

FOURIER COEFFICIENTS OF BOUNDED FUNCTIONS¹

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The results of this paper are divided into two parts. First: inequalities for the Fourier coefficients of any bounded function; second: an approximation theorem for the Fourier development of an arbitrary bounded function.

Inequalities for Fourier coefficients have been discussed in a paper by Professor Szász.² However, his work deals mainly with linear inequalities for complex coefficients. The inequalities to be investigated in this paper are not linear. Nevertheless, they are the best possible, for this reason: given any set of numbers which makes the inequality an equality, there exists a bounded function which has these numbers as its Fourier coefficients.

A simple illustration will clarify this. Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leq 1$. The Fourier coefficients of $f(x)$ are given by the formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

$n = 1, 2, 3, \dots$

Then, it is clear that

$$|a_n| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)| |\cos nx| \, dx \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos nx| \, dx,$$

$$|b_n| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)| |\sin nx| \, dx \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin nx| \, dx$$

since $|f(x)| \leq 1$.

Since the cosine is negative in the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$ and positive in the remaining interval $(-\pi/2, \pi/2)$,

$$\begin{aligned} \int_{-\pi}^{\pi} |\cos x| \, dx &= \int_{-\pi}^{-\pi/2} (-\cos x) \, dx + \int_{-\pi/2}^{\pi/2} \cos x \, dx \\ &\quad + \int_{\pi/2}^{\pi} (-\cos x) \, dx = 4. \end{aligned}$$

Therefore, $|a_1| \leq 4/\pi$. However, if $f_1(x) = -1$, $-\pi < x < -\pi/2$; $f_1(x)$

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² American Journal of Mathematics, vol. 61 (1939).

$= 1$, $-\pi/2 < x < \pi/2$; $f_1(x) = -1$, $\pi/2 < x < \pi$; then

$$\begin{aligned} \int_{-\pi}^{\pi} f_1(x) \cos x \, dx &= \int_{-\pi}^{-\pi/2} (-\cos x) \, dx + \int_{-\pi/2}^{\pi/2} \cos x \, dx \\ &\quad + \int_{\pi/2}^{\pi} (-\cos x) \, dx \end{aligned}$$

and the bound $4/\pi$ will actually be attained for the function $f_1(x)$.

Likewise, $\int_{-\pi}^{\pi} |\cos nx| \, dx$ and $\int_{-\pi}^{\pi} |\sin nx| \, dx$ both equal four, so that $|a_n|$ and $|b_n| \leq 4/\pi$; these bounds will be attained for the functions $\text{signum } \cos nx$ and $\text{signum } \sin nx$ respectively.

But these bounds for the Fourier coefficients cannot be attained simultaneously. If one of the coefficients is too large, the others must be small. For example: b_1 and b_2 must satisfy the following inequality:

$$\left(\frac{\pi}{4} b_1\right)^2 + \frac{\pi}{4} |b_2| \leq 1$$

so that if $b_1 = 4/\pi$ then $b_2 = 0$ while if $b_2 = 4/\pi$ then $b_1 = 0$. Conversely, any numbers b_1, b_2 satisfying this relation will be the Fourier coefficients of some bounded function.

Similar relations hold between the other Fourier coefficients, but they are much more complicated:

$$\begin{aligned} a_n^2 + b_n^2 &\leq \left(\frac{4}{\pi}\right)^2, \\ \left(\frac{\pi}{4} b_1 + 1\right)^3 - 3\left(\frac{\pi}{4} b_1 + 1\right) - 1 \\ &\leq 3\left(\frac{\pi}{4} b_3\right) \leq \left(\frac{\pi}{4} b_1 - 1\right)^3 - 3\left(\frac{\pi}{4} b_1 - 1\right) + 1. \end{aligned}$$

After the inequalities for the Fourier coefficients are obtained, they are used to derive an approximation theorem. But instead of being an approximation to the function itself, it will be an approximation to the Fourier development of the function. More precisely: take a bounded measurable function $f(x)$ whose absolute value is less than one; let its Fourier development be $a_0/2 + \sum (a_n \cos nx + b_n \sin nx)$. Then, for any N there exists a function $g_N(x)$ which takes the values 1 and -1 only, such that the first N Fourier coefficients of $g_N(x)$ are the same as the first N Fourier coefficients of $f(x)$.

