being nilpotent (or in the ideal), where the subscripts tag corresponding images a in A, and so on. Two spectra S and Σ may be subjected to a generalized calculus guided by a distinct master sfield M. A cleft R is generated by one spectrum. If R properly contains a one-sfield spectrum it must be uncleft. The same holds if two spectra (relatively unordered) have overlapping images in M. The singular case of one element generation is easily handled in the case of algebras. The spectra fall into invariant subspaces, and an M without automorphisms is equivalent to one-spectrum subspaces. A spectrum may sometimes be generated by a function $k(x, t)$, where the parameters x generate a sfield and the t generate the equivalence classes. (Received March 22, 1946.)

135. Morgan Ward: *Note on the order of the free distributive lattice.*

If r_n denotes the order of the free distributive lattice on n elements, and if we set $\log_2 r_n$ equal to $2^n\phi(n)$, then for large n, $1/n^{1/2}<\phi(n)<1/4$ so that $\log_2 \log_2 r_n \sim n$. Computational evidence and combinatorial arguments suggest that $n^{1/2}\phi(n)\rightarrow\infty$, but the exact order of $\phi(n)$ is unknown. Incidentally the value of r_6 was computed. It is 7,828352. The method of computation devised easily verified Randolph Church's value 7579 for r_5 (Duke Math. J. vol. 6 (1940) pp. 732–734) but is not powerful enough to evaluate r_7 without prohibitive labor. (Received March 22, 1946.)

Analysis

136. R. H. Bing: *Converse linearity conditions.*

An example is given of a bounded function $f(x)$ $(a<x<b)$ having a derivative on its range and being nonlinear on every subinterval of its range which is such that each point of the graph of $f(x)$ and each point between two points of the graph of $f(x)$ is halfway between some two points of the graph of $f(x)$. (Received March 16, 1946.)

137. Garrett Birkhoff and L. J. Burton: *A weakening of the Hölder conditions for Newtonian force fields.*

Let $\rho(x)$ be a continuous density function of position $x=(x_1, \cdots, x_n)$ near a point $a=(a_1, \cdots, a_n)$ in Euclidean n-space. It is shown that the improper integrals $\int\int\int\rho(x_i-a_i)dR/r^n$ defining the force components for Newtonian attraction exist as improper Riemann multiple integrals (that is, are absolutely integrable) if and only if $\int\int\int\rho d\omega dr/r^{n-1}<+\infty$, where $d\omega$ denotes infinitesimal spherical area. The sufficiency of the usual Hölder conditions for convergence is a weak corollary of this. If $\rho dr d\omega=dm$, the corresponding result for Stieltjes integrals is obtained. (Received March 25, 1946.)

138. D. G. Bourgin: *Approximate isometries.*

The Hilbert space results of Hyers and Ulam (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 288–292) are extended to the spaces $L_p(0, 1)$, $1<p<\infty$. (Received March 22, 1946.)

139. R. C. Buck: *An extension of Carlson's theorem.*

Let $K^*(a, c)$ be the class of functions regular and of order 1 in $R\{z\}\geqq 0$, and of type a on the whole positive real axis and type c on the imaginary axis. If A is a subset of the set I of all integers, denote by $\gamma(A)$ the least number for which the following theorem is true: if $f(z)\in K^*(a, c)$, $c<\gamma(A)$, and if $f(z)$ vanishes in A then it vanishes identically. A theorem of Carlson asserts that $\gamma(I)=\pi$; more generally, if A possesses a density $D(A)$, then $\gamma(A)=\pi D(A)$. In this paper these results are extended to the case where A has a mean density or a Poisson density. For general A it is shown that $\pi \inf D_P(A) \leqq \gamma(A) \leqq \pi \sup D_P(A)$. Results are also obtained on the location of singularities of power series with missing coefficients; in these gap theorems, ordinary density and Pólya minimum density are replaced by more general densities. (Received March 18, 1946.)

140. R. H. Cameron and W. T. Martin: *The behavior of measure and measurability under change of scale in Wiener space.*

In this paper it is shown that the measured (outer measure, inner measure, measurability) of a set in Wiener space is completely independent of its measure (outer measure, inner measure, measurability) after change of scale. In fact, given two functions $f(\lambda)$, $g(\lambda)$ defined on $0<\lambda<\infty$ and completely arbitrary except that $0\leqq f(\lambda) \leqq g(\lambda)\leqq 1$, there exists a set E in Wiener space such that inner meas.$(\lambda E)=f(\lambda)$ and outer meas.$(\lambda E)=g(\lambda)$ for all positive λ. If $f(\lambda)\equiv g(\lambda)$, the set E (which is now measurable in every magnification) can be constructed explicitly (without the use of Zermelo's axiom). (Received March 19, 1946.)

