ON THE THEOREM OF FEJÉR-RIESZ

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1. Statement of results.

Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1}$$

be a function regular for $|z| \leqq 1$. The well known inequality of Fejér and Riesz asserts that

$$\int_D |f(z)|\,|dz| \leqq \frac{1}{2}\int_C |f(z)|\,|dz|, \tag{2}$$

where C is the circumference $|z| = 1$, and D any of its diameters.[1]

For $f(z) = F'(z)$, the inequality (2) takes the form

$$\int_D |F'(z)|\,|dz| \leqq \frac{1}{2}\int_C |F'(z)|\,|dz|, \tag{3}$$

which shows that the total variation of $F(z)$ along D does not exceed half of the total variation of F along C. In this form the inequality remains valid for harmonic functions. Let $z = \rho e^{i\theta}$. If $U(z) = U(\rho, \theta)$ is harmonic for $|z| \leqq 1$, the total variation of F along D does not exceed half of the total variation of F along C.[2] In symbols,

$$\int_D |U_\rho|\,d\rho \leqq \frac{1}{2}\int_C |U_\theta|\,d\theta. \tag{4}$$

Let $V(z) = V(\rho, \theta)$ be the harmonic function conjugate to U. In (4) we may replace U_ρ by $\rho^{-1}V_\theta$. Writing $U_\theta = u$, $V_\theta = v$, we obtain an equivalent form of the inequality (4), namely

$$\int_D \left|\frac{v(z)}{z}\right|\,|dz| \leqq \frac{1}{2}\int_C |u(z)|\,|dz|. \tag{5}$$

Received by the editors October 22, 1945, and, in revised form, November 19, 1945.

[1] L. Fejér and F. Riesz, *Ueber eine funktionentheoretische Ungleichung*, Math. Zeit. vol. 11 (1921) pp. 305–314.

[2] The inequality (4), with 1/2 on the right replaced by an undetermined constant A, was first proved by B. N. Prasad, *On the summability of power series and the bounded variation of power series*, Proc. London Math. Soc. vol. 35 (1933) pp. 407–424. That $C = 1/2$ was shown in F. Riesz, *Eine Ungleichung für harmonische Funktionen*, Monatshefte für Mathematik und Physik vol. 43 (1936) pp. 401–406, and A. Zygmund, *Some points in the theory of trigonometric and power series*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 586–617, especially p. 599.

It is valid for any pair of functions

$$u(z) = u(\rho, \theta) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)\rho^\nu, \tag{6}$$

$$v(z) = v(\rho, \theta) = \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta)\rho^\nu, \tag{7}$$

harmonic and conjugate in $|z| \leqq 1$, provided that $v(0)=0$.

The purpose of this note is to prove the following complement to (4).

THEOREM 1. *Let*

$$U(z) = \frac{A_0}{2} + \sum_{\nu=1}^{\infty} (A_\nu \cos \nu\theta + B_\nu \sin \nu\theta)\rho^\nu \qquad (z = \rho e^{i\theta}) \tag{8}$$

be a function harmonic for $|z| \leqq 1$, and let $U_n(z)$ be the nth partial sum of the series (8). *For every n, the total variation of $U_n(z)$ over D does not exceed $2/\pi$ times the total variation of $U(z)$ over C. The constant $2/\pi$ here cannot be replaced by any smaller number.*

We shall prove this result in the following equivalent form.

THEOREM 1′. *Let $u(z)$ and $v(z)$, given by* (6) *and* (7), *be harmonic for $|z| \leqq 1$ and conjugate (in particular, $v(0)=0$). Let $v_n(z)$ be the nth partial sum of the series* (7). *Then*

$$\int_D \left|\frac{v_n(z)}{z}\right| |dz| \leqq \frac{2}{\pi} \int_C |u(z)| |dz| \tag{9}$$

for all n, the factor $2/\pi$ on the right being the best possible.

Taking $u(z)=zf(z)$, so that $v(z)=-izf(z)$, we deduce from (9) the following corollary.