In general, there will be $N+1$ discontinuities. If $f(x)$ is even or odd, $g_N(x)$ will be even or odd. For example, let $f(x) = (\pi - x)/\pi$,

$0 < x < \pi; f(-x) = -f(x)$. Then its Fourier series will be

$$\frac{2}{\pi} \sum_1^{\infty} \frac{\sin nx}{n},$$

$$g_1(x) = +1, \quad 0 < x < \frac{2\pi}{3}; \quad g_2(x) = +1, \quad 0 < x < \frac{\pi}{2};$$

$$g_1(x) = -1, \quad \frac{2\pi}{3} < x < \pi; \quad g_2(x) = -1, \quad \frac{\pi}{2} < x < \frac{2\pi}{3};$$

$$g_2(x) = +1, \quad \frac{2\pi}{3} < x < \pi.$$

Incidentally, one of the inequalities leads to the following result: If $F(x)$ is a bounded measurable function in absolute value less than $\frac{1}{2}$,

$$\int_0^{\pi} F(x) \cos x \, dx \leq \cos \left(\int_0^{\pi} F(x) \, dx \right).$$

The method used in proving the above inequalities is as follows: Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leq 1$. Let the Fourier development of $f(x)$ be

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx).$$

Consider the expression

$$(1) \quad f_n(\mu, a) \equiv \mu_0 a_0 + \mu_1 b_1 + \mu_2 a_1 + \cdots + \mu_{2n-1} b_n + \mu_{2n} a_n$$

where $\mu_0, \mu_1, \dots, \mu_{2n}$ are arbitrary real parameters. The functions $f_n(\mu, a)$ can be expressed in terms of the function $f(x)$ by using the usual formulae for Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

From these formulae and (1), it is clear that

$$f_n(\mu, a) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) [\mu_0 + \mu_1 \sin x + \mu_2 \cos x + \cdots$$

$$+ \mu_{2n-1} \sin nx + \mu_{2n} \cos nx] \, dx$$

$$\equiv \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) C_n(\mu, x) \, dx,$$

say.

Assuming the μ 's are fixed, the maximum of $f_n(\mu, a)$ will be attained when the integrand $f(x)C_n(\mu, x)$ is as large as possible. But

$$|f(x)C_n(\mu, x)| \leq |C_n(\mu, x)|$$

since $|f(x)| \leq 1$. However, if $f(x)$ is taken as signum $C_n(\mu, x)$; that is, $f(x) = 1$ when $C_n(\mu, x) > 0$ and $f(x) = -1$ when $C_n(\mu, x) < 0$, then

$$|f(x)C_n(\mu, x)| = |C_n(\mu, x)|.$$

The following result is now obvious:

$$(2) \quad f_n(\mu, a) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |C_n(\mu, x)| dx \equiv D_n(\mu),$$

and the equality holds only when

$$f(x) = \text{signum } C_n(\mu, x) \equiv g_n(\mu, x).$$

For example, if $n = 1$ and $\mu_1 = 0$, formula (2) states that

$$\mu_0 a_0 + \mu_2 a_1 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(x) C_1(\mu, x) dx$$

where

$$f_1(x) = \text{signum } (\mu_0 + \mu_2 \cos x).$$

Assume $\mu_0 + \mu_2 \cos x = 0$ has the real roots $\pm \alpha_1$; then $g_1(\mu, x)$ will be defined as follows:

$$\begin{aligned} g_1(\mu, x) &= +1, & -\alpha_1 < x < \alpha_1, \\ g_1(\mu, x) &= -1, & \begin{cases} -\pi < x < -\alpha_1, \\ \alpha_1 < x < \pi, \end{cases} & \text{if } \mu_0 > 0; \\ g_1(\mu, x) &= -1, & -\alpha_1 < x < \alpha_1, \\ g_1(\mu, x) &= +1, & \begin{cases} -\pi < x < -\alpha_1, \\ \alpha_1 < x < \pi, \end{cases} & \text{if } \mu_0 < 0; \end{aligned}$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} |C_1(\mu, x)| dx &= \pm \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(\mu, x) C_1(\mu, x) dx \\ &= \pm \frac{1}{\pi} \left[\int_{-\pi}^{-\alpha_1} -C_1(\mu, x) dx + \int_{-\alpha_1}^{\alpha_1} C_1(\mu, x) dx \right. \\ &\quad \left. + \int_{\alpha_1}^{\pi} -C_1(\mu, x) dx \right] \end{aligned}$$