141. W. B. Caton: *An inequality for analytic functions which are represented by their Fourier-Laguerre series.*

Let $f(z)$ be analytic, and let the nth Fourier-Laguerre coefficient be defined as $a_n=\int_0^\infty l_n(z)f(z)dz$, where $l_n(z)=e^{-z/2}L_n(z)$, with $L_n(z)$ the normalized polynomial of Laguerre of degree n. Now suppose that $-\lim\sup n^{-1/2}\log|a_n|=\kappa>0$, and assume that $f(z)$ is holomorphic in $2R[(-z)^{1/2}]=\kappa$, and that $f(z)=\sum_0^\infty a_n l_n(z)$ for any $z\equiv x+iy$ in this parabola. It is shown that for any γ, $0<\gamma<2^{-1}\kappa$, and any λ, $0\leqq\lambda\leqq\gamma$, one has $|f(z)|\leqq K|z|^{-1/8}\exp\{(x-|z|)/4-(\lambda/2)(|z|+x)^{1/2}\}$ for all values of z for which $|z|-x\leqq\gamma^2-\lambda^2$. The inequality obtained here is the analogue of an inequality of Hille for analytic functions represented by their Hermitian series. See E. Hille, *Contributions to the theory of Hermitian series*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 80–94. (Received March 18, 1946.)

142. W. M. Chen: *Distortion theorems in the theory of pseudo conformal mappings.*

Let P denote a mapping by a pair of two complex variable, $w_k(z_1, z_2)$, $k=1, 2$, which are regular in a domain M. M is assumed to be bounded by two analytic hypersurfaces: $z_1=h(z_2, \exp i\theta_2)$, $z_2=\exp i\theta_1$; $0\leqq\theta_k\leqq 2\pi$, $k=1, 2$. Let B be a proper subdomain of M. Using the integral formula proved by Bergman (Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) p. 851) and some of his results on complex orthogonal functions (Math. Zeit. vol. 36 (1933) p. 171), the author gives an upper (lower) bound for the variation of Euclidean volume of $P(B)$. These bounds depend upon B and the minimum (maximum) in $P(B)$ of the Kernelfunction of $P(M)$. Similar results are obtained for B-area (Bergman, Rendiconti della Reale Accademia Nazionale dei Lincei (6) vol. 19 (1934) p. 474) of a hypersurface $P(S)$, $S'\subset M$, from which bounds for the variation of ordinary area of $P(S)$ follow readily if an additional point function over $P(S)$ is introduced. (Received March 21, 1946.)

143. Nelson Dunford: *Resolutions of the identity.*

A formula of the type $\int\Psi(s)dxE_s^*x^*=x^*\psi(T)x$ is derived for an arbitrary linear operator on an arbitrary Banach space. The formula reduces to the customary one in the case of a self-adjoint operator in Hilbert space. It also throws some light on questions of strong and weak convergence of sequences of functions of certain operators on other Banach spaces. (Received February 23, 1946.)

144. Bernard Epstein: *Integral representation of solutions of the harmonic and wave equations.*

Using the method of integral operators (see Bergman, Math. Zeit. vol. 24 (1926) p. 641; Math. Ann. vol. 99 (1928) p. 629; Bull. Amer. Math. Soc. vol. 49 (1943) p. 163), the author investigates the singularities of some harmonic functions in three variables and "harmonic" vectors $\mathfrak{S}$ (that is, vectors for which $\nabla \cdot \mathfrak{S} = \nabla \times \mathfrak{S} = 0$). Algebraic functions with singularities along circles and other algebraic curves are obtained and investigated. A similar operator is introduced for the wave equation and the functions corresponding to the same "associates" as those previously studied in the harmonic case are investigated. Finally, a pair of operators is introduced which generates vector fields, $\mathfrak{E}$ and $\mathfrak{H}$, which constitute solutions of Maxwell's equations for the electric and magnetic fields in free space. (Received March 20, 1946.)

145. Herbert Federer: *The (ϕ, k) rectifiable subsets of n-space.*

Suppose $H_n^k(A)$ is the k-dimensional Hausdorff measure of the subset A of Euclidean n-space, and $F_n^k(A) = \beta(n, k)^{-1}\int_{G_n}\int_{E_k} N(P_R^k, A, y)dL_k y d\phi_n R$ is the k-dimensional integralgeometric Favard measure of A. Here L_k is Lebesgue measure over k-space, E_k, and ϕ_n is the Haar measure over the group G_n of all orthogonal transformations of n-space such that $\phi_n(G_n) = 1$. $N(P_R^k, A, y)$ is the number (possibly ∞) of points x in A for which $P_R^k(x) = y$. P_R^k is the projecting function on E_n to E_k such that $P_R^k(x) = (z_1, z_2, \cdots, z_k)$ whenever $R(x) = z = (z_1, z_2, \cdots, z_n)$, $R \in G_n$. Further $\beta(n, k) = \Gamma((k+1)/2)\Gamma((n-k+1)/2)/\Gamma(1/2)\Gamma((n+1)/2)$. The following theorem is proved: If A is an H_n^k measurable set for which $H_n^k(A) < \infty$, then $F_n^k(A) \leqq H_n^k(A)$. A necessary and sufficient condition for equality is that A be contained in a countable number of rectifiable (Lipschitzian) k-dimensional surfaces, except for a subset of H_n^k measure zero. Another result is as follows: The Lebesgue area of every continuous nonparametric two-dimensional surface in three space is equal to the Hausdorff measure H_3^2, and also to the Favard measure F_3^2, of the associated point set. (Received February 13, 1946.)

