ADDITION FORMULAS FOR HYPERELLIPTIC FUNCTIONS

BY W. V. PARKER

1. *Introduction.** In a recent paper the writer† discussed the Kummer surface associated with the hyperelliptic curve of the form

$$s^2 = r^5 + ar^4 + br^3 + cr^2 + r.$$

This form for the curve was used by A. L. Dixon‡ in a paper in which he obtained addition formulas for hyperelliptic functions with distinct arguments. Dixon emphasizes the unusual symmetry of this particular form for the hyperelliptic curve. In the present paper duplication formulas are found as well as the addition formulas for distinct arguments. The odd functions used here differ from those of Dixon and are slightly different from the ones ordinarily used. The particular form used here seems desirable because of the symmetry.

2. *Distinct Arguments.* Consider the fixed hyperelliptic curve, H, of genus two, given by the equation

$$s^2 = r^5 + ar^4 + br^3 + cr^2 + r, \tag{1}$$

and let L be a variable curve of the form

$$m_0 s = n_0 r^3 + n_1 r^2 + n_2 r + n_3, \qquad m_0 \neq 0. \tag{2}$$

If $n_0 \neq 0$, H and L intersect in six finite points any four of which are sufficient to determine L and hence to determine the remaining two points.

Denote the six points of intersection of H and L by

$$(\alpha_1, \rho_1),\ (\alpha_2, \rho_2),\ (\beta_1, \sigma_1),\ (\beta_2, \sigma_2),\ (\gamma_1, \tau_1),\ (\gamma_2, \tau_2),$$

and let

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* This paper is a part of a dissertation written in Brown University, 1931. The writer is indebted to A. A. Bennett for many helpful suggestions.

† This Bulletin, vol. 38 (1932), p. 403.

‡ *On hyperelliptic functions of genus two,* Quarterly Journal of Mathematics, vol. 36 (1904), p. 1.

$$
\begin{aligned}
u_1 &= \int_\infty^{\alpha_1} \frac{dr}{s} + \int_\infty^{\alpha_2} \frac{dr}{s}, & u_2 &= \int_\infty^{\alpha_1} \frac{r\,dr}{s} + \int_\infty^{\alpha_2} \frac{r\,dr}{s},\\
v_1 &= \int_\infty^{\beta_1} \frac{dr}{s} + \int_\infty^{\beta_2} \frac{dr}{s}, & v_2 &= \int_\infty^{\beta_1} \frac{r\,dr}{s} + \int_\infty^{\beta_2} \frac{r\,dr}{s},\\
w_1 &= \int_\infty^{\gamma_1} \frac{dr}{s} + \int_\infty^{\gamma_2} \frac{dr}{s}, & w_2 &= \int_\infty^{\gamma_1} \frac{r\,dr}{s} + \int_\infty^{\gamma_2} \frac{r\,dr}{s}.
\end{aligned} \tag{3}
$$

We then have by Abel's Theorem*

$$
\begin{aligned}
u_1 + v_1 + w_1 &\equiv 0 \pmod{\text{period}},\\
u_2 + v_2 + w_2 &\equiv 0 \pmod{\text{period}}.
\end{aligned} \tag{4}
$$

Let

$$
\frac{x_\alpha}{t_\alpha} = \alpha_1 + \alpha_2, \qquad \frac{y_\alpha}{t_\alpha} = \alpha_1\alpha_2,
$$

$$
\eta_\alpha = \frac{\alpha_2\rho_1 - \alpha_1\rho_2}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}, \qquad \zeta_\alpha = \frac{\alpha_1^2\rho_2 - \alpha_2^2\rho_1}{\alpha_1\alpha_2(\alpha_1 - \alpha_2)};
$$

and denote similarly the corresponding expressions in β_1, β_2, σ_1, σ_2; γ_1, γ_2, τ_1, τ_2. These are all periodic functions of the integrals u_1 and u_2. We know that x_α/t_α, and y_α/t_α are even functions† and η_α and ζ_α are odd functions. The functions x_γ/t_γ, y_γ/t_γ, η_γ, ζ_γ can be expressed rationally in terms of x_α, y_α, t_α, η_α, ζ_α, x_β, y_β, t_β, η_β, ζ_β. But x_γ/t_γ, y_γ/t_γ, η_γ, ζ_γ are the same functions of (w_1, w_2) as x_α/t_α, y_α/t_α, η_α, ζ_α are of (u_1, u_2) and x_β/t_β, y_β/t_β, η_β, ζ_β are of (v_1, v_2), and since, from (4),