THEOREM 2. *Let $f(z)$ be a function regular for $|z| \leqq 1$ and let $s_n(z)$ be the nth partial sum of the series* (1). *Then for all n*

$$\int_D |s_n(z)| |dz| \leqq \frac{2}{\pi} \int_C |f(z)| |dz|. \tag{10}$$

In all these results the assumption that the functions are harmonic (or regular) inside *and on C* can obviously be relaxed, and more general results may be obtained from the special ones by routine passages to the limit.[3] Thus, for example, in Theorem 2 we may assume

[3] See also §3, below.

that $f(z)$ is regular for $|z|<1$ and continuous for $|z|\leqq 1$. The partial sums $s_n(z)$ for such an f may be unbounded, and it is of interest to note that in the passage from (2) to (10) the increase of the coefficient on the right (from $1/2$ to $2/\pi=0.64\cdots$) is not significant.

That $2/\pi$ is the best constant in Theorem 1′ (and so in Theorem 1) is easily seen from the example

$$u(z) = 1/2 + \rho R\cos\theta + \rho^2R^2\cos 2\theta + \cdots \qquad (z=\rho e^{i\theta})$$

where R is a fixed positive number less than 1. The integral on the right of (9) is then π. Taking $n=1$ and the segment $(-i, i)$ for D we find for the integral on the left the value $2R$. Since R may be as close to 1 as we wish, the conclusion follows.

As we shall see later, for each $n>1$ the constant $2/\pi$ on the right can be replaced by a constant $C_n<2/\pi$, and clearly $2/\pi=C_1\geqq C_2\geqq\cdots\geqq C_n\geqq\cdots$. One might expect that $C_n\to 1/2$, in accordance with (5). It is however not so, and the difference $C_n-1/2$ stays above a positive number. This fact has close connection with the Gibbs' phenomenon.

2. **Proof of Theorem 1′.** We shall use the formula

$$(11)\qquad v_n(\rho,\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi} u(1,\theta+t)\rho\Delta_n(\rho,t)dt,$$

where

$$(12)\qquad \Delta_n(\rho,t) = \sum_{\nu=1}^{n}\rho^{\nu-1}\sin\nu t.$$

Taking, as we may, for D the segment $(-1,+1)$ of the real axis, we get

$$\int_D\left|\frac{v_n(z)}{z}\right||dz| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi}|u(1,t)|\left\{\int_{-1}^{+1}|\Delta_n(\rho,t)|d\rho\right\}dt,$$

and our problem reduces to showing that

$$(13)\qquad \int_{-1}^{+1}|\Delta_n(\rho,t)|d\rho \leqq 2, \qquad \text{for } n=1,2,\cdots,\ -\pi\leqq t\leqq\pi.$$

That the integral on the left here is bounded as a function of n would be a simple matter to show. Complications arise when we want to show that the integral in question does not exceed 2.

We shall need the following lemma.

LEMMA. *Let $0 \leqq t \leqq \pi$. For every μ positive and odd, the values of the trigonometric polynomials*

$$\sin t/1 + \sin 3t/3 + \cdots + \sin \mu t/\mu,$$

$$\sin t/1 + \sin 3t/3 + \cdots + \sin (\mu - 2)t/(\mu - 2) + \sin \mu t/2\mu$$

are contained between 0 *and* 1.

Taking temporarily this lemma for granted, we shall proceed with the proof of the inequality (13). Let us consider the auxiliary function

$$\Delta_n^*(\rho, t) = \sum_{\nu=1}^{n}{}' \rho^{\nu-1} \sin \nu t = \sum_{\nu=1}^{n-1} \rho^{\nu-1} \sin \nu t + \frac{\rho^{n-1} \sin nt}{2}. \tag{14}$$

We easily find that

$$\Delta_n^*(\rho, t) = \frac{\dfrac{\sin t}{2}\left[2(1 - \rho^n \cos nt) - \rho^{n-1} \dfrac{\sin nt}{\sin t}(1 - \rho^2)\right]}{1 - 2\rho \cos t + \rho^2}.$$

We note that, for $0 \leqq \rho \leqq 1$,

$$\begin{aligned} 2(1 - \rho^n \cos nt) - \rho^{n-1} \frac{\sin nt}{\sin t}(1 - \rho^2) &\geqq 2(1 - \rho^n) - n\rho^{n-1}(1 - \rho^2) \\ &= (1 - \rho)[2(1 + \rho + \cdots + \rho^{n-1}) - \rho^{n-1}n(1 + \rho)] \\ &\geqq (1 - \rho)(2n\rho^{n-1} - 2n\rho^{n-1}) = 0. \end{aligned}$$

Thus Δ_n^* (unlike Δ_n) is non-negative for $0 \leqq t \leqq \pi$, and so nonpositive in the interval $(-\pi, 0)$. This remark applies, in particular, to the trigonometric polynomial

$$\Delta_n^*(1, t) = \sum_{\nu=1}^{n}{}' \sin \nu t.$$

Let us now fix t, $0 \leqq t \leqq \pi$. In the proof of (13) we shall consider various special cases.

