# THE EIGHTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Eighth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Friday and Saturday, December 27–28, 1901. A single day's session now often barely suffices for the adequate presentation of the programme of an ordinary meeting of the Society. It was therefore decided last year to extend the annual meeting to cover two days, in the hope of relieving considerably the severe compression occasioned by the constantly increasing number of papers presented. The energetic response of the Society's productive forces has, however, been such as to leave the problem precisely where it was before. The reading of twenty-seven or more mathematical papers in two days is, it appears, a possible feat, but hardly one for sober commendation. It is becoming a serious question whether it will not be necessary in the near future to adopt a practice of selection, permitting the presentation, even then in rather condensed form, of more important papers only, those of lesser interest or value being read in severely brief abstract or by title. This plan would possess certain advantages in point of dignity over the present hurried procedure; it would also permit better opportunity for intelligent discussion of important tendencies and larger interests. The annual meeting and the summer meeting stand in especial need of some such relief.

During the past year, the Society has in all respects maintained a vigorous growth. The membership has increased from 357 to 375, the number of papers read from 112 to 136. The total attendance of members at all meetings was 231, and the number attending at least one meeting was 131. The Council has been enlarged and an assistant secretary has been added to the administrative forces. The presidential term has been extended to two years, and the date of the presidential address has been changed to the more appropriate occasion of the last annual meeting of the presidential term. An especially important event was the deposit of the library of the Society in the charge of Columbia University, by which the books become generally accessible to our members. A catalogue will be issued and rules for the use of the library will be formulated as soon as practicable. Professor Pomeroy Ladue, who has faithfully discharged the duties of Librarian of the Society since 1895, has resigned this position, being succeeded by Professor D. E. Smith. Resolutions expressing appreciation of Professor Ladue's long and valuable services were adopted by the Council.

The attendance of members at the annual meeting was fifty-nine, slightly surpassing the highest previous record of all meetings and marking an increase of fifty per cent. over any previous annual meeting. The following members were recorded as present:

Dr. Grace Andrews, Mr. C. H. Ashton, Dr. G. A. Bliss, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Professor E. S. Crawley, Dr. W. S. Dennett, Dr. William Findlay, Professor G. E. Fisher, Professor T. S. Fiske, Mr. W. B. Ford, Dr. A. S. Gale, Professor Fanny C. Gates, Dr. G. B. Germann, Dr. W. A. Granville, Professor Harris Hancock, Professor James Harkness, Professor J. N. Hart, Professor Edwin Haviland, Jr., Dr. E. R. Hedrick, Dr. A. A. Himowich, Dr. E. V. Huntington, Professor Harold Jacoby, Mr. S. A Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Professor Pomeroy Ladue, Professor W. W. Landis, Professor Gustave Legras, Dr. C. T. Lewis, Dr. Emory McClintock, Dr. Isabel Maddison, Dr. Emily N. Martin, Professor Mansfield Merriman, Professor E. H. Moore, Professor F. Morley, Dr. J. L. Patterson, Mr. J. C. Pfister, Professor James Pierpont, Dr. I. E. Rabinovitch, Professor J. K. Rees, Professor E. D. Roe, Dr. Arthur Schultze, Professor I. J. Schwatt, Professor Charlotte A. Scott, Dr. S. E. Slocum, Professor P. F. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Dr. W. M. Strong, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor L. C. Wait, Professor A. G. Webster, Professor J. B. Webb, Professor R. S. Woodward.

Morning and afternoon sessions were held on each day. The President of the Society, Professor Eliakim Hastings Moore, occupied the chair, being relieved at the later sessions by Vice-President Professor Thomas S. Fiske. The Council announced the election of the following persons to membership in the Society: Professor R. E. Allardice, Stanford University; Dr. Grace Andrews, Columbia University; Mr. S. E. Brasefield, Michigan State Agricultural College; Mr. W. E. Brooke, University of Minnesota; Professor T. C. Esty, University of Rochester; Mr. L. L. Jackson, State Normal School, Brockport, N. Y. Seven applications for admission to the Society were received. Reports were received from the Secretary, Treasurer, Librarian, and Auditing Committee. These reports will appear in the Annual Register.

An enjoyable social feature of the meeting was the dinner held on Friday evening at Hotel Marlborough. Falling between the two days of the meeting, this was a remarkably successful occasion. Fifty persons were present, including several representatives of the American Physical Society, which met on Friday.

