BIBLIOGRAPHY

1. M. R. Hestenes, *An analogue of Green's theorem in the calculus of variations*, Duke Math. J. vol. 8 (1941) pp. 300–311.
2. W. T. Reid, *Green's lemma and related results*, Amer. J. Math. vol. 63 (1941) pp. 563–574.
3. M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1939.

HARVARD UNIVERSITY

ON A CERTAIN TYPE OF NONLINEAR INTEGRAL EQUATIONS

MARK LOTKIN

1. **Introduction.** The object of this paper is to prove that the nonlinear integral equation

$$\phi(x) = \lambda \left[f(x) + \sum_{i=1}^{m} \int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \phi(s_1), \cdots, \phi(s_i)) ds_1 \cdots ds_i \right] \tag{1}$$

has at least one eigenvalue, provided the functionals

$$G_i(x, v) = \int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, v(s_1), \cdots, v(s_i)) ds_1 \cdots ds_i \tag{2}$$

are fully continuous, and the F_i satisfy a certain linear integrodifferential equation. The solution of (1) is shown to be equivalent to that of a variational problem containing infinitely many parameters. The latter problem, however, can be solved easily by the method of Rayleigh-Ritz, which consists in approaching the solution of the variational problem by a sequence of variational problems containing only a finite number of parameters. The convergence of this procedure is assured by a convergence theorem of Friedrich Riesz.

2. **Preparatory remarks.** Let I be the closed interval $a \leqq x \leqq b$, and L^2 the class of all functions having Lebesgue integrable squares on I with a norm not larger than N^2. Let, further, $\{v_n(x)\}$ $(n = 1, 2, 3, \cdots)$

Received by the editors May 18, 1944.

be a set of functions in L^2 and $\bar{v}(x)$ a function such that[1]

$$\lim_{n\to\infty} \int v_n(x)w(x)dx = \int \bar{v}(x)w(x)dx \tag{3}$$

for any arbitrary w of integrable square on I or, what is equivalent, for any arbitrary $w \in L^2$; that is, we assume the weak convergence of $\{v_n\}$, in the following denoted by $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$. We next show that $\bar{v} \in L^2$. Since the right-hand side of (3) is assumed to exist for every $w \in L^2$, it follows by a known result (see, for example, Kaczmarz-Steinhaus, *Theorie der Orthogonalreihen*) that $\bar{v}$ is of integrable square over I. Hence, if $w = \bar{v}$, (3) becomes

$$\int \bar{v}^2(x)dx = \lim_{n\to\infty} \int v_n(x)\bar{v}(x)dx,$$

and since by Schwarz's inequality

$$\left[\int v_n(x)\bar{v}(x)dx\right]^2 \leqq N^2 \left[\int \bar{v}^2(x)dx\right],$$

we obtain, as claimed,

$$\int \bar{v}^2(x)dx \leqq N^2. \tag{4}$$

We now assume that the functional $G_i(x, v)$ be fully continuous, that is, that

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\bar{x}, \bar{v}), \qquad i = 1, 2, \cdots, m, \tag{5}$$

for any $\{x_n\} \in I$ and $v_n \in L^2$ for which $\lim_{n\to\infty} x_n = \bar{x}$ and $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$.

The introduction of a closed orthonormal system of functions $\{w_\nu(x)\} \in L^2$ associates with each v_n an infinite sequence of numbers

$$c_{n\nu} = \int v_n(x)w_\nu(x)dx, \qquad \nu = 1, 2, \cdots,$$

such that

$$\sum_{\nu=1}^{\infty} c_{n\nu}^2 = \int v_n^2(x)dx \leqq N^2$$

for every $n \geqq 1$. The class L^2 of functions v then corresponds to a class $\mathfrak{H}^2$ of vectors $\mathfrak{v} \equiv (c_1, c_2, \cdots)$ with $c_\nu = \int v(x)w_\nu(x)dx$. The transition

[1] All integrations are to be extended over the interval I.