Case (1). $\Delta_n(\rho, t) \geqq 0$, $\Delta_n(\rho, t+\pi) \leqq 0$ for $0 \leqq \rho \leqq 1$.

Thus

$$\begin{aligned} \int_{-1}^{+1} | \Delta_n(\rho, t) | \, d\rho &= \int_0^1 [\Delta_n(\rho, t) - \Delta_n(\rho, t + \pi)] d\rho \\ &= \int_0^1 \left\{ \sum_{\nu \leqq n} \rho^{\nu-1} \sin \nu t - \sum_{\nu \leqq n} (-1)^\nu \rho^{\nu-1} \sin \nu t \right\} d\rho \\ &= 2 \sum_{\nu \leqq n, \nu \text{ odd}} \frac{\sin \nu t}{\nu} \leqq 2, \end{aligned}$$

by the lemma.

Case (ii). $\Delta_n(\rho_0, t) < 0$, for some ρ_0, $0 \leqq \rho_0 \leqq 1$. We may assume that $0 < \rho_0 < 1$.

This cannot happen for $n=1$. Thus $n \geqq 2$. We shall show that in this case, for all ρ, $0 \leqq \rho \leqq 1$:

(a) $\Delta_n(\rho, t+\pi) < 0$.

(b) $|\Delta_n(\rho, t)|$ is majorized both by $\Delta_{n-1}(\rho, t)$ and $\Delta_{n+1}(\rho, t)$ (in particular, the latter quantities must be non-negative).

For suppose that

$$\Delta_n(\rho, t) = \sin t + \rho_0 \sin 2t + \cdots + \rho_0^{n-1} \sin nt < 0.$$

We know that for all ρ, $0 \leqq \rho \leqq 1$,

$$\begin{aligned} \Delta_n^*(\rho, t) &= \sin t + \rho \sin 2t + \cdots \\ &\quad + \rho^{n-2} \sin (n-1)t + 2^{-1}\rho^{n-1} \sin nt \geqq 0, \\ \Delta_{n+1}^*(\rho, t) &= \sin t + \rho \sin 2t + \cdots \\ &\quad + \rho^{n-1} \sin nt + 2^{-1}\rho^n \sin (n+1)t \geqq 0. \end{aligned}$$

A comparison of these inequalities shows that

$$\sin nt < 0, \qquad \sin (n+1)t > 0. \tag{15}$$

Thus, since $\Delta_n = \Delta_n^* + 2^{-1}\rho^{n-1} \sin nt$, we have

$$\begin{aligned} 2^{-1}\rho^{n-1} \sin nt &\leqq \Delta_n(\rho, t) \leqq \Delta_n^*(\rho, t), \\ |\Delta_n(\rho, t)| &\leqq |\Delta_n^*(\rho, t)| + |2^{-1}\rho^{n-1} \sin nt| \\ &= \Delta_n^*(\rho, t) - 2^{-1}\rho^{n-1} \sin nt = \Delta_{n-1}(\rho, t). \end{aligned} \tag{16}$$

Similarly, the formula $\Delta_n = \Delta_{n+1}^* - 2^{-1}\rho^n \sin(n+1)t$ and (15) imply

$$\begin{aligned} -2^{-1}\rho^n \sin (n+1)t &\leqq \Delta_n(\rho, t) \leqq \Delta_{n+1}^*(\rho, t), \\ |\Delta_n(\rho, t)| &\leqq |\Delta_{n+1}^*(\rho, t)| + |2^{-1}\rho^n \sin (n+1)t| \\ &= \Delta_{n+1}^*(\rho, t) + 2^{-1}\rho^n \sin (n+1)t = \Delta_{n+1}(\rho, t). \end{aligned} \tag{17}$$

Thus assertion (b) is proved. To prove (a) we replace t by $t+\pi$ in the equations

$$\begin{aligned} \Delta_n(\rho, t) &= \Delta_n^*(\rho, t) + 2^{-1}\rho^{n-1} \sin nt, \\ \Delta_n(\rho, t) &= \Delta_{n+1}^*(\rho, t) - 2^{-1}\rho^n \sin (n+1)t. \end{aligned}$$

If n is even, the first of these resulting equations, coupled with the inequalities $\Delta_n^*(\rho, t+\pi) \leqq 0$, $\sin nt < 0$, shows that $\Delta_n(\rho, t+\pi) < 0$. If n is odd, we similarly argue with the second equation. Thus (a) is proved.