$$
w_1 \equiv -(u_1 + v_1),
$$

and

$$
w_2 \equiv -(u_2 + v_2),
$$

these relations give us expressions for the functions of (u_1+v_1, u_2+v_2) in terms of the functions of (u_1, u_2) and (v_1, v_2), the sign being positive for even functions and negative for odd functions. Furthermore ζ_α can be expressed rationally in terms of x_α, y_α, t_α, η_α, and similarly η_α can be expressed rationally in terms of x_α, y_α, t_α, ζ_α. For the functions are connected by the relations

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* Appell et Goursat, *Théorie des Fonctions Algébriques*, Chap. 9.

† Dixon, loc. cit.

$$
\begin{aligned}
yt\eta^2 &= ayt + xy + zt, \\
2yt\eta\zeta &= byt - y^2 - t^2 - xz, \qquad (5) \\
yt\zeta^2 &= cyt + yz + xt,
\end{aligned}
$$

where

$$
\frac{z}{t} = \frac{2F(\alpha_1, \alpha_2) - 2\rho_1\rho_2}{(\alpha_1 - \alpha_2)^2},
$$

and

$$
\begin{aligned}
2F(\alpha_1, \alpha_2) = \alpha_1^2\alpha_2^2(\alpha_1 + \alpha_2) &+ 2a\alpha_1^2\alpha_2^2 \\
&+ b\alpha_1\alpha_2(\alpha_1 + \alpha_2) + 2c\alpha_1\alpha_2 + \alpha_1 + \alpha_2.
\end{aligned}
$$

We see from these relations that interchanging a with c and y with t also interchanges η with ζ.

If we eliminate s between (1) and (2) we get the equation

$$
\begin{aligned}
& n_0^2 r^6 + (2n_0n_1 - m_0^2)r^5 + (n_1^2 + 2n_0n_2 - am_0^2)r^4 \\
& \quad + (2n_1n_2 + 2n_0n_3 - bm_0^2)r^3 \qquad (6) \\
& \quad + (n_2^2 + 2n_1n_3 - cm_0^2)r^2 + (2n_2n_3 - m_0^2)r + n_3^2 = 0,
\end{aligned}
$$

whose roots are $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. Hence we have

$$
\begin{aligned}
\frac{x_\alpha}{t_\alpha} + \frac{x_\beta}{t_\beta} + \frac{x_\gamma}{t_\gamma} &= \frac{m_0^2 - 2n_0n_1}{n_0^2}, \\
\frac{y_\alpha}{t_\alpha}\,\frac{y_\beta}{t_\beta}\,\frac{y_\gamma}{t_\gamma} &= \frac{n_3^2}{n_0^2}. \qquad (7)
\end{aligned}
$$

Since (α_1, ρ_1), (α_2, ρ_2), (β_1, σ_1), (β_2, σ_2) are on L, we have

$$
\begin{aligned}
m_0\rho_1 &= n_0\alpha_1^3 + n_1\alpha_1^2 + n_2\alpha_1 + n_3, \\
m_0\rho_2 &= n_0\alpha_2^3 + n_1\alpha_2^2 + n_2\alpha_2 + n_3, \qquad (8) \\
m_0\sigma_1 &= n_0\beta_1^3 + n_1\beta_1^2 + n_2\beta_1 + n_3, \\
m_0\sigma_2 &= n_0\beta_2^3 + n_1\beta_2^2 + n_2\beta_2 + n_3.
\end{aligned}
$$