from L^2 to $\mathfrak{H}^2$ implies the substitution in $G_i(x_n, v_n)$ of $v_n(x)$ by its equivalent $\mathfrak{v}(x) \sim \sum_{\nu=1}^{\infty} c_{n\nu} w_\nu(x)$, and (5) now changes to

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\bar{x}, \bar{v}) \equiv P_i(\bar{x}, \bar{c}_1, \bar{c}_2, \cdots)$$

for any $\{x_n\} \in I$ and $\{v_n\} \in \mathfrak{H}^2$ for which $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} c_{n\nu} = \bar{c}_\nu$, $\nu = 1, 2, \cdots$. From the full continuity of the G_i thus follows the full continuity (*Vollstetigkeit*)[2] of the P_i. It is just as easily seen that the converse also holds true.

3. **Construction of a solution of the integral equation.** In the G_i we now admit as arguments v_n only aggregates of the form $v_n(x) = \sum_{\nu=1}^{n} c_{n\nu} w_\nu(x)$ with $\sum_{\nu=1}^{n} c_{n\nu}^2 = N^2$ for every fixed n. The functional

$$(6) \qquad J(v_n) = 2\left[\int f(x) v_n(x) dx + \sum_{i=1}^{m} e_i \int G_i(x, v_n) v_n(x) dx\right]$$

—here the e_i denote finite numbers to be determined later—is now a continuous function of the $c_{n\nu}$ and so has at least one minimum. Let $c_{n\nu} = a_{n\nu}$ $(\nu = 1, 2, \cdots, n)$ be the minimal coordinates:

$$(7) \quad \min J(v_n) = d_n = J(\phi_n), \quad \phi_n(x) = \sum_{\nu=1}^{n} a_{n\nu} w_\nu(x), \qquad \sum_{\nu=1}^{n} a_{n\nu}^2 = N^2.$$

As a consequence of (7) we have[3]

$$(8) \ \frac{\partial}{\partial c_{n\nu}}\left[J(v_n) + \frac{1}{\lambda_n}\left(N^2 - \sum_{\mu=1}^{n} c_{n\mu}^2\right)\right]_{c_{n\nu}=a_{n\nu}} = 0, \qquad \nu = 1, 2, \cdots, n.$$

However,

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int f w_\nu dx + \sum_i e_i \int\int \cdots \int K_i \left(v_n \frac{\partial F_i}{\partial c_{n\nu}} + w_\nu F_i\right) dx ds_1 \cdots ds_i\right].$$

We must now make the following assumption: *The F_i satisfy the linear integrodifferential equations*

$$(9) \quad \int\int \cdots \int K_i \left[e_i v_n \frac{\partial F_i}{\partial c_{n\nu}} - (1 - e_i) w_\nu F_i\right] dx ds_1 \cdots ds_i = 0$$

[2] See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, 1912, p. 177. A fully continuous function $P(x, c_1, c_2, \cdots)$, where $x \in I$ and $\sum_{r=1}^{\infty} c_r \leq N^2$, is bounded.

[3] In (8), $1/\lambda_n$ denotes Lagrange's multiplier for the extremum problem under consideration. As will be shown subsequently, $\lambda_n \neq 0$ for every n.

for all arguments $v_n = \sum_{(\nu)} c_{n\nu} w_\nu$, *identically in the* $c_{n\nu}$.

In this case we obtain

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int\left(f + \sum_i \int \cdots \int K_i F_i ds_1 \cdots ds_i\right) w_\nu dx\right],$$

and (8) leads to

$$(10) \qquad a_{n\nu} = \lambda_n \int G(x, \phi_n) w_\nu dx, \qquad \nu = 1, 2, \cdots, n,$$

with

$$G(x, \phi_n) = f(x) + \sum_i \int \cdots \int K_i(x, s_1, \cdots, s_i)$$
$$\cdot F_i(s_1, \cdots, s_i, \phi_n(s_1), \cdots, \phi_n(s_i)) ds_1 \cdots ds_i.$$

On account of $a_{n\nu} = \int \phi_n w_\nu dx$ the relations (10) may be written as

$$(11) \qquad \int (\phi_n - \lambda_n G(x, \phi_n)) w_\nu dx = 0 \qquad \text{for } \nu = 1, 2, \cdots, n.$$