$$= \pm \frac{1}{\pi} [\mu_0(4\alpha - 2\pi) + \mu_2^4 \sin \alpha] = \pm \frac{4}{\pi} \left[\mu_0 \left(\alpha - \frac{\pi}{2} \right) + \mu_2 \sin \alpha \right].$$

If μ_0 is positive, the upper sign holds; if μ_0 is negative, the lower sign is used.

But $C_1(\mu, \alpha_1) = 0$; that is, $\mu_0 + \mu_2 \cos \alpha_1 = 0$ or

$$\cos \alpha = -\mu_0/\mu_2,$$

$$\alpha = \arccos(-\mu_0/\mu_2), \quad \alpha - \pi/2 = \arcsin \mu_0/\mu_2,$$

$$\sin \alpha = (1 - \mu_0^2/\mu_2^2)^{1/2},$$

so that

$$D_1(\mu) = \pm \frac{4}{\pi} [\mu_0 \arcsin(\mu_0/\mu_2) + \mu_2(1 - \mu_0^2/\mu_2^2)^{1/2}].$$

Therefore

$$(3) \quad \mu_0 a_0 + \mu_2 a_1 \leq \pm \frac{4}{\pi} [\mu_0 \arcsin(\mu_0/\mu_2) + \mu_2(1 - \mu_0^2/\mu_2^2)^{1/2}]$$

where a_0 and a_1 are the Fourier coefficients of any bounded function. It is easy to see that the equality sign holds only when a_0 and a_1 equal the Fourier coefficients of $g_1(\mu, x)$.

Similar results hold for any n . Assume $C_n(\mu, x) = 0$ has $2n$ real roots α_i where $-\pi < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots \leq \alpha_{2n} \leq \pi$. Call $-\pi = \alpha_0$ and $\pi = \alpha_{2n+1}$.

Assume $C_n(\mu, x)$ is positive between α_0 and α_1 . Then $g_n(\mu, x)$ will be defined as follows: $g_n(\mu, x) = (-1)^i$, $\alpha_i < x < \alpha_{i+1}$, $i = 0, 1, \dots, 2n+1$. If $C_n(\mu, x)$ is negative, then $g_n(\mu, x) = (-1)^{i+1}$, $\alpha_i < x < \alpha_{i+1}$. Therefore,

$$D_n(\mu) = \frac{1}{\pi} \int_{-\pi}^{\pi} g_n(\mu, x) C_n(\mu, x) dx = \pm \frac{1}{\pi} \sum_0^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x) dx$$

so that from (2)

$$(4) \quad \mu_0 a_0 + \mu_1 b_1 + \mu_2 a_1 + \dots + \mu_{2n-1} b_n + \mu_{2n} a_n \leq D_n(\mu).$$

Let

$$(5) \quad a_i = x_{2i}, \quad b_i = x_{2i-1}.$$

Consider the $2n+1$ dimensional space of points whose coordinates are the x_j ($j=0, 1, 2, \dots, 2n+1$). The inequality (4) can be written as

$$\sum_{j=0}^{2n+1} \mu_j x_j \leq D_n(\mu).$$

This implies that, for any bounded measurable function, the point P whose coordinates are given by (5) must lie on one side of the hyperplane whose equation is

$$(6) \quad \sum_0^{2n+1} \mu_j x_j = D_n(\mu).$$

The point P will lie on the plane if (4) becomes an equality. This will occur when and only when $f(x)$ has its first $2n+1$ Fourier coefficients equal to those of $g_n(\mu, x)$.

Since the μ_j were arbitrary, the same result holds for any set $\mu_0, \mu_1, \dots, \mu_{2n+1}$. Therefore, the point P , whose coordinates are (x_j) , $j=0, 1, \dots, 2n+1$, must lie on one side of every plane in the family of planes defined by the equation (6).