From (8) we get at once

$$
\begin{aligned}
\eta_\alpha m_0 &= \frac{x_\alpha}{t_\alpha} n_0 + n_1 - \frac{t_\alpha}{y_\alpha} n_3, \quad \zeta_\alpha m_0 = -\frac{y_\alpha}{t_\alpha} n_0 + n_2 + \frac{x_\alpha}{y_\alpha} n_3, \qquad (9) \\
n_\beta m_0 &= \frac{x_\beta}{t_\beta} n_0 + n_1 - \frac{t_\beta}{y_\beta} n_3, \quad \zeta_\beta m_0 = -\frac{y_\beta}{t_\beta} n_0 + n_2 + \frac{x_\beta}{y_\beta} n_3.
\end{aligned}
$$

If now we let $n_0 = t_\alpha t_\beta p_0$ and $n_3 = y_\alpha y_\beta p_3$ and write

$$\begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} = (x, y), \qquad \begin{vmatrix} \eta_\alpha & 1 \\ \eta_\beta & 1 \end{vmatrix} = (\eta, 1), \text{ etc.},$$

we get

$$(10)\quad (\eta, 1)m_0 = (x, t)p_0 + (y, t)p_3, \quad (\zeta, 1)m_0 = (t, y)p_0 + (x, y)p_3.$$

Writing

$$m_0 = \begin{vmatrix} (x, t) & (y, t) \\ (t, y) & (x, y) \end{vmatrix},$$

we have

$$p_0 = \begin{vmatrix} (\eta, 1) & (y, t) \\ (\zeta, 1) & (x, y) \end{vmatrix}, \quad p_3 = \begin{vmatrix} (x, t) & (\eta, 1) \\ (t, y) & (\zeta, 1) \end{vmatrix}.$$

From (9) we get

$$(11)\quad \begin{aligned} 2n_1 &= m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha) - p_0(x_\alpha t_\beta + x_\beta t_\alpha), \\ 2n_2 &= m_0(\zeta_\alpha + \zeta_\beta) + p_0(y_\alpha t_\beta + y_\beta t_\alpha) - p_3(x_\alpha y_\beta + x_\beta y_\alpha) \end{aligned}$$

It is interesting to note here that interchanging y with t, and η with ζ, interchanges p_0 with p_3, n_0 with n_3, n_1 with n_2, and leaves m_0 unaltered.

If we substitute the value for $2n_1$ from (11) in (7) we get

$$(12)\quad \begin{aligned} \frac{x_\gamma}{t_\gamma} &= \frac{m_0^2 - t_\alpha t_\beta p_0[m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha)]}{t_\alpha^2 t_\beta^2 p_0^2}, \\ \frac{y_\gamma}{t_\gamma} &= \frac{p_3^2 y_\alpha y_\beta}{p_0^2 t_\alpha t_\beta}. \end{aligned}$$

Since the points (γ_1, τ_1), (γ_2, τ_2) are also on L we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3.$$

If now we let $x_{\alpha+\beta}$ etc. denote the same functions of (u_1+v_1, u_2+v_2) as x_α etc. are of (u_1, u_2) and x_β etc. are of (v_1, v_2), we have the following addition formulas (valid for distinct arguments):

$$
(13)\qquad
\begin{aligned}
x_\gamma : y_\gamma : t_\gamma = x_{\alpha+\beta} : y_{\alpha+\beta} : t_{\alpha+\beta} = \Bigg\{ & \begin{vmatrix} (x,t) & (y,t) \\ (t,y) & (x,y) \end{vmatrix}^2 \\
& - t_\alpha t_\beta \begin{vmatrix} (\eta,1) & (y,t) \\ (\zeta,1) & (x,y) \end{vmatrix} \left[(\eta_\alpha + \eta_\beta) \begin{vmatrix} (x,t) & (y,t) \\ (t,y) & (x,y) \end{vmatrix} \right. \\
& \left. + (y_\alpha t_\beta + y_\beta t_\alpha) \begin{vmatrix} (x,t) & (\eta,1) \\ (t,y) & (\zeta,1) \end{vmatrix} \right] \Bigg\} : y_\alpha y_\beta t_\alpha t_\beta \begin{vmatrix} (x,t) & (\eta,1) \\ (t,y) & (\zeta,1) \end{vmatrix}^2 \\
& : t_\alpha^2 t_\beta^2 \begin{vmatrix} (\eta,1) & (y,t) \\ (\zeta,1) & (x,y) \end{vmatrix}^2,
\end{aligned}
$$