Equations (10) show that the $|\lambda_n|$ have a common positive lower bound: multiplication of (10) by $a_{n\nu}$ and summation for $\nu = 1, \cdots, n$ result in

$$(12) \qquad N^2 = \lambda_n \int G(x, \phi_n) \phi_n dx.$$

But since $G(x, \phi_n)$ is a fully continuous function of the $a_{n\nu}$ for $x \in I$ and $\phi_n \in H^2$ it is bounded: there exists a $\delta > 0$ such that

$$|G(x, \phi_n)| \leqq N/(b-a)^{1/2}\delta \quad \text{for every } n.$$

Therefore

$$\left|\int G(x, \phi_n)\phi_n dx\right| \leqq \left[\int G^2(x, \phi_n) dx\right]^{1/2} \left[\int \phi_n^2 dx\right]^{1/2} \leqq N^2/\delta,$$

whence $|\lambda_n| \geqq \delta > 0$ for every n.

Now the $\int \phi_n^2 dx$ all have the same value N^2. This property of the sequence $\{\phi_n\}$ guarantees the existence of a $\Phi(x)$—defined almost everywhere in I and possessing a Lebesgue integrable square—which is the W-lim of a suitably chosen subsequence $\{\phi_{\bar{n}}\}$ of $\{\phi_n\}$,[4]

[4] Friedrich Riesz, *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Ann. vol. 69 (1910) p. 467. The sequence $\{\bar{n}\}$ is determined by Hilbert's diagonal method (see, for example, Hellinger-Toeplitz, Encyklopädie der mathematischen Wissenschaften vol. II C 13, p. 1405).

$$(13) \qquad W\text{-}\lim_{\bar{n}\to\infty} \phi_{\bar{n}}(x) = \bar{\Phi}(x);$$

because the system $\{w_\nu\}$ is closed $\bar{\Phi}$ is determined uniquely almost everywhere in I. On account of (4), $\int \bar{\Phi}^2(x)dx \leqq N^2$.

We are now going to show that

$$(14) \qquad \lim_{\bar{n}\to\infty} \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx = \int G(x, \bar{\Phi})\bar{\Phi}dx.$$

Since

$$\left|\int G(x, \bar{\Phi})\bar{\Phi}dx - \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx\right| \leqq \left|\int G(x, \bar{\Phi})(\bar{\Phi} - \phi_{\bar{n}})dx\right| + \left|\int (G(x, \bar{\Phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx\right|,$$

and the first expression on the right hand side—by (13)—may be made as small as desired by taking $\bar{n}$ sufficiently large, only the second term remains to be considered. Now

$$\left|\int (G(x, \bar{\Phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx\right| \leqq N\left[\int (G(x, \bar{\Phi}) - G(x, \phi_{\bar{n}}))^2dx\right]^{1/2},$$

and so (14) will be proved if we can show that $\lim_{\bar{n}\to\infty}\int(G(x, \bar{\Phi}) - G(x, \phi_{\bar{n}}))^2dx = 0$. This, however, follows immediately from the convergence theorem of Lebesgue.[5] The sequence $L_{\bar{n}} \equiv (G(x,\bar{\Phi}) - G(x, \phi_{\bar{n}}))^2$ obviously satisfies the conditions of that theorem: (a) $L_{\bar{n}}$ is Lebesgue integrable; (b) Since $|G(x, v)| \leqq N/(b-a)^{1/2}\delta$, $|L_{\bar{n}}| \leqq 4N^2/(b-a)\delta^2$ for every $\bar{n}$; (c) Because of the full continuity of $G(x, v)$, $\lim_{\bar{n}\to\infty}(G(x, \bar{\Phi}) - G(x, \phi_{\bar{n}})) = 0$. Therefore $\bar{L} = 0$, which proves (14).