146. K. O. Friedrichs: *An inequality for potential functions.*

Let ϕ denote potential functions of the N variables $x_1, \cdots, x_N$ in a domain R of a rather general type. To every such domain R there is a constant Γ such that the inequality $\int_R(\partial\phi/\partial x_1)^2 dx \leqq \Gamma\int_R\left[(\partial\phi/\partial x_2)^2 + \cdots + (\partial\phi/\partial x_N)^2\right]dx$ holds provided that ϕ satisfies the side condition $\int_R(\partial\phi/\partial x_1)dx = 0$. The existence of the right-hand member implies in particular the existence of the left-hand member. For $N=2$, the inequality reduces to one between real and imaginary parts of an analytic function. The main tool of the proof is the identity $\int_R\sum_{k=1}^N(\partial\Omega_k/\partial x_k)(\partial\phi/\partial x_1)^2 dx = \int_R\left[\sum_{\mu=2}^N A_\mu(\partial\phi/\partial x_\mu)(\partial\phi/\partial x_1) + \sum_{\mu,\nu=2}^N B_{\mu\nu}(\partial\phi/\partial x_\mu)(\partial\phi/\partial x_\mu)\right]dx$, where A_μ and $B_{\mu\nu}$ are linear combinations of the first derivatives of the functions Ω_k, $k = 1, \cdots, N$. These functions Ω_k are to vanish at the boundary, to possess bounded derivatives, and to satisfy $(\partial\Omega_1/\partial x_1) + \cdots + (\partial\Omega_N/\partial x_N) \geqq 1$ in the neighborhood of the boundary. Such functions Ω_k are constructed for domains whose boundary possesses a continuous normal except for a finite number of corners or edges. (Received March 29, 1946.)

147. K. O. Friedrichs: *On a theorem of Lichtenstein.*

Let ϕ denote quadratically integrable functions of the N variables $x_1, \cdots, x_N$ which possess quadratically integrable Laplacian derivatives $\Delta\phi$ defined in an appropriate generalized sense. Then ϕ possesses all first and second derivatives in the same generalized sense. To every sphere S_0 and subsphere S there is a constant Γ such that the inequality $\int_S \Delta\phi^2 dx \leqq \int_{S_0} \Delta\phi^2 dx + \Gamma \int_{S_0} \sum_{k=1}^{N} (\partial\phi/\partial x_k)^2 dx$ holds. This theorem is used to generalize to N dimensions the theorem of Lichtenstein that the function $\phi(x, y) = (2\pi)^{-1} \int_R (\log r) \rho(x', y') dx' dy'$, with $r^2 = (x-x')^2 + (y-y')^2$, possesses almost everywhere quadratically integrable derivatives if ρ is quadratically integrable. (Received March 29, 1946.)

148. K. O. Friedrichs: *On the boundary-value problems of the theory of elasticity and Korn's inequality.*

Let $[u_1, \cdots, u_N]$ denote functions of the N variables $[x_1, \cdots, x_N]$ in a domain R of a rather general type. To any such domain R there is a constant κ such that the inequality $\int_R \sum_{\kappa,\lambda} (u_{\kappa,\lambda} - u_{\lambda,\kappa})^2 dx = \int_R \sum_{\kappa,\lambda} (u_{\kappa,\lambda} + u_{\lambda,\kappa})^2 dx$, $u_{k,l} = (\partial u_k/\partial u_l)$, holds provided that the functions u either vanish at the boundary of R or satisfy the side conditions $\int_R (u_{k,l} - u_{l,k}) dx = 0$, $k, l = 1, \cdots, N$. This inequality was formulated by A. Korn in 1907 for $N = 3$. A new more direct proof is given in the present paper. Korn's inequality is a decisive tool in the treatment of boundary value and characteristic value problems of elasticity by minimum problems. These problems can then be completely treated without using the characteristic singularity of the differential equation. (Received March 29, 1946.)

