, u)/(u, u)_1$, $g_2(v_i)\geqq g_1(v_i)$. Since $\min g_2(v_i)=\mu_n$, and $\min g_1(v_i)$ λ_n, then $\mu_n\geqq\lambda_n$.

With these preliminaries, the proof about the asymptotic behavior of λ_n is standard.

An example of the use of this new definition of λ_n is that it affords a method of obtaining upper bounds and approximations to λ_n.

Take $v_i = v_i(x, a_1, a_2, \cdots, a_r)$, with r parameters, such that v_i for $i = 1, 2, \cdots, n$ satisfies the boundary conditions, that is, it is a vector in function space. Then form

$$u = c_1 v_1 + \cdots + c_n v_n,$$

and normalize u, getting

$$(u, u) = \sum c_i c_j (v_i, v_j) = 1.$$

We can now calculate max $D(u, u)$, for

$$D(u, u) = \sum c_i c_j D(v_i, v_j),$$

and hence we get

$$\frac{\partial[\sum c_i c_j D(v_i, v_j)]}{\partial c_i} = \sigma \frac{\partial[\sum c_i c_j (v_i, v_j)]}{\partial c_i},$$

which together with the equation

$$\sum c_i c_j (v_i, v_j) = 1$$

determine the c_i.

Hence max $D(u, u) = h(a_1, a_2, \cdots, a_r) \geqq \lambda_n$. Hence to approximate λ_n, we minimize the function $h(a_1, a_2, \cdots, a_r)$. Good approximations depend of course on the choice of the original v_i. It is to be noticed that this is just an extension of the ordinary method of computing λ_0.

Brown University