The previous example will clarify this. Equation (3) implies that if, for any $f(x)$, a_0 is plotted as the x coordinate and a_1 as the y coordinate of a point, then such points will lie on the origin side of the straight line whose equation is

$$\mu_0 x + \mu_2 y = D_1(\mu).$$

The point will lie on the straight line if and only if $f(x)$ has its Fourier coefficients a_0 and a_1 equal to those of $g_1(\mu, x)$.

Since the μ 's were arbitrary, the same result holds for any set μ_0, μ_2 . Therefore, the point P , whose coordinates are (a_0, a_1) must lie on one side of every line in the family of straight lines defined by the equation

$$\mu_0 x + \mu_2 y = D_1(\mu).$$

This implies that the point P will lie inside the region enveloped by these straight lines.

The usual procedure in finding the equation of this envelope is, first, to differentiate partially with respect to the parameter, and second, to eliminate the parameters between the resulting equation and the original equation. The partial differentiation results in the following two equations:

$$x = \frac{\partial D}{\partial \mu_0} = \pm (4/\pi) \arcsin (\mu_0/\mu_2), \quad y = \frac{\partial D}{\partial \mu_2} = \pm (4/\pi)(1 - \mu_0^2/\mu_2^2)^{1/2}.$$

After the elimination of the parameters, the following equation is obtained for the envelope:

$$y = (4/\pi) \cos (\pi x/4).$$

This implies that

$$a_1 \leq (4/\pi) \cos (\pi a_0/4).$$

Equality will hold, as before, if $f(x)$ has the same first two Fourier coefficients as any function $g_1(\mu, x)$.

In the general case, the equation of the envelope is obtained by the same method. When formula (6) is differentiated partially with respect to the parameters, the following equations result:

$$x_j = \frac{\partial D_n(\mu)}{\partial \mu_j}, \quad j = 0, 1, \dots, 2n + 1.$$

But

$$\frac{\partial D_n(\mu)}{\partial \mu_j} = \pm \frac{1}{\pi} \frac{\partial}{\partial \mu_j} \left[\sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x) dx \right]$$

where the α_i , the roots of $C_n(\mu, x) = 0$, also depend on the μ_j . Now

$$\begin{aligned} \frac{\partial}{\partial \mu_j} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x) dx &= \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \frac{\partial C_n(\mu, x)}{\partial \mu_j} dx \\ &+ (-1)^i C_n(\mu, \alpha_{i+1}) \frac{\partial \alpha_{i+1}}{\partial \mu_j} - (-1)^i C_n(\mu, \alpha_i) \frac{\partial \alpha_i}{\partial \mu_j} \end{aligned}$$

by the standard rule for differentiating under the integral sign.

However $C_n(\mu, \alpha_{i+1}) = C_n(\mu, \alpha_i) = 0$ and

$$\begin{aligned} \frac{\partial C_n(\mu, x)}{\partial \mu_j} &= \sin (j + 1)x/2 \text{ if } j \text{ is odd,} \\ &= \cos (jx/2) \text{ if } j \text{ is even,} \end{aligned}$$

so that

$$\begin{aligned} (7) \quad x_j &= \pm \frac{1}{\pi} \sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin (j + 1)x/2 dx \text{ if } j \text{ is odd,} \\ &= \pm \frac{1}{\pi} \sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \cos (jx/2) dx \text{ if } j \text{ is even,} \end{aligned}$$

where the signs are either all positive or all negative.

Instead of eliminating the parameters μ , it is simpler to eliminate the parameters α , thus obtaining the equation of the envelope $\phi_n(x_j) = 0$ which implies the inequality

$$(8) \quad \phi_n(a, b) \leq 0.$$

Since $\phi_n(x_j)$ is obtained as the envelope of planes, it is obvious that

the region bounded by it is the convex hull of a family of half-spaces. Again, the inequality will hold if and only if $f(x)$ has its first $2n+1$ Fourier coefficients $(a_0, a_1, b_1, \dots, a_n, b_n)$ equal to the corresponding Fourier coefficients of any one of the functions $g_n(\mu, x)$.