Using (a) and (b), we see that in case (ii)

$$\int_{-1}^{+1} |\Delta_n(\rho, t)|\, d\rho \leqq \int_0^1 \{\Delta_{n\pm 1}(\rho, t) - \Delta_n(\rho, t+\pi)\} d\rho$$
$$= \int_0^1 \left\{ \sum_{\nu=1}^{n\pm 1} \rho^{\nu-1} \sin \nu t - \sum_{\nu=1}^{n} (-1)^\nu \rho^{\nu-1} \sin \nu t \right\} d\rho.$$

If n is even, we take the sign $+$ in $n\pm 1$, and the last integral becomes

$$2\left\{ \sum_{\nu\leqq n-1,\nu \text{ odd}} \frac{\sin \nu t}{\nu} + \frac{1}{2}\, \frac{\sin (n+1)t}{n+1} \right\} \leqq 2,$$

by the lemma. If n is odd ($n \geqq 3$), we take the sign $-$ in $n\pm 1$, and the integral in question takes the form

$$2\left(\sum_{\nu\leqq n-2,\nu \text{ odd}} \frac{\sin \nu t}{\nu} + \frac{1}{2}\, \frac{\sin nt}{n} \right) \leqq 2,$$

again by the lemma. Thus (13) holds in case (ii).

Case (iii). $\Delta_n(\rho_0, t+\pi) > 0$ for some ρ_0, $0 \leqq \rho_0 \leqq 1$.

To prove that (13) holds in this case, we observe that, since $\Delta_n(\rho, t)$ is odd in t,

$$\Delta_n(\rho_0, t+\pi) = -\Delta_n(\rho_0, \pi - t) = -\Delta_n(\rho_0, t'),$$

where $t' = \pi - t$. Thus $0 \leqq t' \leqq \pi$, $\Delta_n(\rho_0, t')$ is negative, and we are in case (ii). It follows that (13) holds if we replace there t by t'. But

$$\begin{aligned} \int_{-1}^{+1} |\Delta_n(\rho, t)|\, d\rho &= \int_{-1}^{+1} |\Delta_n(\rho, t-\pi)|\, d\rho \\ &= \int_{-1}^{+1} |\Delta_n(\rho, t')|\, d\rho \leqq 2, \end{aligned} \tag{18}$$

and (13) holds again.

Cases (i), (ii), (iii) exhaust all possibilities, since the simultaneous occurrence of the inequalities $\Delta_n(\rho_0, t) < 0$, $\Delta_n(\rho_1, t+\pi) > 0$ is excluded by case (ii). Thus for the completion of the proof of Theorem 1′ we need only prove the lemma.

In estimating the polynomial

$$\begin{aligned} &\frac{\sin t}{1} + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{\mu} \\ &= \int_0^t (\cos u + \cos 3u + \cdots + \cos \mu u) du \\ &= \int_0^t \frac{\sin (\mu+1)u}{2 \sin u}\, du, \end{aligned} \tag{19}$$

we may assume that $0 \leqq t \leqq \pi/2$, since replacing t by $\pi - t$ does not change the value of the polynomial. It is clear that the maximum of the last integral is attained for $t = \pi/(\mu+1)$, and is equal to

$$(20)\quad \int_0^{\pi/(\mu+1)} \frac{\sin (\mu+1)u}{2 \sin u} du = \int_0^{\pi} \frac{\sin u}{2(\mu+1) \sin (u/\mu+1)} du.$$

For fixed u, $0 \leqq u \leqq \pi$, and $\mu \geqq 1$, the minimum of the last denominator is attained when $\mu = 1$. For this particular value of μ the last integral becomes $2^{-1}\int_0^\pi \cos(u/2)du = 1$, and one-half of the lemma is proved.