We must now distinguish between these two cases:

I. There exists a $\delta' > 0$ such that $|\int G(x, \bar{\Phi})\bar{\Phi}dx| \geqq N^2/\delta'$;

II. $\int G(x, \bar{\Phi})\bar{\Phi}dx = 0$.

CASE I. By (12), $\lambda_{\bar{n}} = N^2/\int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx$, so that by (14)

$$(15) \qquad \lim_{\bar{n}\to\infty} \lambda_{\bar{n}} = \bar{\lambda} = \frac{N^2}{\int G(x, \bar{\Phi})\bar{\Phi}dx}$$

exists; because of I, $|\bar{\lambda}| \leqq \delta'$.

[5] If a sequence of Lebesgue integrable functions $L_n(x)$ possessing a common bound has a limit function $\bar{L}(x)$, then $\bar{L}$, too, is Lebesgue integrable and $\lim_{n\to\infty}\int L_n(x)dx = \int \bar{L}(x)dx$.

If we now apply equations (11) to indices $\bar{n}$ only and then take the $\lim_{\bar{n}\to\infty}$ we obtain

$$\int (\bar{\phi}(x) - \bar{\lambda}G(x, \bar{\phi}))w_\nu(x)dx = 0 \qquad \text{for } \nu = 1, 2, \cdots .$$

Since the system of the $\{w_\nu\}$ is closed we may deduce $\bar{\phi} - \bar{\lambda}G(x, \bar{\phi}) = 0$, that is,

$$\bar{\phi}(x) = \bar{\lambda}\Big[f(x) + \sum_{i=1}^{m}\int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \bar{\phi}(s_1), \cdots, \bar{\phi}(s_i))ds_1 \cdots ds_i\Big]$$

almost everywhere in I. We have thus obtained a solution $\bar{\phi}(x)$ of (1) belonging to the finite eigenvalue $\bar{\lambda}$.

The previously derived relationship $\int\bar{\phi}^2(x)dx \leqq N^2$ may now be improved: replacing $G(x, \bar{\phi})$ in (15) by its equal $(1/\bar{\lambda})\bar{\phi}$ leads to $\int\bar{\phi}^2(x)dx = N^2$.

CASE II. We write equations (10) for indices $\bar{n}$ only:

$$\int G(x, \phi_{\bar{n}})w_\nu dx = \frac{1}{\lambda_{\bar{n}}} a_{\bar{n}\nu}, \qquad \nu = 1, 2, \cdots, \bar{n},$$

and increasing $\bar{n}$ beyond any bound we obtain, since $\lim_{\bar{n}\to\infty}\lambda_{\bar{n}} = \infty$ and $|a_{\bar{n}\nu}| \leqq N$,

$$\int G(x, \bar{\phi})w_\nu dx = 0, \qquad \nu = 1, 2, \cdots .$$

In this case $\bar{\phi}(x)$ may be considered a solution of (1) belonging to $\lambda = \infty$.

4. **The variational problem.** We see, then, that $\bar{\phi}$ is always a solution of (1). This function possesses another important property: If $\bar{\mathfrak{H}}^2$ denotes the class of all $\mathfrak{v}(x) \sim \sum_{(\nu)} c_\nu w_\nu(x)$ with $\sum_{(\nu)} c_\nu^2 = N^2$, and

$$\bar{\mathfrak{v}}(x) \sim \sum_{\nu=1}^{\infty} \bar{a}_\nu w_\nu(x) \quad \text{with} \quad \bar{a}_\nu = \int \bar{\phi}(x)w_\nu(x)dx,$$

then $\bar{\mathfrak{v}}$ minimizes $J(\mathfrak{v})$.

To prove this we notice first that $J(\mathfrak{v}_n)$ results from $J(\mathfrak{v}_{n+1})$ if we put $c_{n+1} = 0$. Let d_n be the minimum of $J(\mathfrak{v}_n)$ in $\bar{\mathfrak{H}}^2$. Then obviously $d_n \geqq d_{n+1}$. Let, further, d be the minimum of $J(\mathfrak{v})$ for $\mathfrak{v} \in \bar{\mathfrak{H}}^2$; then $d_n \geqq d$ for every n. Therefore, if $\bar{d} = \lim_{\bar{n}\to\infty} d_{\bar{n}} = J(\bar{\mathfrak{v}})$, we get $\bar{d} \geqq d$ or $\bar{d} = d + \eta$ with $\eta \geqq 0$. We shall show that $\eta = 0$.