A few illustrations will clarify this method. Let $n=2$ and $\mu_0=\mu_2=\mu_4=0$. Then the α_i will be the roots of the equation

$$\mu_1 \sin x + \mu_3 \sin 2x = 0,$$

and from formula (7)

$$x_1 = \pm \frac{1}{\pi} \sum \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin x \, dx = \pm (-4/\pi \cos \alpha_3),$$

since $\alpha_1 = -\alpha_3$, and

$$x_3 = \pm \frac{1}{\pi} \sum \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin 2x \, dx = \pm (2/\pi)(1 - \cos 2\alpha_3).$$

Therefore $(\pi x_1/4)^2 + \pi x_3/4 = 1$ or $(\pi x_1/4)^2 - \pi x_3/4 = 1$ according as the plus or minus signs are used. But since x_1 was the coordinate for b_1 and x_3 for b_3 , the following inequality results:

$$(\pi b_1/4)^2 + \pi |b_3|/4 \leq 1.$$

From the result proved above: that the Fourier coefficients of $f(x)$ satisfy certain inequalities or, stating it geometrically, lie in certain convex regions, the approximation theorem follows. The theorem can be stated as follows:

Let $f(x)$ be a real, bounded, measurable function in absolute value less than one. Let its Fourier development be

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx).$$

Then for every N , there exists a step-function $g_N(x)$ which takes on the values of $+1$ and -1 only, such that the first N Fourier coefficients of $g_N(x)$ are the same as the first N Fourier coefficients of $f(x)$, that is,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g_N(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g_N(x) \sin nx \, dx,$$

$n = 0, 1, \dots, N.$

Again, let $a_i = x'_{2i}$, $b_i = x'_{2i-1}$. Then, as before, the point P' whose coordinates are $(x'_0, x'_1, \dots, x'_{2N})$ must satisfy the inequality (8):

$\phi_N(x'_j) \leq 0$, or, geometrically, the point P' must lie inside the convex region bounded by the hypersurface whose equation is

$$\phi_N(x_0, x_1, x_2, \dots, x'_{2N}) = 0.$$

Suppose now that P' lies on the hypersurface, that is, its coordinates satisfy the equation

$$\phi_N(x'_0, x'_1, x'_2, \dots, x'_{2N}) = 0.$$

Since $\phi_N(x_j) = 0$ was obtained as the envelope of the family of hyperplanes represented by equation (6), $\sum \mu_j x_j = D_N(\mu)$, there will be a plane of the family tangent to the surface at P' . But since P' lies on this plane, whose equation can be written

$$\mu'_0 x_0 + \mu'_1 x_1 + \mu'_2 x_2 + \dots + \mu'_{2N} x_{2N} = D_N(\mu'),$$

its coordinates must equal those of the function $g_N(\mu', x)$. Therefore, $g_N(\mu'_0, \dots, \mu'_{2N}, x)$ is the function described in the theorem.

If P' lies inside the hypersurface, consider the points in $2N+3$ dimensional space whose coordinates are $(x'_0, x'_1, x'_2, \dots, x'_{2N}, x_{2N+1}, x_{2N+3})$ where x_{2N+1} is an arbitrary variable. They will lie on the straight line through $(x'_0, x'_1, x'_2, \dots, x'_{2N}, 0, 0)$ perpendicular to the hyperplane $x_{2N+1} = x_{2N+2} = 0$.

Since $\phi_{N+1}(x_j) = 0$ is a closed convex surface for all n , the above straight line will intersect it in some point whose coordinates are $(x'_0, x'_1, x'_2, \dots, x'_{2N}, x'_{2N+1}, x'_{2N+2})$. Just as before, there will exist a step-function $g_{N+1}(x)$ whose Fourier development is $a'_0/2 + \sum (a'_n \cos nx + b'_n \sin nx)$ and such that $a'_i = x'_{2i}$, $b'_i = x'_{2i-1}$; $i = 0, 1, 2, \dots, N+1$.

This function satisfies the conditions of the theorem for

$$x'_{2i} = a_i = a'_i, \quad x'_{2i-1} = b_i = b'_i.$$