149. K. O. Friedrichs and W. R. Wasow: *On singular perturbations of nonlinear oscillations.*

Many questions in the theory of elastic and electric vibrations lead to systems of differential equations of the form $dX_i/dt = F_i(X_1, \cdots, X_n)$ $(i = 1, 2, \cdots, n-1)$, $\epsilon dX_n/dt = F_n(X_1, \cdots, X_n)$. Of particular interest in this connection is the determination of periodic solutions for small values of the parameter ϵ. The authors prove that if for $\epsilon = 0$ a periodic solution of such a system is given, then there exists for every small ϵ a unique periodic solution in its neighborhood, provided $\partial F_n/\partial X_n \neq 0$, and a certain nondegeneracy condition is satisfied. The latter states in essence that the given periodic solution for $\epsilon = 0$ is an isolated periodic solution. This result justifies the use of the perturbation method for the calculation of periodic solutions of such differential systems. These systems are of a singular type in as much as the order is reduced for $\epsilon = 0$. Although an approach related to that of Poincaré's classical investigations has been used, this singular character of the system necessitates considerable additions and modifications. (Received March 21, 1946.)

150. W. S. Gustin: *Convex boundary sets.*

Two topics are discussed in this paper. 1. A set E in a euclidean ν-space is a *convex boundary set* if E is a subset of the boundary of a convex set in space. Some characterizations of convex boundary sets are obtained. 2. A set E in a euclidean ν-space is of *order* τ (*exact order* τ) if every euclidean subspace of dimension $\nu - 1$ intersects E in at most τ (exactly τ) points. It is shown that: the closure of a connected set of finite order is of finite order; the closure of a connected set of order ν is of order ν; a connected set of exact order ν is totally disconnected. These two topics are linked by the theorem that a connected set of order ν in a euclidean ν-space is a convex boundary set. (Received March 21, 1946.)

151. P. R. Halmos. *Invariant measures.*

Necessary and sufficient conditions are found for the existence of an invariant σ-finite measure m^* stronger than m, for one-to-one measurable transformations T of a measure space X with σ-finite measure m. Hopf's theorem on the existence of a finite m^* is freed from its unnecessary nonsingularity condition; an example is constructed of an incompressible transformation which preserves no finite measure. The extent of uniqueness of the invariant measure (in case it exists) is determined. A structure theorem for nonpathological transformations is derived: it asserts that the general measurable transformation is the product of a measure preserving transformation and a purely dissipative transformation. (Received March 18, 1946.)

152. M. R. Hestenes: *Theorem of Lindeberg in the calculus of variations.*

The present paper is concerned with an analogue of the theorem of Lindeberg for the parametric problem of Bolza. It is shown that if C_0 is a nonsingular arc along which the strengthened condition of Weierstrass holds relative to an integral $I(C)$ and if near C_0 the E-function of $I(C)$ dominates the E-function of a second integral $J(C)$, then there is a constant $b>0$ such that given a constant $\epsilon>0$ one can select a neighborhood $\mathfrak{F}$ of C_0, such that $I(C)-I(C_0)\geqq b|J(C)-J(C_0)|-\epsilon$ for every admissible arc C in $\mathfrak{F}$. A number of consequences of this and similar results are given. It is shown in particular that the problems of Bolza and Mayer are equivalent, that parametric and nonparametric problems are equivalent, and that properimetric and non-isoperimetric problems are equivalent. (Received March 22, 1946.)

153. Einar Hille: *Analysis in a noncommutative Banach algebra without unit element.*

In the analytical theory of a complex (B)-algebra $\mathfrak{A}$ with unit element, the resolvent $R(\lambda; x)$ is basic. If no unit element exists, one may introduce instead the function $A(\lambda; x)$ with $x+\lambda A(\lambda; x)=xA(\lambda; x)=A(\lambda; x)x$. Thus $A(\lambda; x)$ is the adverse (quasi-inverse) of x/λ when it exists. This leads to a definition of resolvent sets and spectra which is consistent with usual conventions when $\mathfrak{A}$ has a unit element. $A(\lambda; x)$ is a holomorphic function of λ in each of the components of the resolvent set for fixed x and an F-differentiable function of x for fixed λ. If $f(\lambda)$ is holomorphic in a simply connected domain of the λ-plane containing the spectrum of x and if $f(0)=0$, then we may define $f(x)=-(1/2\pi i)\oint f(\lambda)A(\lambda; x)d\lambda/\lambda$ and this definition is consistent with the definition in terms of $R(\lambda; x)$ when $\mathfrak{A}$ has a unit element. Here $f(x)$ is F-differentiable and the mapping $f(\lambda)\rightarrow f(x)$ is a continuous isomorphism taking λ into x and is uniquely determined by these properties. (Received February 13, 1946.)