Since d is the lower bound of $J(\mathfrak{v})$ in $\bar{\mathfrak{H}}^2$ there exists a $\mathfrak{p}(x) \sim\sum_{(\nu)}p_\nu w_\nu(x)$ in $\bar{\mathfrak{H}}^2$ so that $J(\mathfrak{p})=\bar{d}-\theta\eta$ with $0<\theta\leqq 1$. If, now, $\epsilon>0$ be chosen as small as desired, there is, because of the full continuity of $J(\mathfrak{v})$, a $\delta>0$ and an index r such that $|J(\mathfrak{v})-J(\mathfrak{p})|<\epsilon$ for every $\mathfrak{v}\in\bar{\mathfrak{H}}^2$, so long as $|c_\nu-p_\nu|<\delta$ for $\nu=1, 2, \cdots, r$. We take $\epsilon=\theta\eta$ and choose $r'\geqq r$ large enough to have $\sum_{\nu=1}^{r'}p_\nu^2=N'^2>N^2\cdot N^2/(N+\delta)^2$. Then the vector $\bar{\mathfrak{p}}(x)=\sum_{\nu=1}^{r'}\bar{p}_\nu w_\nu(x)$ with $\bar{p}_\nu=(N/N')p_\nu$ belongs to $\bar{\mathfrak{H}}^2$, and since $N'>N^2/(N+\delta)$,

$$|\bar{p}_\nu - p_\nu| = |p_\nu|\cdot(N/N'-1) \leqq N(N/N'-1) < \delta$$

for $\nu=1, 2, \cdots, r'$. We may, therefore, conclude that $|J(\bar{\mathfrak{p}})-J(\mathfrak{p})| <\theta\eta$ or $J(\bar{\mathfrak{p}})<J(\mathfrak{p})+\theta\eta=\bar{d}$. But $d_{r'}\leqq J(\bar{\mathfrak{p}})$, and so $d_{r'}<\bar{d}$.

By now choosing $\bar{n}$, $\bar{n}\geqq r'$, such that $d_{\bar{n}}\leqq d_{r'}$ we get $d_{\bar{n}}\leqq\bar{d}$, a relation which contradicts the fact that the sequence $d_{\bar{n}}$ converges to $\bar{d}$ from above. Thus we see that $\eta=0$ or $\bar{d}=J(\bar{\mathfrak{v}})$.

5. **Solution of the integrodifferential equation.** It is easy to verify that equations (9) are fulfilled if we put $e_i=1/(i+1)$, K_i continuous and

$$K_i(x, s_1, \cdots, s_k, \cdots, s_i) = K_i(s_k, s_1, \cdots, x, \cdots, s_i), \qquad k = 1, 2, \cdots, i,$$

$$F_i(s_1, \cdots, s_i, u_1, \cdots, u_i) = a_i u_1 \cdots u_i$$

for $i=1, 2, \cdots, m$.

It remains to be shown that functionals of the type

$$Q(x, v) = \int \cdots \int K(x, s_1, \cdots, s_i)v(s_1)\cdots v(s_i)ds_1\cdots ds_i$$

are fully continuous for $x\in I$ and $v\in L^2$. Let us, therefore, assume that $\{x_n\}\in I$, $\{v_n\}\in L^2$, $\lim_{n\to\infty}x_n=\bar{x}$, and W-$\lim_{n\to\infty}v_n=\bar{v}$. Then

$$\begin{aligned}|Q(\bar{x}, \bar{v}) - Q(x_n, v_n)| &\leqq |Q(\bar{x}, \bar{v}) - Q(\bar{x}, v_n)| + |Q(\bar{x}, v_n) - Q(x_n, v_n)| \\ &= \left|\int\cdots\int K(\bar{x}, s_1, \cdots, s_i)[\bar{v}(s_1)\cdots\bar{v}(s_i) - v_n(s_1)\cdots v_n(s_i)]ds_1\cdots ds_i\right| \\ &\quad + \left|\int\cdots\int [K(\bar{x}, s_1, \cdots, s_i) - K(x_n, s_1, \cdots, s_i)]v_n(s_1)\cdots v_n(s_i)ds_1\cdots ds_i\right|.\end{aligned}$$

