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ON THE SUMMATION OF MULTIPLE FOURIER SERIES. III[1]

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Let $f(x)=f(x_1, \cdots, x_k)$ be a function of the Lebesgue class L, which is periodic in each of the k-variables, having the period 2π. Let

$$a_{\nu_1\cdots\nu_k} = \frac{1}{(2\pi)^k} \cdot \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} f(x) \exp \left\{ - i(\nu_1 x_1 + \cdots + \nu_k x_k)\right\} dx_1 \cdots dx_k,$$

where $\{\nu_k\}$ are all integers. Then the series $\sum a_{\nu_1}\cdots{}_{\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k)$ is called the multiple Fourier series of the function $f(x)$, and we write

$$f(x) \sim \sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k).$$

Let the numbers $(\nu_1^2 + \cdots + \nu_k^2)$, when arranged in increasing order of magnitude, be denoted by $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, and let

$$C_n(x) = \sum a_{\nu_1\cdots\nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k),$$

where the sum is taken over all $\nu_1^2 + \cdots + \nu_k^2 = \lambda_n$,

$$\phi(x, t) = \sum C_n(x) \exp(-\lambda_n t),$$

$$S_R(x) = \sum_{\lambda_n \leqq R^2} C_n(x), \qquad \lambda_n \leqq R^2 < \lambda_{n+1}.$$

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[1] Papers I and II with the same title are to appear in Proc. London Math. Soc.

Also, let $R_k(\lambda_s)$ and $r_k(\lambda_s)$ represent respectively the number of solutions of $\nu_1^2+ \cdots +\nu_k^2 \leqq \lambda_s$ and of $\nu_1^2+ \cdots +\nu_k^2 = \lambda_s$.

The object of this note is to study the convergence of multiple Fourier series, when summed up spherically by Bochner's method, that is, of the series $\sum C_n(x)$. We prove the following results.

THEOREM I. *If*

$$\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2} |a_{\nu_1 \cdots \nu_k}|^2 < \infty,$$

then the series $\sum_{n=0}^{\infty} C_n$ *converges at every point of continuity of* $f(x)$.

THEOREM II. *If*

$$\sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2+\epsilon} |a_{\nu_1 \cdots \nu_k}|^2 < \infty, \qquad \epsilon > 0,$$

then the series $\sum_{n=0}^{\infty} C_n$ *converges absolutely.*

The following result of Bochner[2] is used in the proof of the above theorems.

LEMMA. *At a point of continuity of* $f(x)$, $\phi(x, t)$ *tends to a limit as* t *tends to zero.*

PROOF OF THEOREM I. We shall first prove that

$$\lim_{R\to\infty} S_R(x) = \lim_{t\to+0} \phi(x, t), \tag{1}$$

whenever the limit on the right exists. Next, by the application of the above lemma, we deduce that at a point where $f(x)$ is continuous, $\sum C_n(x)$ is convergent.

Now

$$\begin{aligned} S_R(x) - \phi(x, t) &= \sum_{s=0}^{n} C_s[1 - \exp(-\lambda_s t)] - \sum_{s=n+1}^{\infty} C_s \exp(-\lambda_s t) \\ &\equiv \sum J_1 - J_2, \end{aligned} \tag{2}$$

say. We have,

$$\begin{aligned} J_1 &= \sum_{s=0}^{n} C_s[1-\exp(-\lambda_s t)] \\ &= \sum_{s=0}^{n} [1-\exp(-\lambda_s t)] \sum a_{\nu_1 \cdots \nu_k} \exp[i(\nu_1 x_1 + \cdots + \nu_k x_k)] \\ &= \sum a_{\nu_1 \cdots \nu_k} \exp i(\nu_1 x_1 + \cdots + \nu_k x_k)[1-\exp(-\nu_1^2 - \cdots - \nu_k^2)t], \end{aligned}$$

[2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207.

where the third sum runs over $\lambda_s=\nu_1^2+\cdots+\nu_k^2$ and the last sum runs over $\nu_1^2+\cdots+\nu_k^2\leqq\lambda_n$, so that

$$
\begin{aligned}
|J_1| &\leqq \sum | a_{\nu_1\cdots\nu_k}\{1-\exp(-\nu_1^2-\cdots-\nu_k^2)t\}| \\
&\leqq [\sum (\nu_1^2+\cdots+\nu_k^2)^{k/2}|a_{\nu_1\cdots\nu_k}|^2 \\
&\quad \times \sum\{1-\exp(-\nu_1^2-\cdots-\nu_k^2)t\}^2(\nu_1^2+\cdots+\nu_k^2)^{-k/2}]^{1/2} \\
&\leqq O(1)\cdot t\left[\sum_{s=0}^{n} r_k(\lambda_s)\lambda_s^{2-k/2}\right]^{1/2},
\end{aligned} \tag{3}
$$

where the first sum runs over $\nu_1^2+\cdots+\nu_k^2\leqq\lambda_n$.