154. Alfred Horn: *The asymptotic behavior of solutions of integral equations.* Preliminary report.

In the case of the Volterra integral equation $u(x)=\lambda\int_\alpha^x K(x, y)u(y)dy+f(x)$, where $K(x, x)=-A(x)<0$ for $\alpha\leqq x\leqq\beta$, the author proves that if K and f are of class C^{n+1}, then $u(x)=\exp\left(-\lambda\int_\alpha^x A(t)dt\right)\sum_{i=0}^{n-1}F_i(x)/\lambda^i+\sum_{i=1}^{n}G_i(x)/\lambda^i+v(x, \lambda)/\lambda^n$, where $v(x, \lambda)=-G_n(\alpha)\exp\left(-\lambda\int_\alpha^x A(t)dt\right)+R(x, \lambda)/\lambda$, and $R(x, \lambda)$ is uniformly bounded for $\alpha\leqq x\leqq\beta$, $|\arg\lambda|\leqq\pi/2-\delta$. For F_i and G_i are independent of n and λ, and are obtained by solving recurrent differential equations and Volterra equations of the first kind respectively (each G dependent on the previous F_i and G_i). $G_1(x)$ is identically zero if and only if $f(x)$ is a multiple of $K(x, \alpha)$. Hence the resolvent kernel $R(x, y, \lambda)$ has a similar representation with $G_1(x, y) \equiv 0$. Also $G_2(x, y) \equiv 0$ if and only if $K(x, y)$ is of the form $-B(x)C(y)$, in which case $R = B(x)C(y) \exp\left(-\lambda\int_y^x B(t)C(t)\,dt\right)$. Using this representation, it is shown that if K and f are of class C^2, $K(x, x) < 0$, $f(\alpha) = 0$, then the solution of $u(x)/\lambda = \int_\alpha^x K(x, y)u(y)dy + f(x)$ approaches the solution of $\int_\alpha^x K(x, y)u(y)dy + f(x) = 0$ uniformly in $\alpha + \eta \leqq x \leqq \beta$, $|\arg \lambda| \leqq \pi/2 - \delta$. (Received March 13, 1946.)

155. R. D. James and W. H. Gage: *A generalized integral.* Preliminary report.

Let $D_+^2 g$ and $D_-^2 g$ denote the upper and lower limits, respectively, as $h \to 0$ of $\{g(x+h) - 2g(x) + g(x-h)\}/h^2$. In this paper a three-point integral, $\int_{a,b,c} f(x)dx$, is defined in terms of major and minor functions $M(x)$ and $m(x)$, which are continuous and satisfy the conditions: (1) $M(a) = M(c) = m(a) = m(c) = 0$; (2) $D_-^2 M \geqq f(x)$, $D_-^2 M > -\infty$, $D_+^2 M \leqq f(x)$, $D_+^2 m < +\infty$ for all x in (a, c). Certain properties of the integral are studied, and, in particular, it is shown that the function defined by $F(x) = -\int_{a,x,c} f(x)dx$ has, almost everywhere in (a, c), a generalized second derivative equal to $f(x)$. (Received March 12, 1946.)

156. S. Minakshisundaram and Otto Szász: *On absolute convergence of multiple Fourier series.*

The authors consider integrable functions of k variables, periodic of period 2π in each variable, and establish a close relationship between the mean modulus of continuity of a function and the absolute convergence of its Fourier series, thus generalizing certain results for simple Fourier series (cf. Szász, Trans. Amer. Math. Soc. vol. 42 (1936) pp. 366–396). In particular if the function satisfies a Lipschitz condition of degree α, and if $a_{n_1 \cdots n_k}$ are its Fourier coefficients, then $\sum |a_{n_1 \cdots n_k}|^\beta < \infty$ for $\beta > 2k/(k+2\alpha)$, while the series may become divergent for $\beta = 2k/(k+2\alpha)$. (Received March 5, 1946.)

157. M. E. Munroe: *A second note on weak differentiability of Pettis integrals.*

In a recent paper (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 167–174) the author proved that if Ω is any compact metric space containing nondenumerably many points and $C(\Omega)$ is the Banach space of all continuous functionals over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure. Using a different construction, the same result is obtained, assuming only that Ω contains infinitely many points. It then follows that if M is an abstract (M)-space with unity, a necessary and sufficient condition that every Pettis integral defined to M be almost everywhere weakly differentiable is that M be finite dimensional. (Received March 13, 1946.)

158. Leopoldo Nachbin: *On linear expansions.* I.

Trigonometric series $\sum a_n \cos nx + b_n \sin nx$ possess a number of properties embodied in the very well known theorems of Cantor-Lebesgue, Denjoy-Lusin, and so on. Mutual relations of these properties are investigated for general orthogonal systems. (Received February 25, 1946.)

159. George Piranian: *A summation matrix with a governor.*

For every formal series (1) $\sum a_n$ the relation $p_n\{a\}=\sum_{k=0}^{n}|a_k|$ defines a Nörlund sequence p_n provided (2) $\lim_{n\to\infty} p_n\{a\}/\sum_{r=0}^{n} p_r\{a\}=0$. The series (1) is defined to be summable (G) to S if condition (2) is satisfied and the series is summable $(N; p_n\{a\})$ to S. The method (G) is regular; it is included in the Abel method; and if $f(z)$ is a polynomial with real coefficients, the series $\sum(-1)^n f(n)$ is summable (G). (Received March 13, 1946.)