Because of the continuity of K the second term on the right-hand side may be made arbitrarily small by choosing n sufficiently large. In order to show that the same applies also to the first term we continue as follows:

$$\left| \int \cdots \int K[\bar{v}(s_1) \cdots \bar{v}(s_i) - v_n(s_1) \cdots v_n(s_i)]ds_1 \cdots ds_i \right|$$

$$= \left| \int \cdots \int K \sum_{k=1}^{i} \bar{v}(s_1) \cdots \bar{v}(s_{k-1})[\bar{v}(s_k) - v_n(s_k)] \cdot v_n(s_{k+1}) \cdots v_n(s_i) ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left| \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right) \cdot \bar{v}(s_1) \cdots \bar{v}(s_{k-1}) v_n(s_{k+1}) \cdots v_n(s_i) ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left\{ \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 \cdot ds_1 \cdots ds_{k-1} ds_{k+1} \cdots ds_i \right\}^{1/2} \cdot N^{i-1}.$$

Since $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$, $\lim_{n\to\infty}(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k) = 0$, and since $|\int K(\bar{x}, s_1, \cdots, s_i)[\bar{v}(s_k) - v_n(s_k)]ds_k| \leqq 2N(b-a) \cdot \max |K|$, we see that the sequence of Lebesgue integrable functions $(\int K(\bar{x}, s_1, \cdots, s_i) \cdot [\bar{v}(s_k) - v_n(s_k)]ds_k)^2$ has a common bound and the limit function zero. By Lebesgue's convergence theorem we may conclude that

$$\lim_{n\to\infty} \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 ds_1 \cdots ds_{k-1} ds_{k+1} \cdots ds_i = 0$$

for $k = 1, 2, \cdots, i$, so that the proof of the full continuity of $Q(x, v)$ is now complete.

6. A special case. The deductions of §4 are therefore applicable to the integral equation

$$\phi(x) = \lambda \left[f(x) + \sum_{i=1}^{m} a_i \int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot \phi(s_1) \cdots \phi(s_i) ds_1 \cdots ds_i \right].$$

If we assume that $a_m=1$, $a_i=0$ for $i=1, 2, \cdots, m-1$, that is, if we consider

$$\phi(x) = \lambda\left[f(x) + \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot\phi(s_1) \cdots \phi(s_m)ds_1 \cdots ds_m\right],$$

we know that it has at least one solution, and that this solution may belong to a finite or an infinite eigenvalue. The homogeneous equation

$$\phi(x) = \lambda \int \cdots \int K_m(x, s_1, \cdots, s_m)\phi(s_1) \cdots \phi(s_m)ds_1 \cdots ds_m,$$

however, has always at least one finite eigenvalue. In this case namely (see (6))

$$J(v_n) = 2e_m \int G_m(x, v_n)v_n(x)dx,$$

so that

$$d_n = \frac{2}{m+1}\int G_m(x, \phi_n)\phi_n(x)dx = \frac{2}{m+1}\cdot\frac{N^2}{\lambda_n}$$

or

$$\lambda_n \cdot d_n = (2/(m+1))N^2.$$

But since the functional

$$\frac{1}{2(m+1)}J(v) = \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot v(s_1) \cdots v(s_m)dxds_1 \cdots ds_m,$$

$K_m \not\equiv 0$, $v \in \overline{H}^2$, certainly has a minimum $\bar{d}$ differing from zero,

$$\bar{\lambda} = \lim_{\bar{n}\to\infty} \lambda_{\bar{n}} = \frac{2N^2}{\bar{d}(m+1)}$$

is finite.

CARLETON COLLEGE