A GEOMETRIC THEOREM AND ITS APPLICATION TO BIORTHOGONAL SYSTEMS

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1. **The geometric theorem.** Let E_n denote an n-dimensional Euclidean space. A set of points in E_n may constitute an E_{n-1}; if so, it will be called a plane in E_n. We shall sometimes denote planes by $p_1, p_2, \cdots$. Also, if $x_1, \cdots, x_{n-1}$ are linearly independent vectors emanating from a common point O in E_n, the plane through O and containing these vectors will be denoted by $E(x_1, \cdots, x_{n-1})$. A plane divides E_n into two closed half-spaces each having the plane as boundary, but otherwise having no points in common. If S is a point set in E_n, a plane p is called a supporting plane of S if (1) S lies in one of the closed half-spaces determined by p, and (2) the distance between S and p is zero.

THEOREM 1. *Let S be a bounded and closed point set in E_n. Let O be a point of E_n, and suppose that the set consisting of O and S does not lie in any plane. Then there exist n linearly independent vectors $x_1, \cdots, x_n$ emanating from O, with terminal points $P_1, \cdots, P_n$ in S, and n planes $p_1, \cdots, p_n$ satisfying the following conditions: For each i* (a) *p_i contains P_i;* (b) *p_i is parallel to $E(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$;* (c) *$p_i$ is a supporting plane of the set consisting of O and S.*

PROOF. Consider any n points $P_1, \cdots, P_n$ in S. Let x_i be the vector from O to P_i. Form the parallelepiped determined by the vectors $x_1, \cdots, x_n$. As a figure in E_n this parallelepiped has a certain content, which is a function of $P_1, \cdots, P_n$, say $V(P_1, \cdots, P_n)$. Because of the hypothesis concerning O and S, there is at least one choice of $P_1, \cdots, P_n$ for which the parallelepiped is nondegenerate; V therefore assumes a positive value. Now V is a continuous function of the variables $P_1, \cdots, P_n$, each of which ranges over the compact set S. It follows that V assumes a positive absolute maximum value. Throughout the remainder of the proof we shall use $P_1, \cdots, P_n$ to denote a set of points in S at which V attains its absolute maximum. The fact that the maximum is positive then implies that the vectors $x_1, \cdots, x_n$ are linearly independent.

Let p_i be the plane through P_i parallel to $E(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. The parallelepiped of maximum content lies between these planes, each of a pair of opposite faces lying in one of the planes.

Presented to the Society, November 30, 1946; received by the editors November 12, 1946.

We have to prove that p_i is a supporting plane of the set consisting of O and S, that is (since O is not on p_i), that O and any point of S both lie in the same one of the two closed half-spaces determined by p_i. Suppose that for some point Q of S this were not the case. Consider the parallelepiped determined by the vectors $x_1, \cdots, x_{i-1}, OQ, x_{i+1}, \cdots, x_n$. Its content would evidently be greater than that of the parallelepiped determined by $x_1, \cdots, x_n$, in contradiction to the maximal property of the latter. For, the n-dimensional content of a parallelepiped is equal to the $(n-1)$-dimensional content of one of its faces multiplied by the distance between the plane of this face and that of the face opposite. The two parallelepipeds which we are comparing have a face through O in common, but the distance from O to p_i is less than the distance from O to the parallel plane through Q. The proof of Theorem 1 is now complete.[1]

2. **Normed linear spaces.** Let X denote a real normed linear space.[2] In this section we shall prepare the way for the proof of the theorem in the next section by discussing certain matters pertaining to n-dimensional linear subspaces of X. Let Y_n denote such a subspace, and let $y_1, \cdots, y_n$ be n linearly independent elements of Y_n, so that any y in Y_n is uniquely representable in the form

$$y = e_1 y_1 + \cdots + e_n y_n. \tag{2.1}$$

If in an n-dimensional Euclidean space E_n we introduce a rectangular coördinate system, we may set up a correspondence between the element y of Y_n, as given by (2.1), and the point with coördinates $(e_1, \cdots, e_n)$ in E_n. This correspondence is one-to-one and bicontinuous. We state the following lemmas, leaving the proof of the first one to the reader.