160. Maxwell Reade: *On averages of Newtonian potentials.*

The problem considered in this paper is that of determining relations that exist between the mass distribution associated with a Newtonian potential and the mass distribution associated with a particular average (mean-value) of the potential; this problem had been considered by Thompson (Bull. Amer. Math. Soc. vol. 41 (1935)) only for circular (spherical) mean-values. Although more general averages are considered in the paper, the following illustrates the results of the paper. Let $\mu(e)$ be a distribution of positive mass, defined for all Borel sets e contained in a fixed, bounded closed plane set F, such that $\mu(F)<\infty$. Define $\mu(E)\equiv\mu(E\cdot F)$, for all Borel sets E in the plane W. Let $U(x, y)$ be the Newtonian potential due to $\mu(E)$ at (x, y), and define $V(x, y)\equiv(1/4h^2)\int_{-h}^{h}\int_{-h}^{h}U(x+\xi, y+\eta)d\xi d\eta$. Then it is proved that $V(x, y)$ is a Newtonian potential with associated mass distribution $\sigma(E)\equiv(1/4h^2)\int_{-h}^{h}\int_{-h}^{h}\mu(E_{x,y})dxdy$, where $E_{x,y}$ is the image of E when the origin is translated to (x, y). Moreover, if $\delta(x, y)$ is the density of $\sigma(E)$ at (x, y), it is proved that $\delta(x, y)=\mu(Q_{x,y})/(4h^2)$ holds almost everywhere; here $Q_{x,y}$ is the square with vertices $(x\pm h, y\pm h)$. It should be noted that the method used here is different from that used by Thompson; the method used here depends upon the approximation methods developed by Riesz (Acta Math. vol. 54 (1930)). (Received March 21, 1946.)

161. L. A. Ringenberg: *On the extension of interval functions.*

In this paper a necessary and sufficient condition is obtained for a function of intervals to have a completely additive extension. This result was proved in a previous paper (*The extension of rectangle functions* by Reichelderfer and Ringenberg, Duke Math. J. vol. 8 (1941) pp. 231–242) using the results of Radon. The present proof is based on the theory of outer measure in the sense of Carathéodory; it permits a simple characterization of the extension. The second part of the paper is concerned with B-extensions. The properties of the B-extension were suggested by a result of Burkill on the extension of the indefinite integral of a function of intervals. Burkill obtained his extension using absolute continuity and integrability as sufficient conditions, while the present paper gives conditions which are both necessary and sufficient. (Received March 16, 1946.)

162. E. H. Rothe: *Gradient mappings in Hilbert space.*

Mappings of the form $y=x+F(x)$ of the real Hilbert space into itself are considered under the assumption that F is the gradient of a scalar $I(x)$. Necessary conditions and sufficient conditions for I are given in order that F be completely continuous. If these conditions are satisfied an approximation theorem for such mappings is proved which does not hold if F is not a gradient. The second part of the paper contains applications to nonlinear integral equations, especially to the eigenvalue problem. Since in case of the classical linear integral equation the symmetry of the kernel is the necessary and sufficient condition for the corresponding integral operator to be a gradient, the equation $x+F(x)=0$, where F is a gradient, appears as a natural nonlinear generalization of the linear symmetric kernel case. (Received March 21, 1946.)

163. A. C. Schaeffer and D. C. Spencer: *The coefficients of schlicht functions.* III. Preliminary report.

For functions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ which are regular and schlicht in the unit circle the coefficients $a_2, a_3, \cdots, a_n$ define a point in $2n-2$ dimensional real euclidean space. The properties of $f(z)$ restrict the point $(a_2, a_3, \cdots, a_n)$ to a well defined region of this space which is called the nth region of variability V_n. A method has been found which gives V_n implicitly for general n and explicitly in terms of elementary functions in the case $n=3$. Let $F(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ be a real function whose gradient is not zero in a domain containing V_n. There exist functions F which have a maximum in V_n at an everywhere dense set of boundary points. By using a variational method developed by the authors in previous papers it may be shown that any function $f(z)$ maximizing F satisfies a differential equation of a certain type. There is a one-to-one correspondence between solutions $f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$ of this differential equation which are regular in $|z|<1$ and boundary points $(a_2, a_3, \cdots, a_n)$ of V_n. (Received March 23, 1946.)

164. Robert Schatten (National Research Fellow): *On projections with bound* 1.

The present notation is that of Bull. Amer. Math. Soc. Abstract 51-9-162. A Banach space $\mathfrak{B}$ is proved to be unitary if, and only if, $\mathfrak{M}\otimes_\gamma\mathfrak{M}^*\subset\mathfrak{B}\otimes_\gamma\mathfrak{M}^*$, for every two-dimensional linear manifold $\mathfrak{M}\subset\mathfrak{B}$. (Received February 13, 1946.)

