POWER SERIES THE ROOTS OF WHOSE PARTIAL SUMS LIE IN A SECTOR[1]

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If the roots of the partial sums of a power series $f(z)=\sum a_n z^n$ lie in a sector with vertex at the origin and aperture $\alpha < 2\pi$, the power series cannot have a positive finite radius of convergence.[2] But if $f(z)$ is an entire function, the roots of its partial sums may lie in such a sector. The question arises: what restrictions are imposed on $f(z)$ by the requirement that α be sufficiently small, say $\alpha < \pi$? According to a theorem of Pólya the order of $f(z)$ must be not greater than 1 if the radius of convergence of the power series is positive.[3] Without this assumption the investigation which follows shows that *if $\alpha < \pi$, $f(z)$ is an entire function of order* 0. This result was obtained by Pólya for the case in which $\alpha = 0$.[4]

LEMMA. *If the complex numbers $z_1, \cdots, z_n$ $(z_1 \cdots z_n \neq 0)$ lie in a sector with vertex at the origin and aperture $\alpha < \pi$, then*

$$(1) \qquad \frac{n \cos \alpha/2}{\left| \sum_{k=1}^{n} z_k^{-1} \right|} \leqq \left| z_1 \cdots z_n \right|^{1/n} \leqq \frac{1}{n} \sec \alpha/2 \left| \sum_{k=1}^{n} z_k \right|.$$

When $\alpha = 0$ equality occurs if and only if $z_1 = \cdots = z_n$. When $\alpha > 0$ equality occurs if and only if n is even and $n/2$ of the numbers are equal to $re^{i\phi}$ $(r > 0;\ 0 \leqq \phi < 2\pi)$ and the other $n/2$ numbers are equal to $re^{i(\phi+\alpha)}$.

Suppose first that the sector is $-\alpha/2 \leqq \operatorname{am} z \leqq \alpha/2$. Let the n numbers be

$$z_k = \left| z_k \right| e^{i\theta_k}, \qquad -\alpha/2 \leqq \theta_k \leqq \alpha/2;\ k = 1, \cdots, n.$$

Since

$$(2) \qquad \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \left| z_k \right| \cos \theta_k + i \sum_{k=1}^{n} \left| z_k \right| \sin \theta_k,$$

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[1] Presented to the Society, April 27, 1940.

[2] This follows from Jentzsch's theorem: every point on the circle of convergence of a power series is a limit point of roots of its partial sums. See R. Jentzsch, *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Acta Mathematica, vol. 41 (1917), p. 219; E. C. Titchmarsh, *Theory of Functions*, 1932, p. 238.

[3] G. Pólya, *Ueber Annäherung durch Polynome deren sämtliche Wurzeln in einen Winkelraum fallen*, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1913, pp. 325–330.

[4] G. Pólya, *Ueber Annäherung durch Polynome mit lauter reellen Wurzeln*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 279–295.

$$(3) \qquad \left| \sum_{k=1}^{n} z_k \right| \geqq \sum_{k=1}^{n} | z_k | \cos \theta_k \geqq \cos \alpha/2 \sum_{k=1}^{n} | z_k |.$$

Now

$$(4) \qquad \frac{1}{n} \sum_{k=1}^{n} | z_k | \geqq | z_1 \cdots z_n |^{1/n}.$$

Consequently

$$\frac{1}{n} \left| \sum_{k=1}^{n} z_k \right| \geqq \cos \alpha/2 \, | z_1 \cdots z_n |^{1/n}.$$

Since the numbers $z_1^{-1}, \cdots, z_n^{-1}$ also lie in the sector $-\alpha/2 \leqq \text{am } z \leqq \alpha/2$, we have

$$\frac{1}{n} \left| \sum_{k=1}^{n} z_k^{-1} \right| \geqq \cos \alpha/2 \, | z_1^{-1} \cdots z_n^{-1} |^{1/n}.$$

Combining the last two inequalities, (1) results.

When $\alpha = 0$, (1) reduces to the well known relation among the harmonic, geometric and arithmetic means of n positive numbers. Here equality occurs if and only if $z_1 = \cdots = z_n$.

