THE GENERALIZATION OF A LEMMA OF M. S. KAKEYA

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We shall prove the following:

LEMMA. *It is always possible to find the unique polynomial*

$$\phi^*(z) = \sum_{k=0}^{2s} \gamma_k^* z^k$$

of degree $2s$ *possessing the following properties*:

I. $$\phi^*(z) = ci^2(z)\tau(z)\tau^*(z), \qquad c = \text{const.},$$

the polynomial $i(z)$ *of degree* $\sigma \leqq s$ *having all roots in the domain* $|z| > 1$:

$$i(z) = \prod_{i=1}^{\sigma} (z - a_i), \qquad |a_i| > 1, \quad i = 1, 2, \cdots, \sigma,$$

and the polynomial $\tau(z)$ *being of degree* $\nu = s - \sigma$:

$$\tau(z) = \prod_{i=1}^{\nu} (z - \alpha_i), \qquad \tau^*(z) = z^\nu \bar{\tau}\left(\frac{1}{z}\right) = \prod_{i=1}^{\nu} (1 - z\bar{\alpha}_i).$$

II. *It is subject to the conditions*

$$\omega_i(\phi^*) = \sum_{k=0}^{2s} \gamma_k^* c_k^{(i)} = d_i, \qquad i = 0, 1, \cdots, s,$$

the given linear functionals ω_i *being such that every polynomial* $\phi(z)$ *of degree* $n \geqq 2s$ *for which*

$$\omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c_k^{(i)} = 0, \quad (i = 0, 1, \cdots, s), \qquad \phi(z) = \sum_{k=0}^{n} \gamma_k z^k,$$

has $s+1$ *roots at least in the domain* $|z| < 1$.

In the particular case when

$$\omega_i(\phi) = \phi^{(i)}(z_k), \qquad |z_k| < 1,$$

this lemma has been proved by M. S. Kakeya [1];[1] without being aware of his result we have proved this lemma in the case[2]

[1] Numbers in brackets refer to the bibliography at the end.

[2] In [1] and [2] one may find the application of this lemma to some extremal problems.

$$\omega_i(\phi) = \frac{1}{i!}\left(\frac{d^i\phi}{dz^i}\right)_{z=0}, \qquad i = 0, 1, \cdots, s.$$

In order to prove this lemma in the most general case we consider the following extremal problem:

PROBLEM. *To find the minimum of the integral*

$$L(b) = \int_0^{2\pi} |\, t(z)\,|^2 b(\theta)d\theta, \qquad z = e^{i\theta}, \tag{1}$$

$t(z)$ being the given polynomial of degree s with $t(0) \neq 0$ and $b(\theta)$ being a trigonometric polynomial of order $n \geqq 2s$:

$$b(\theta) = R\left\{z^n\bar{\phi}\left(\frac{1}{z}\right)\right\} = R\sum_{k=0}^{n} \bar{\gamma}_k e^{i(n-k)\theta}, \qquad z = e^{i\theta},$$

subject to the conditions[3]

$$\omega_i(b) = \omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c_k^{(i)} = d_i, \quad i = 0, 1, \cdots, s. \tag{2}$$

The fundamental property of our functionals ω_i yields at once that every trigonometric polynomial $b(\theta)$ subject to the conditions

$$\omega_i(b) = 0, \qquad i = 0, 1, \cdots, s,$$

has in $(0, 2\pi)$ no more than $2(n-s-1)$ changes of sign. It is clear that there exists a solution of our problem. Further, the necessary conditions for an extremum are

$$\operatorname{sgn} b^*(\theta)\,|\, t(z)\,|^2 = R\sum_{k=n-2s}^{\infty} A_k z^k, \qquad z = e^{i\theta},$$

whence we find at once that the Fourier expansion of $\operatorname{sgn} b^*(\theta)$ is of the form

$$\operatorname{sgn} b^*(\theta) = R\sum_{k=n-s}^{\infty} B_k z^k, \qquad z = e^{i\theta}.$$

We have shown in [2] that every trigonometric polynomial with this property must be of the form

$$b^*(\theta) = R\{\bar{c}z^{n-2s+\nu}q^2(z)\}\tau(z)\bar{\tau}(1/z), \qquad z = e^{i\theta},$$

$q(z)$ being a polynomial of degree $\sigma \leqq s$ all of whose roots lie in the domain $|z| < 1$, and $\tau(z)$ being a polynomial of degree $\nu = s - \sigma$.

[3] The functionals ω_i are the same as above.

The polynomial $b^*(\theta)$ for which the minimum is attained is unique. If there were two such polynomials, $b_1^*(\theta)$ and $b_2^*(\theta)$, then we would have

$$L(b_1^*) \leqq L\left(\frac{b_1^* + b_2^*}{2}\right) \leqq L(b_1^*);$$

then $b_1^*(\theta)$ and $b_2^*(\theta)$ would change sign at the same points, that is, the polynomial

$$b_1^*(\theta) - b_2^*(\theta) = R\{z^{n-2s+\nu}q^2(z)\}\{\bar{c}_1 | \tau_1(z)|^2 - \bar{c}_2 | \tau_2(z)|^2\}, \qquad z = e^{i\theta},$$

would have at least $2(n-\nu)$ changes of sign in $(0, 2\pi)$; but since

$$\omega_i(b_1^* - b_2^*) = 0, \qquad i = 0, 1, \cdots, s,$$

the polynomial $b_1^*(\theta) - b_2^*(\theta)$ cannot have more than $2(n-s-1)$ changes of sign in $(0, 2\pi)$; this contradiction proves the unicity of the polynomial solving our problem. *Thus we find that there exists the unique polynomial* $b^*(\theta)$ *minimizing* (1) *under conditions* (2) *and it must be of the form*

$$\begin{aligned} b^*(\theta) &= R\{\bar{c}z^{n-2s+\nu}q^2(z)\tau(z)\bar{\tau}(1/z)\} \\ &= R\{\bar{\gamma}_0^* z^n + \bar{\gamma}_1^* z^{n-1} + \cdots + \bar{\gamma}_{2s}^* z^{n-2s}\}, \qquad z = e^{i\theta}. \end{aligned}$$

Since the real parts of two polynomials coincide on the unit circle, these polynomials are identical, that is,

$$\bar{c}z^{n-2s}q^2(z)\tau(z)\tau^*(z) = \bar{\gamma}_0^* z^n + \bar{\gamma}_1^* z^{n-1} + \cdots + \bar{\gamma}_{2s}^* z^{n-2s},$$

whence we find finally

$$\phi^*(z) = \gamma_0^* + \gamma_1^* z + \cdots + \gamma_{2s}^* z^{2s} = ci^2(z)\tau(z)\tau^*(z),$$

where

$$i(z) = q^*(z) = z^\sigma \bar{q}(1/z).$$

Thus we have found the polynomial $\phi^*(z)$ satisfying all the conditions of our lemma.

Bibliography

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2. J. Geronimus, *On a problem of F. Riesz and on the generalized problem of Tchebycheff-Korkine-Zolotareff*, Bulletin de l'Académie des Sciences de l'URSS., vol. 3 (1939), pp. 279–288.

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