165. Y. C. Shen: *Interpolation to certain analytic functions by rational functions.*

Let $f(z)$ be a function of class S_2, that is, a function analytic in the interior K of the unit circle $|z|=1$, integrable together with its square on K, and (hence) capable of the integral representation $f(z)=\pi^{-1}\iint_K(f(t)/(1-z\bar{t})^2)dS$, $|z|<1$. Let a_{ni}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, be a set of preassigned points on K such that (i) a_{ni} have no limit point on K; (ii) $a_{ni}\neq 0$; and (iii) $a_{ni}\neq a_{nj}$ if $i\neq j$. Finally, let $f_n(z)$ be the rational function of the form $\sum_{i=1}^n A_{ni}/(1-\bar{a}_{ni}z)^2$, found by interpolation to $f(z)$ at the points a_{ni}: $f_n(a_{ni})=f(a_{ni})$, $i=1, 2, \cdots, n$. The following results are obtained: (A) *If* $\lim_{n\to\infty} n\prod_{i=1}^n|a_{ni}|^2=0$, *then, for every function* $f(z)$ *of class* S_2, *the sequence* $f_n(z)$ *converges to* $f(z)$ *on* K, *uniformly on any point set on* K. (B) *The condition in* (A) *can be replaced by the condition* $\lim_{n\to\infty}\prod_{i=1}^n|a_{ni}|=0$, *if, for each* n, *the* n *points* a_{ni} *all lie on a circle* C_n *orthogonal to the unit circle and passing through a fixed point* P *on* K. The restrictions (i), (ii), and (iii) on a_{ni} can be easily removed. (Received March 27, 1946.)

166. P. A. Smith: *Foundations of Lie groups.*

Let G_r be an r-parameter local group with coordinate system $a^1, \cdots, a^r$. Assume that the composition function ab can be represented in the form (written vectorially): $ab=a+b+|a|F(a, b)$ where $|a|=(\sum(a^i)^2)^{1/2}$ and $|F|\to 0$ when $|a|\to 0$, $|b|\to 0$. Then G_r is a local Lie group—that is, G_r admits a coordinate system in which ab is analytic. (Received March 28, 1946.)

167. Fred Supnick: *A class of functions with associated circle manifolds.*

The author treats functions of the form (1) $\sum(\pm 1)/(rn^2+sn+t)$ summed over a subset of the integers, where r, s and t are functions of one or two variables. These functions include the first and second derivatives of the log of the gamma function, $\pi^2/\sin^2 \pi x$, $(\psi(x)-\psi(y))/(x-y)$ and a number of definite integrals and double series, for example, $\sum_{n=2}^{\infty}\sum_{s=0}^{\infty}1/[(ps+r)^n-1]$. Let $C(x)$, $C(y)$, $C(z)$ be three mutually tangent circles not all tangent at one point. Let $C(1)$ and $C(-1)$ be the circles tangent to $C(x)$, $C(y)$, $C(z)$ ($C(1)$ being the smaller one if they are not both equal). Let $C(n)$ and $C(-n)$ be the circles tangent to $C(n-1)$, $C(y)$, $C(z)$ and $C(1-n)$, $C(y)$, $C(z)$ respectively, for $n=1, 2, \cdots$; $C(x)=C(0)$. The circles $C(x)$, $C(y)$, $C(z)$ are called the generators. A non-null subset of the set $\{C(i)\}$ $(i=0, \pm 1, \pm 2, \cdots)$ is called a circle manifold. In this paper it is proved that each function $f(x)$ of the class (1) has an associated circle manifold such that the value of $f(x)$ is $\sum(\pm$ radii) of the circle manifold. The radii of the generators are given as explicit functions of x. Also certain areal inequalities relating to circle manifolds are discussed. (Received March 16, 1946.)

168. Fred Supnick: *On certain limiting sequences of polygons.*

Let a polygonal line P_0 joining the noncollinear vertices u, v, w in the stated order be given. Let a polygonal line P_n be obtained from P_{n-1} by "cutting off" the corners of P_{n-1} according to a given rule R for $n=1, 2, \cdots$. Let R be such that the sequence of polygonal lines P_i $(i=1, 2, \cdots)$ has a limit P. Suppose it is required that P should have a first derivative at each (non-end) point. The question arises—how must R be defined? In this paper it is shown how R can and in certain cases must be defined so that P should have a derivative at each point. Other properties are also considered. On the other hand, the elements of the sequence can be obtained by building triangles on each side of the successive polygonal lines. The sequence P_i does not always have a limit in this case. If it does, it might be "pathological." Certain properties of the latter are derived. (Received March 18, 1946.)