$$
\begin{aligned}
\eta_\gamma = -\eta_{\alpha+\beta} = \frac{m_0}{t_\alpha t_\beta p_0} - \frac{p_0(x_\alpha t_\beta + x_\beta t_\alpha)}{2m_0} - \frac{p_0^2 t_\alpha t_\beta}{m_0 p_3} \\
- \frac{p_3(y_\alpha t_\beta + y_\beta t_\alpha)}{2m_0} - \frac{\eta_\alpha + \eta_\beta}{2},
\end{aligned}
$$

$$
\begin{aligned}
\zeta_\gamma = -\zeta_{\alpha+\beta} = \frac{m_0}{y_\alpha y_\beta p_3} - \frac{p_3(x_\alpha y_\beta + x_\beta y_\alpha)}{2m_0} - \frac{p_3^2 y_\alpha y_\beta}{m_0 p_0} \\
- \frac{p_0(y_\alpha t_\beta + y_\beta t_\alpha)}{2m_0} - \frac{\zeta_\alpha + \zeta_\beta}{2}.
\end{aligned}
$$

These last two equations may be written

$$
\eta_\gamma = -\eta_{\alpha+\beta} = \frac{m_0}{t_\alpha t_\beta p_0} + \frac{p_0 t_\alpha t_\beta (\zeta, 1)}{p_3 (y, t)} - \frac{\eta_\alpha y_\alpha t_\beta - \eta_\beta y_\beta t_\alpha}{(y, t)},
$$

$$
\zeta_\gamma = -\zeta_{\alpha+\beta} = \frac{m_0}{y_\alpha y_\beta p_3} + \frac{p_3 y_\alpha y_\beta (\eta, 1)}{p_0 (t, y)} - \frac{\zeta_\alpha y_\beta t_\alpha - \zeta_\beta y_\alpha t_\beta}{(t, y)}.
$$

From (12) we get

$$
\frac{x_\gamma}{y_\gamma} = \frac{m_0^2 - t_\alpha t_\beta p_0 [m_0(\eta_\alpha + \eta_\beta) + p_3(y_\alpha t_\beta + y_\beta t_\alpha)]}{y_\alpha y_\beta t_\alpha t_\beta p_3^2}.
$$

If now we interchange y with t, η with ζ and p_0 with p_3 we get

$$
\frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - y_\alpha y_\beta p_3 [m_0(\zeta_\alpha + \zeta_\beta) + p_0(y_\alpha t_\beta + y_\beta t_\alpha)]}{y_\alpha y_\beta t_\alpha t_\beta p_0^2}.
$$

This is exactly the form obtained for x_γ/t_γ by using the coefficient of r from (6) rather than the coefficient of r^5. A somewhat more symmetric though much longer form for x_γ/t_γ is obtained by taking one-half the sum of these two expressions.

3. *The Coincidence Case, Duplication Formulas.* If (β_1, σ_1) coincides with (α_1, ρ_1) and also (β_2, σ_2) coincides with (α_2, ρ_2), that is, if $u_1=v_1$ and $u_2=v_2$, the above formulas become indeterminate in the sense that they do not give the expressions for the functions of $(2u_1, 2u_2)$ in terms of the functions of (u_1, u_2) simply by setting $v_1=u_1$ and $v_2=u_2$. In order to obtain the formulas in this case as expeditiously as possible we determine the curve L so that it is tangent to H at each of the points (α_1, ρ_1) and (α_2, ρ_2). We then have by Abel's theorem

$$2u_1 + w_1 \equiv 0 \text{ (mod period)}, \quad 2u_2 + w_2 \equiv 0 \text{ (mod period)}.$$