Similarly, assuming, as we may, that $\mu \geqq 3$, we get

$$\sin t + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{2\mu} = \int_0^t \frac{\sin \mu u}{2 \tan u} du \leqq \int_0^{\pi/\mu} \frac{\sin \mu u}{2 \tan u} du$$
$$\leqq \int_0^{\pi/\mu} \frac{\sin \mu u}{2 \sin u} du \leqq 1$$

by the result just obtained. This completes the proof of the lemma.[4]

3. **Additional remarks.** (i) Let C_n be the least number such that

$$\int_D \left| \frac{v_n(z)}{z} \right| |dz| \leqq C_n \int_C |u(z)| |dz|$$

for all functions $u(z)$ harmonic in $|z| \leqq 1$. We know that $C_1 = 2/\pi$, and the argument just completed clearly shows that $C_n < 2/\pi$ for $n = 2, 3, \cdots$. The numbers $C_1 \geqq C_2 \geqq C_3 \geqq \cdots$ tend to a limit $C^* \geqq 1/2$. Combining the estimates in cases (i), (ii), (iii) with the inequalities of the lemma, we find that

$$(21)\quad \begin{aligned} C^* &\leqq \lim_{m\to\infty} \frac{1}{\pi} \int_0^{\pi/m} \frac{\sin mu}{\sin u} du = \lim_{m\to\infty} \frac{1}{\pi} \int_0^{\pi/m} \frac{\sin mu}{u} du \\ &= \frac{1}{\pi} \int_0^{\pi} \frac{\sin u}{u} du. \end{aligned}$$

On the other hand, let R be any positive number less than 1 and let

$$u(z) = u_R(z) = \pi^{-1}(1/2 + R\rho \cos \theta + R^2\rho^2 \cos 2\theta + \cdots), \quad z = \rho e^{i\theta},$$

so that $\int_C |u(z)| |dz| = 1$. If t, $0 < t < \pi$, is the angle of D with the positive real axis, then

[4] The author is indebted to a referee for pointing out a slip in the initial proof of the lemma.

$$\begin{aligned}\int_D \left|\frac{v_n(z)}{z}\right| |dz| &= \int_{-1}^{+1} |\Delta_n(\rho, t)| d\rho \\ &\geqq \int_0^{+1} \{\Delta_n(\rho, t) - \Delta_n(\rho, t+\pi)\} d\rho \\ &= 2\left\{R\frac{\sin t}{1} + R^3\frac{\sin 3t}{3} + \cdots + R^\mu \frac{\sin \mu t}{\mu}\right\},\end{aligned}$$

where μ is the largest odd integer not greater than n. Taking R as close to 1 as we wish, we see that the last sum comes arbitrarily close to

$$2\left(\frac{\sin t}{1} + \frac{\sin 3t}{3} + \cdots + \frac{\sin \mu t}{\mu}\right), \tag{22}$$

which is a partial sum of the Fourier series of the function $(\pi/2)$ sign t $(-\pi < t < \pi)$. This function has a jump at $t=0$ so that the partial sums (22) must display Gibbs' phenomenon. This also follows from (19) and (20), which show that

$$C^* \geqq \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u} du.$$

Comparing this with (21) we see that

$$C^* = \frac{1}{\pi}\int_0^\pi \frac{\sin u}{u} du = .589490 \cdots .$$

(ii) Let $U(\rho, \theta)$ be the Poisson integral of a function $F(t)$, $0 \leqq t < 2\pi$, of bounded variation and not constant. Thus

$$U(\rho, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{1 - 2\rho\cos(t-\theta) + \rho^2} F(t) dt.$$

Since the total variation of $U(z)$ on the circle $|z| = \rho < 1$ tends to the total variation of $F(t)$ over $(0, 2\pi)$, Theorem 1 gives

$$\int_D \left|\frac{d}{d\rho} U_n(z)\right| d\rho \leqq \frac{2}{\pi}\int_0^{2\pi} |dF(t)|. \tag{23}$$

This, of course, may be obtained directly, by applying to the formula

$$\frac{d}{d\rho} U_n(\rho, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Delta_n(\rho, \theta - t) dF(t)$$

the estimates for $\int_{-1}^{+1} |\Delta_n(\rho, t)| d\rho$. This direct approach shows that if, for simplicity, we take for D the segment $(-1, +1)$, then the sign of equality occurs in (23) if and only if (a) $n=1$, (b) $F(t)$ is $Cx(t)$, where C is any constant, and $x(t)$ equals $+1$ inside $(-\pi/2, +\pi/2)$ and equals -1 inside the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$. In other words, $U(\rho, \theta)$ must be

$$C\sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{\cos (2\nu - 1)\theta}{2\nu - 1} \rho^{2\nu-1}.$$

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