169. Fred Supnick: *On Vitali's covering theorem.*

A synthetic proof (non-Vitalian methods) of the following special case of the Vitali covering theorem is given: Let some one maximal circle C_1 be packed into a Jordan domain J, C_2 into $J-C_1$ and C_n into $J-\sum_{i=1}^{n-1}C_i$ for $n=1, 2, \cdots$. Then $m(\sum C_i)=m(J)$. Vitali's theorem and some generalizations thereof relating to sphere packing problems are also discussed. (Received March 16, 1946.)

170. H. S. Wall: *Bounded J-fractions.*

A J-fraction is bounded and has norm N if the associated J-form is bounded and has norm N. There is determined a convex set K contained in the circle $|z|=N$ such that the J-fraction converges uniformly over every domain whose distance from K is positive. The J-matrix has a unique bounded reciprocal for any z not in K. If the coefficients in the J-fraction are real, the set K reduces to a real interval contained in the interval $(-N, +N)$. (Received March 5, 1946.)

171. Alexander Weinstein: *On a generalization of an axially symmetric potential.*

This paper deals with the differential equation (*) $y\Delta\phi+p\phi_y=0$, previously considered by H. Bateman (*Partial differential equations*, Cambridge University Press, 1932, pp. 408, 409) and L. Bers and A. Gelbart (Quarterly of Applied Mathematics vol. 1 (1943) pp. 168–188; Bull. Amer. Math. Soc. vol. 50 (1944) p. 56). For all real $p \geqq 1$, $\phi(x, y)$ is interpreted as the potential of a $(p+2)$-dimensional axially symmetric flow with a stream function $\psi(x, y)$ given by $\psi_x = -y^p\phi_y$, $\psi_y = y^p\phi_x$. It is shown that the fundamental solution $\phi_0(x, y)$ of (*), with a singularity at (x_0, y_0), can be expressed in terms of Bessel functions, and that the corresponding $\psi_0(x, y)$ is many-valued, except for $y_0 = 0$. An application of the divergence theorem yields the period of ψ_0 and leads to the determination of the various limits of the integral $\int_0^\infty \exp(-|x|s)J_q(as)J_{q+1}(ys)ds$ for $x \to 0$ and $y \to b$, where q denotes a non-negative real number. Corresponding results can be formulated in terms of generalized Laplace integrals. (Received February 28, 1946.)

172. J. E. Wilkins: *A note on the general summability of functions.*

There is a result of Titchmarsh (*Introduction to the theory of Fourier integrals*, Oxford, 1937) which gives sufficient conditions for the relation $\int K(x, y, \delta)f(y)dy = f(x) + o(1)$ as $\delta \to 0$. In case $K(x, y, \delta)$ has the form $\{\Gamma(\lambda+1)/2\pi^{1/2}\Gamma(\lambda+1/2)\}\cos^{2\lambda}(1/2)(y-x)$, where $\lambda = 1/\delta$, then Natanson (*On some estimations connected with singular integral of C. de la Vallée Poussin*, C. R. (Doklady) Acad. Sci. URSS. vol. 45 (1944) pp. 274–277) has shown that the remainder term $o(1)$ may be written as $\delta f''(x) + o(\delta)$. The author proposes to provide an extensive generalization of Titchmarsh's theorem to give sufficient conditions for the existence of an asymptotic expansion for the integral of $K(x, y, \delta)f(y)$. The author will also apply the general theory thus developed to several interesting kernels $K(x, y, \delta)$, and in particular will obtain the asymptotic expansion of which Natanson gave the first two terms. (Received March 21, 1946.)

173. H. J. Zimmerberg: *Two general classes of definite integral systems.*

In this paper the notions of definite integral systems considered by Reid (Trans. Amer. Math. Soc. vol. 33 (1931) pp. 475–485), Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) and the author (Bull. Amer. Math. Soc. Abstract 52-3-77) are extended to integral systems written in matrix form $y(x) = \lambda\{A(x)y(a) + B(x)y(b) + \int_a^b K(x, t)y(t)dt\}$, where the $n \times n$ matrix $K(x, t) \equiv H(x, t)S(t)$ and the $n \times 2n$ matrix $\|A(x)B(x)\| \equiv \|H(x, a)H(x, b)\|\mathcal{G}$, $\mathcal{G}$ denoting a $2n \times 2n$ constant matrix. These integral systems include the system of integral equations to which a system of first-order linear definite differential equations containing the characteristic parameter linearly in the two-point boundary conditions is equivalent. It is also shown that an integral system of the above form is equivalent to a system of $3n$ homogeneous equations of Fredholm type. (Received February 27, 1946.)

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174. Nathaniel Coburn: *Pressure-volume relations and the geometry of the net of characteristics in two-dimensional supersonic flows.*

In a previous paper (Quarterly of Applied Mathematics vol. 3 (1945) pp. 106–116), the author showed that for the Karman-Tsien pressure-volume relation in the two-dimensional supersonic flow of a perfect, irrotational, compressible fluid, the net of characteristics (Mach lines) consists of a Tschebyscheff (fish) net. In the present paper,