AREOLAR MONOGENIC FUNCTIONS

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1. Introduction. There have been several modifications of the definition of the derivative of a complex function of a complex variable which lead to theories of non-analytic functions. These generalizations were initiated by Riemann (1851) and Picard (1892) and followed with others by Pompeiu, Kasner and Cioranescu. The general derivatives of Riemann and Cioranescu depend on direction and have an infinite set of values at a given point, hence Kasner gave to the class of non-analytic functions considered the name polygenic functions to distinguish them from classical analytic, monogenic functions.[1]

The conditions for classical monogenity have been much reduced by Looman-Menchoff [7, pp. 9–16; 9, pp. 198–201].[2] We shall similarly reduce the restrictions for the existence of the Cioranescu single-valued areolar derivative and show that under those reduced conditions the real and imaginary parts of the areolar monogenic function are biharmonic. Finally the class of areolar monogenic functions so determined will be simply characterized in terms of the Pompeiu derivative.

2. The Cioranescu and Pompeiu derivatives. Let $f(z)=f(x, y)=u(x, y)+iv(x, y)$ be defined in a domain D of the complex variable $z=x+iy$. Construct a rectangle in D at a point z of D whose vertices in positive order are z, z_1, z', z_2. If z is taken as the pole of a polar coordinate system (ρ, ϕ) then $z_1-z=\rho_1e^{i\phi}$, $z_2-z=\rho_2e^{i(\phi+\pi/2)}$ and $z'-z=(\rho_1^2+\rho_2^2)^{1/2}e^{i(\phi+\alpha)}$ where $\alpha=\tan^{-1}\rho_2/\rho_1$. We now form the quotient

$$\Delta^2f(z) = \frac{f(z') - f(z_1) - f(z_2) + f(z)}{(z_1 - z)(z_2 - z)} \tag{2.1}$$

and consider the limit of $\Delta^2f(z)$ as ρ_1 and ρ_2 approach zero with ϕ held

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[1] See E. R. Hedrick, *Non-analytic functions of a complex variable*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 75–96 for an extensive bibliography. The author is indebted to the referee for the following observation. "Calugareano studied the second derivative of a polygenic function for only one rectilinear path of approach; Nicolesco studied it for any two different rectilinear paths of approach and Cioranescu considered the limit for two mutually perpendicular, rectilinear paths. Kasner and DeCicco have studied the geometry of the second derivative for a general curvilinear path of approach."

[2] Numbers in brackets refer to the references cited at the end of the paper.

constant. If $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives at z, the limit exists and we have

$$
\begin{aligned}
\underset{z_1 \to z,\, z_2 \to z}{L} \Delta^2 f(z) &= D^2 f(z) \\
&= (f_{xx} - f_{yy} - 2if_{xy})/4 - (f_{xx} - f_{yy} + 2if_{xy})e^{-4i\phi}/4 \\
&= [(u_{xx} - u_{yy} + 2v_{xy}) + i(v_{xx} - v_{yy} - 2u_{xy})]/4 \\
&\quad - [(u_{xx} - u_{yy} - 2v_{xy}) + i(v_{xx} - v_{yy} + 2u_{xy})]e^{-4i\phi}/4.
\end{aligned} \tag{2.2}
$$

The necessary and sufficient conditions that $D^2 f(z)$ shall be independent of ϕ are that

$$
u_{xy} = -(v_{xx} - v_{yy})/2, \qquad v_{xy} = (u_{xx} - u_{yy})/2. \tag{2.3}
$$

If (2.3) are satisfied then $D^2 f(z) = v_{xy} - iu_{xy}$ is the Cioranescu derivative.

If now $u(x, y)$ and $v(x, y)$ are assumed to have fourth order partial derivatives at z then from (2.3) we have $\nabla^4 u(x, y) = \nabla^4 v(x, y)$ where $\nabla^4 = \nabla^2(\nabla^2)$ and ∇^2 is the Laplacian operator.

We shall make the following definition.

Definition. A complex function $f(z) = u(x, y) + iv(x, y)$ defined in a domain D will be said to be areolar monogenic at a point (x, y) of D if $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives at (x, y) which satisfy (2.3). We shall call $f(z)$ an areolar monogenic function in D if these conditions are satisfied at every point of D.

The Pompeiu areolar derivative is defined as

$$
\frac{Df(z)}{Dw} = \underset{\delta=0}{L} \frac{1}{2iw} \int_s f(z)dz \tag{2.4}
$$

where s is any simple rectifiable closed curve of diameter δ which with its interior σ is in D and w is the area of σ. If $u(x, y)$ and $v(x, y)$ are continuous with their first and second order partial derivatives we have

$$
\frac{Df}{Dw} = [(u_x - v_y) + i(v_x + u_y)]/2 \tag{2.5}
$$

and

$$
\frac{D^2 f}{Dw^2} = [-v_{xy} + (u_{xx} - u_{yy})/2 + i\{u_{xy} + (v_{xx} - v_{yy})/2\}]/2. \tag{2.6}
$$

As remarked by Cioranescu [2, p. 29] we see by (2.6) that the class of areolar monogenic functions is the class of solutions of the differential equation $D^2 f/Dw^2 = 0$.