If equality occurs in (1) it also occurs in (3) and (4). By (4), $|z_1| = \cdots = |z_n|$. By (3), $\cos \theta_k = \cos \alpha/2$; hence $\theta_k = \pm\alpha/2$ $(k = 1, \cdots, n)$. By (2), $\sum_{k=1}^{n} \sin \theta_k = 0$. Therefore if $\alpha > 0$, n must be even, and $n/2$ of the numbers equal $re^{-i\alpha/2}$, while the other $n/2$ numbers equal $re^{i\alpha/2}$. Conversely, when these conditions are satisfied, equality is attained in (1).

If the numbers are in the sector $\phi \leqq \text{am } z \leqq \text{am } (\phi + \alpha)$, we apply the transformation

$$z' = e^{-i(\alpha/2 + \phi)} z,$$

which rotates this sector into the sector $-\alpha/2 \leqq \text{am } z \leqq \alpha/2$ without affecting the value of any member of (1).

THEOREM. *If, for each $n \geqq n_0$, the roots of the partial sum of degree n of the formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ lie in some sector with vertex at the origin and aperture $\alpha < \pi$,*[5] *then $f(z)$ is an entire function of order* 0.

The case in which $f(z)$ is a polynomial is trivial and is excluded from

[5] While α is independent of n, we do not require that there shall be one sector which contains the roots of all the partial sums of degree $n \geqq n_0$; the lines bounding the sector may be different for different values of n.

consideration. We shall suppose $a_0 \neq 0$; otherwise a power of z could be removed from $f(z)$ without affecting the theorem. Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k, \qquad n \geqq n_0.$$

By the Gauss-Lucas theorem the roots of $f_n'(z)$ are also in the sector which contains the roots of $f_n(z)$, and the only roots of $f_n'(z)$ that lie on the boundary of the sector are multiple roots of $f_n(z)$; hence $f_n'(0) \neq 0$. Repeated applications of this argument yield the result that $a_k \neq 0$ $(k=0, 1, \cdots)$.

According to the lemma, if $z_1, \cdots, z_n$ denote the zeros of $f_n(z)$,

$$(5) \qquad nc\left|\frac{a_0}{a_1}\right| \leqq \left|\frac{a_0}{a_n}\right|^{1/n} \leqq \frac{1}{nc}\left|\frac{a_{n-1}}{a_n}\right|, \qquad c = \cos \alpha/2;\ n \geqq n_0.$$

From the first two members of this inequality it follows that $|a_n|^{-1/n} \to \infty$ with n. Therefore $f(z)$ is an entire function. If ρ is its order,

$$(6) \qquad \frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log |a_n|^{-1}}{n \log n}.$$

From the last two members of (5) we have

$$(7) \qquad \begin{aligned} &\frac{1}{n}\log\frac{1}{|a_n|} - \frac{1}{n-1}\log\frac{1}{|a_{n-1}|} \\ &\qquad \geqq \frac{1}{n(n-1)}\log|a_0| + \frac{1}{n-1}\log nc. \end{aligned}$$

Let $m = \max(n_0, 4)$, $n > m$. Substituting $n = m, m+1, \cdots, n$ in (7), and adding, we obtain

$$\begin{aligned} \frac{1}{n}\log\frac{1}{|a_n|} \geqq \frac{1}{m-1}\log\frac{1}{|a_{m-1}|} &+ \log|a_0|\sum_{s=m}^{n}\frac{1}{s(s-1)} \\ &+ \sum_{s=m}^{n}\frac{\log s}{s-1} + \log c\sum_{s=m}^{n}\frac{1}{s-1}. \end{aligned}$$

Now

$$\sum_{s=m}^{n}\frac{\log s}{s-1} > \sum_{s=m}^{n}\frac{\log(s-1)}{s-1} > \frac{1}{2}\log^2 n - \frac{1}{2}\log^2(m-1),$$

$$\sum_{s=m}^{n}\frac{1}{s-1} > \log n - \log(m-1).$$

Consequently

$$\frac{1}{n} \log \frac{1}{|a_n|} > A + \frac{1}{2} \log^2 n + \log c \log n, \tag{8}$$

where A is bounded as $n \to \infty$. Comparing (6) and (8), we conclude that $\rho = 0$.

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