Since the roots of (6) are now $\alpha_1, \alpha_1, \alpha_2, \alpha_2, \gamma_1, \gamma_2$, we have

$$\frac{x_\gamma}{t_\gamma} = \frac{m_0^2 - 2n_0n_1}{n_0^2} - \frac{2x_\alpha}{t_\alpha}, \quad \frac{y_\gamma}{t_\gamma} = \frac{n_3^2 t_\alpha^2}{n_0^2 y_\alpha^2}, \tag{14}$$

where the ratios n_0/m_0, n_1/m_0, n_2/m_0, n_3/m_0 are to be determined from the equations*

$$\begin{aligned}(15)\ & yt\eta m_0 = xyn_0 + ytn_1 - t^2n_3, \quad yt\zeta m_0 = -y^2n_0 + ytn_2 + xtn_3,\\ & (5xy^2 + 4ay^2t - 2cyt^2 - xt^2)m_0\\ &\qquad = 6(y^2t\zeta + xy^2\eta)n_0 + 4y^2t\eta n_1 - 2yt^2\zeta n_2,\\ & (5y^3 - 3by^2t - 2cxyt - x^2t + yt^2)m_0\\ &\qquad = 6y^3\eta n_0 - 4y^2t\zeta n_1 - 2(xyt\zeta + y^2t\eta)n_2.\end{aligned}$$

The computation here is greatly simplified by introducing the function z/t as defined in (5) which is expressed rationally in terms of x/t, y/t, η, ζ through the relations given there. Making use of these relations and writing

$$\begin{aligned} p_0 &= (y^2 - t^2)\eta + (xy - zt)\zeta, & p_2 &= t\eta^2 + z\eta\zeta + y\zeta^2,\\ p_1 &= y\eta^2 + x\eta\zeta + t\zeta^2, & p_3 &= (xt - yz)\eta + (t^2 - y^2)\zeta,\end{aligned} \tag{16}$$

we get immediately

$$\begin{aligned} & m_0 = 2ytp_1, \quad n_0 = tp_0, \quad n_1 = tp_3 - xp_0 + 2ytp_1\eta,\\ & n_2 = yp_0 - xp_3 + 2ytp_1\zeta, \quad n_3 = yp_3.\end{aligned} \tag{17}$$

* Since there is no possibility of confusion the subscript α will be omitted in what is to follow.

If now $x_{2\alpha}$ etc. denote the same functions of $(2u_1, 2u_2)$ as x_α etc. are of (u_1, u_2), and we substitute the expressions from (17) in (14), we get the following duplication formulas:

$$x_\gamma : y_\gamma : t_\gamma = x_{2\alpha} : y_{2\alpha} : t_{2\alpha} = (4ytp_1p_2 - 2p_0p_3) : p_3^2 : p_0^2 .$$

Since the points (γ_1, τ_1) and (γ_2, τ_2) are on L, we have

$$\eta_\gamma m_0 = \frac{x_\gamma}{t_\gamma} n_0 + n_1 - \frac{t_\gamma}{y_\gamma} n_3, \quad \zeta_\gamma m_0 = -\frac{y_\gamma}{t_\gamma} n_0 + n_2 + \frac{x_\gamma}{y_\gamma} n_3 .$$

Hence we have

$$\eta_\gamma = -\eta_{2\alpha} = \frac{[(4ytp_1p_2 - 2p_0p_3)t - xp_0^2]p_3 + p_0(tp_3^2 - yp_0^2)}{2ytp_0p_1p_3} + \eta,$$

$$\zeta_\gamma = -\zeta_{2\alpha} = \frac{[(4ytp_1p_2 - 2p_0p_3)y - xp_3^2]p_0 + p_3(yp_0^2 - tp_3^2)}{2ytp_0p_1p_3} + \zeta,$$

where p_0, p_1, p_2, p_3 are as defined in (16).

For certain choices of (α_1, ρ_1) and (α_2, ρ_2) in the coincidence case, (γ_1, τ_1) will coincide with (γ_2, τ_2) and the curve L will be tangent to H at each of three points. In fact (α_1, ρ_1) can be chosen arbitrarily subject to the condition that $\rho_1 \neq 0$ and then (α_2, ρ_2) can be determined in a finite number of ways so that the curve L which is tangent to H at (α_1, ρ_1) and (α_2, ρ_2) will also be tangent at a third point. If we take a fixed point on H and determine a curve L through this point and any three of the points where H meets the r-axis, this curve L will meet H in another pair of points such that the L which is tangent to H at each of these is also tangent at the given fixed point. Furthermore all such tangent curves may be obtained in this way.*

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* For proof of this statement see a paper by the writer in this Bulletin, vol. 37 (1931), p. 557.