3. **Green's lemma.** Most discussions of monogeneity lead to the evaluation of $\int_{C(R)} f(z)dz$ where the integral is a line integral around a closed curve in D which in our case we shall assume is a rectangle with sides parallel to the coordinate axes. This integral is then transformed by Green's lemma into integrals over the area of the rectangle. A reduction of conditions under which

$$(3.1)\quad \int_{C(R)} f(z)dz = -\int\int_R (u_y + v_x)dxdy + i\int\int_R (u_x - v_y)dxdy$$

then leads to reduced conditions for monogeneity. We shall give three conditions for (3.1) to hold for all rectangles $C(R)$ in D.

(3.2) *Condition* C. If $u(x, y)$, $v(x, y)$, u_x, u_y, v_x and v_y are continuous in D, then (3.1) holds for all rectangles in D.

(3.3) *Condition* A. If $F(R) = \int_{C(R)} udy$, $G(R) = -\int_{C(R)} udx$ with similar conditions for $v(x, y)$ and F and G are absolutely continuous functions of point sets in D, then (3.1) holds for all rectangles $C(R)$ in D.

This is condition (A) of Evans [4, p. 32].

(3.4) *Condition* M. If u_x, u_y, v_x and v_y exist almost everywhere in D and are summable and, moreover, at each point of D except for at most a finite or denumerably infinite set, the functions $u(x, y)$ and $v(x, y)$ have finite Dini derivatives with respect to x and y, then (3.1) holds for all rectangles $C(R)$ in D.

This condition among others was given by Menchoff [5, p. 29].

4. **The principal theorem.** We are now in a position to prove the following theorem.

THEOREM 1. *If $u(x, y)$ and $v(x, y)$ satisfy conditions* C, A *or* M *in D and the equations*

$$(4.1)\qquad u_x - v_y = U(x, y), \qquad u_y + v_x = V(x, y)$$

hold almost everywhere (everywhere under condition C*) in D, where $U(x, y)$ and $V(x, y)$ are conjugate harmonic in D, then $f(z) = u(x, y) + iv(x, y)$ is areolar monogenic in D.*

PROOF. Under the conditions C, A or M the differential equations (4.1) are equivalent to the integral equations

$$(4.2)\qquad \begin{aligned} \int_{C(R)} vdx + udy &= \int\int_R U(x, y)dxdy, \\ \int_{C(R)} vdy - udx &= \int\int_R V(x, y)dxdy. \end{aligned}$$

These are the integral equations considered by Evans [4]. Let

$$
\begin{aligned}
\mu(x, y) &= \mu(M) = -\frac{1}{2\pi}\iint_D \log\frac{1}{MP}\, U(P)d\sigma_P, \\
\nu(x, y) &= \nu(M) = -\frac{1}{2\pi}\iint_D \log\frac{1}{MP}\, V(P)d\sigma_P.
\end{aligned} \tag{4.3}
$$

Since $U(x, y)$ and $V(x, y)$ are harmonic in D they satisfy a Hölder condition [6, p. 153] and the first and second partial derivatives of $\mu(M)$ and $\nu(M)$ exist at M and

$$
\nabla^2\mu(x, y) = U(x, y), \qquad \nabla^2\nu(x, y) = V(x, y). \tag{4.4}
$$

The general solution of (4.2) is

$$
u(x, y) = \frac{\partial\mu}{\partial x} + \frac{\partial\nu}{\partial y} + \frac{\partial\psi}{\partial y}, \qquad v(x, y) = \frac{\partial\nu}{\partial x} - \frac{\partial\mu}{\partial y} + \frac{\partial\psi}{\partial x}, \tag{4.5}
$$

where $\psi(x, y)$ is (in D) an arbitrary harmonic function. Now by the theorem of Evans [4, p. 33], (4.5) is a solution of (4.1) almost everywhere in D when $U(x, y)$ and $V(x, y)$ are only bounded and measurable in the Lebesgue sense. In our case with $U(x, y)$ and $V(x, y)$ harmonic in D, (4.5) is a solution of (4.1) everywhere in D and any solution of (4.1) satisfying C, A, or M can be given the form (4.5). We have from (4.5) and (4.4)

$$
u_x - v_y = \nabla^2\mu(x, y) = U(x, y), \quad u_y + v_x = \nabla^2\nu(x, y) = V(x, y) \tag{4.6}
$$

at all points (x, y) of D and therefore $\mu(x, y)$ and $\nu(x, y)$ satisfy

$$
\nabla^4\mu(x, y) = \nabla^2 U(x, y) = 0, \qquad \nabla^4\nu(x, y) = \nabla^2 V(x, y) = 0 \tag{4.7}
$$