Now,

$$
\begin{aligned}
\sum_{s=0}^{n} r_k(\lambda_s)\lambda_s^{2-k/2} &= \sum_{s=0}^{n-1} R_k(\lambda_s)\{\lambda_s^{2-k/2}-\lambda_{s+1}^{2-k/2}\}+R_k(\lambda_n)\lambda_n^{2-k/2} \\
&= O\left(\int_0^{\lambda_n} xdx\right)+O(\lambda_n^2) \\
&= O(\lambda_n^2).
\end{aligned} \tag{4}
$$

Hence, from (3), we obtain,

$$
|J_1| = O(t\lambda_n). \tag{5}
$$

Again,

$$
\begin{aligned}
|J_2| &= \left|\sum_{s=n+1}^{\infty} C_s\exp(-\lambda_s t)\right| \\
&\leqq \lambda_n^{-k/4}\sum_{s=n+1}^{\infty}\lambda_s^{k/4}|C_s\exp(-\lambda_s t)| \\
&\leqq \lambda_n^{-k/4}[\sum(\nu_1^2+\cdots+\nu_k^2)^{k/2}|a_{\nu_1\cdots\nu_k}|^2 \\
&\qquad\qquad \times\sum\exp\{-2(\nu_1^2+\cdots+\nu_k^2)t\}]^{1/2} \\
&\leqq \epsilon_n^{1/2}(t\lambda_n)^{-k/4}
\end{aligned} \tag{6}
$$

(in the last two sums $\nu_1^2+\cdots+\nu_k^2$ runs from λ_{n+1} to ∞), where

$$
\sum(\nu_1^2+\cdots+\nu_k^2)^{k/2}|a_{\nu_1\cdots\nu_k}|^2=\epsilon_n
$$

($\nu_1^2+\cdots+\nu_k^2$ runs from λ_{n+1} to ∞), and $\epsilon_n\rightarrow 0$ as $n\rightarrow\infty$, since $\sum e^{-2\nu^2 t}=O(t^{-1/2})$ as $t\rightarrow 0$. Thus, we have, from (5) and (6),

$$S_R(x) - \phi(x, t) = O(t\lambda_n) + o\left[\epsilon_n^{1/2}(t\lambda_n)^{-k/4}\right]. \tag{7}$$

If t is so chosen that $t\lambda_n = \delta_n = \epsilon_n^{1/k}$, then,

$$S_R(x) - \phi(x, t) = O(\delta_n) + O(\epsilon_n^{1/2} \cdot \delta_n^{-k/4}) = o(1), \qquad \text{as } n \to \infty.$$

PROOF OF THEOREM II.

$$\begin{aligned}
\sum | C_s(x) | &\leqq \sum | a_{\nu_1 \cdots \nu_k} | \\
&\leqq \left\{ \sum (\nu_1^2 + \cdots + \nu_k^2)^{k/2+\epsilon} | a_{\nu_1 \cdots \nu_k} |^2 \right\}^{1/2} \\
&\quad \times \left\{ \sum (\nu_1^2 + \cdots + \nu_k^2)^{-k/2-\epsilon} \right\}^{1/2} \\
&= O(1) \sum (r_k(\lambda_s)\lambda_s^{-k/2-\epsilon})^{1/2} \\
&= O\left(\left(\int^{\infty} R_k(x) x^{-k/2-1-\epsilon} dx\right)^{1/2}\right) \\
&= O\left(\left(\int^{\infty} x^{-1-\epsilon} dx\right)^{1/2}\right) < \infty.
\end{aligned}$$

On using Hölder's inequality instead of Schwarz's in (3) and (6), we can easily generalize Theorem I as follows:

If

$$\sum (\nu_1^2 + \cdots + \nu_k^2)^{k(p-1)/2} | a_{\nu_1 \cdots \nu_k} |^p < \infty,$$

where $1 < p \leqq 2$, *then* $\sum C_n$ *converges at every point of continuity of* $f(x)$.

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