LEMMA 2.1. *The points of E_n which correspond to the elements y of Y_n for which $\|y\| \leqq 1$ form a bounded and closed convex set S which is symmetric about the origin. If $\|y\| < 1$, the corresponding point of E_n is an interior point of S.*

LEMMA 2.2. *Let S be the set referred to in Lemma 2.1. Let P_0 be a point on the boundary of S, and p_0 a plane of support of S through P_0. Denote by p the plane parallel to p_0 and through the origin of E_n. Let y_0 be the element of Y_n corresponding to P_0, and denote by M the set of*

[1] The referee brought to my attention the fact that a theorem similar to Theorem 1, for the special case in which S is a convex body with the point O as a center of symmetry, was announced by M. M. Day, in Bull. Amer. Math. Soc. Abstract 51-11-237.

[2] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 53.

elements of Y_n corresponding to points of p. Then there exists a linear functional f_0 defined on X with the properties (1) $\|f_0\|=1$, (2) $f_0(y_0)=1$, (3) $f_0(y)=0$ *for each y in M.*

PROOF. Since p_0 does not pass through the origin, it may be described by an equation of the form $a_1e_1+\cdots+a_ne_n=1$. The points of p will then satisfy the equation $a_1e_1+\cdots+a_ne_n=0$. By means of the correspondence (2.1) we define a linear functional ϕ_0 on Y_n by the equation $\phi_0(y)=a_1e_1+\cdots+a_ne_n$. It is easily proved, by using the last sentence of Lemma 2.1 and the fact that p_0 is a supporting plane of S, that $|\phi_0(y)|\leqq 1$ if $\|y\|\leqq 1$. Note that $\phi_0(y_0)=1$. Consequently, by the Hahn-Banach theorem,[3] there exists a linear functional f_0 defined on X, having $\|f_0\|=1$, and coinciding with ϕ_0 on Y_n. This functional evidently meets the requirements of Lemma 2.2.

3. **Biorthogonal systems.** Consider a real normed linear space X, and the conjugate space X^* of linear functionals defined on X. A pair of ordered sets $\{x_1, x_2, \cdots, x_n\}\subset X$ and $\{f_1, \cdots, f_n\}\subset X^*$ is called a biorthogonal system if $f_i(x_j)=1$ when $i=j$ and $f_i(x_j)=0$ when $i\neq j$ $(i, j=1, 2, \cdots, n)$. If in addition $\|x_i\|=\|f_i\|=1$ $(i=1, 2, \cdots, n)$, we shall call the pair of sets a biorthonormal system.

THEOREM 2. *Let Y_n be an n-dimensional linear subspace of X. Then there exists a biorthonormal system $\{x_1, \cdots, x_n\}$, $\{f_1, \cdots, f_n\}$ with $x_1, \cdots, x_n$ in Y_n.*

PROOF. Introduce the mapping of Y_n on E_n, and the set $S\subset E_n$ as defined in §2, Lemma 2.1. With O the origin in E_n, let $P_1, \cdots, P_n$ and $p_1, \cdots, p_n$ be points and planes related to S as described in §1, Theorem 1 (here O and S do not lie in any plane). Let $x_1, \cdots, x_n$ be the elements of Y_n corresponding to $P_1, \cdots, P_n$. The points P_j, $j\neq i$, are in the plane through O parallel to p_i. By Lemma 2.2 it follows that there exists a linear functional f_i such that $\|f_i\|=1, f_i(x_i)=1$, $f_i(x_j)=0$, $j\neq i$. This completes the proof of the theorem.

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[3] Banach, loc. cit. Théorème 2, p. 55.