at all points (x, y) of D. Solutions $\mu(x, y)$ and $\nu(x, y)$ of (4.7) are analytic in D and since $\mu(x, y)$, $\nu(x, y)$ and $\psi(x, y)$ are all solutions of $\nabla^4 w = 0$ their first partial derivatives are also biharmonic and therefore analytic in D. Therefore $u(x, y)$ and $v(x, y)$ by (4.5) are biharmonic and analytic in D. Now from (4.1) we derive that

$$
\begin{aligned}
2u_{xy} + (v_{xx} - v_{yy}) &= U_y + V_x = 0, \\
2v_{xy} - (u_{xx} - u_{yy}) &= -U_x + V_y = 0
\end{aligned}
$$

and therefore $f(z)$ is areolar monogenic in D.

COROLLARY. *If $u(x, y)$ and $v(x, y)$ satisfy the conditions* C, A *or* M *in D and the Pompeiu derivative exists for all rectangles $C(R)$ in D and $Df/DR = [U(x, y) + iV(x, y)]/2 = F(z)/2$ almost everywhere in D and $F(z)$ is monogenic in the ordinary sense, then $D^2f(z)/DR^2 = 0$ at all points of D and $f(z)$ is areolar monogenic.*

The following theorem is an analog of the first theorem of Menchoff [7, p. 9] relative to ordinary monogeneity.

THEOREM 2. *The function $f(z)=u(x, y)+iv(x, y)$ is areolar monogenic in a domain D if $u(x, y)$ and $v(x, y)$ satisfy condition* C *in D and if the second partial derivatives of $u(x, y)$ and $v(x, y)$ exist, are finite and satisfy the conditions* (2.3) *everywhere in D except for a point set E which consists of at most a finite or denumerable infinity of points.*

PROOF. Let $U(x, y)=u_x-v_y$, $V(x, y)=u_y+v_x$. Then by (2.3) we have on $D-E$

$$\begin{aligned} U_x - V_y &= -2v_{xy} + u_{xx} - u_{yy} = 0, \\ U_y + V_x &= 2u_{xy} + v_{xx} - v_{yy} = 0. \end{aligned} \tag{4.8}$$

By the theorem of Menchoff [7, p. 9] $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions in D and $F(z)=U(x, y)+iV(x, y)$ is a monogenic function in the ordinary sense. Therefore, by Theorem 1, $f(z)$ is areolar monogenic.

COROLLARY. *If $u(x, y)$ and $v(x, y)$ satisfy condition* C *in D and the second Pompeiu derivative exists, is finite and equal to zero at all points of D except for at most a finite or denumerably infinite set, then $f(z)$ is areolar monogenic in D.*

5. **An analog of Morera's theorem.** We have the following analog of the theorem of Morera.

THEOREM 3. *A necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that it be continuous, and for all circles $C(x_0, y_0; r)$ with center (x_0, y_0) and radius r in D*

$$\int_C f(z)dz = \pi r^2 i[U(x_0, y_0) + iV(x_0, y_0)] \tag{5.1}$$

where $U(x, y)$ and $V(x, y)$ are given conjugate harmonic functions in D.

PROOF. *Necessity.* If $f(z)$ is areolar monogenic in D we let

$$u_x - v_y = U(x, y), \qquad u_y + v_x = V(x, y) \tag{5.2}$$

and, by (5.1) and (2.3), $U(x,y)$ and $V(x, y)$ are conjugate harmonic in D. Therefore we have

$$\begin{aligned} \int_C f(z)dz &= \int_C udx - vdy + i\int_C vdx + udy \\ &= i\int\int_A [U(x, y) + iV(x, y)]dxdy. \end{aligned} \tag{5.3}$$

Now by the mean value property of harmonic functions we have from (5.2) that (5.1) is satisfied for all circles $C(x_0, y_0; r)$ in D.

Sufficiency. From (5.1) by the mean value property of harmonic functions we have (5.3) and therefore

$$\begin{aligned} \int_C v dx + u dy &= \int\int_A U(x, y) dx dy, \\ \int_C v dy - u dx &= \int\int_A V(x, y) dx dy. \end{aligned} \tag{5.4}$$

Now by an argument similar to that for Theorem 1, $f(z)$ is areolar monogenic.

COROLLARY. *The class of ordinary monogenic functions is a subclass of areolar monogenic functions* [1, pp. 264–265].

In this case $U(x, y) + iV(x, y) \equiv 0$ in D and by (5.1), $\int_C f(z) dz = 0$ for all circles $C(x_0, y_0; r)$ in D. The solution of (5.4) with the right-hand member zero is given by (4.6) with $\mu \equiv \nu \equiv 0$. Therefore $u(x, y)$ and $v(x, y)$ are conjugate harmonic in D and $f(z)$ is monogenic in the classical sense.

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