At the annual election, which was held on Saturday morning, the following officers and members of the Council were chosen:

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| *Vice-Presidents,* | Professor MAXIME BÔCHER,<br>Professor FRANK MORLEY. |
| *Secretary,* | Professor F. N. COLE. |
| *Treasurer,* | Dr. W. S. DENNETT. |
| *Librarian,* | Professor D. E. SMITH. |

*Committee of Publication,*

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

*Members of the Council to serve until December, 1904,*

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| Professor POMEROY LADUE, | Professor P. F. SMITH, |
| Professor G. A. MILLER, | Professor E. B. VAN VLECK. |

The following papers were presented at the meeting:

(1) Dr. VIRGIL SNYDER: "Further types of unicursal sextic scrolls."

(2) Dr. EMORY McCLINTOCK: "On the nature and use of the functions employed in the recognition of quadratic residues."

(3) Professor HAROLD JACOBY: "A theorem concerning the method of least squares."

(4) Professor HARRIS HANCOCK: "The theory of maxima and minima of functions of several variables."

(5) Professor Sir R. S. BALL: "Recent researches in the theory of screws."

(6) Dr. H. F. STECKER: "On surfaces whose geodetic lines are represented by curves of the second degree when represented conformally upon the plane."

(7) Professor CHARLOTTE A. SCOTT: "A recent method for treating the intersections of plane curves."

(8) Dr. EDWARD KASNER: "Two principles in the theory of multiple forms."

(9) Mr. D. R. CURTISS: "On the invariants of a homogeneous quadratic differential equation of the second order."

(10) Professor ARNOLD EMCH: "Some applications of the theory of assemblages."

(11) Professor G. A. MILLER: "On a method for constructing all the groups of order $p^m$."

(12) Dr. S. E. SLOCUM: "Note on the transformation of a group into its canonical form."

(13) Dr. E. R. HEDRICK: "On the characteristics of differential equations."

(14) Professor CHARLOTTE A. SCOTT: "On the circuits of plane curves."

(15) Professor F. MORLEY and Mr. A. B. COBLE: "On plane quartic curves."

(16) Professor MAXIME BÔCHER: "On the real solutions of systems of two homogeneous linear differential equations of the first order."

(17) Professor E. H. MOORE: "On the projective axioms of geometry."

(18) Dr. E. R. HEDRICK: "Remarks on the sufficient conditions in the calculus of variations."

(19) Dr. L. P. EISENHART: "Note on isotropic congruences."

(20) Dr. L. P. EISENHART: "Lines of length zero on surfaces."

(21) Dr. W. B. FITE: "Concerning the class of a group of order $p^m$ that contains an operator of order $p^{m-2}$ or $p^{m-3}$, $p$ being a prime."

(22) Dr. EDWARD KASNER: "A characteristic property of the parabolic curve of the $n$th order."

(23) Dr. CARL GUNDERSEN: "On the content or measure of assemblages of points."

(24) Mr. J. W. YOUNG: "On the holomorphisms of a group."

(25) Professor P. F. SMITH: "On the resolution of orthogonal transformations."

(26) Mr. SAUL EPSTEEN: "Proof that the group of an irreducible linear differential equation is transitive."

(27) Dr. W. B. FORD: "On the uniform convergence of Fourier's series."

Professor Ball was introduced by Professor E. W. Brown, Mr. Curtiss by Professor Bôcher, Dr. Gunderson by Professor Fiske. Mr. Young's paper was presented to the Society through Dr. Stecker, Dr. Epsteen's through Professor Lovett. In the absence of the authors, the papers of Dr. Fite and Mr. Young were read by Dr. Stecker, and the papers of Professor Jacoby, Professor Emch, Professor Miller, Dr. Eisenhart, Professor Smith, and Dr. Epsteen were read by title. Abstracts of the papers follow below.

Dr. Snyder's paper considered sextic scrolls as generated by lines joining corresponding points of two unicursal curves, the sum of whose orders is six. Either or both directing curves may be replaced by a multiple curve of lower order, or by curves of higher order which intersect in one or more self-corresponding points. Different types exist according to the configuration of the multiple points and of the assumed points which define the correspondence. This list together with that presented at the Ithaca meeting, comprises 66 types. The only further types are those which may arise from limiting cases of these, or from imaginary nodal curves. Neither of these forms have been considered.

The integer $n$ is or is not a quadratic residue of the prime $k$ according as the value of Legendre's function $(n/k) \equiv n^{\frac{1}{2}(k-1)} \pmod{k}$ is $+1$ or $-1$. The evaluation of $(n/k)$ is performed by depressing the numerical value of $n$, if greater than $k$, and by clearing it of powers of 2, so as to make $n$ odd and less than $k$. The "law of quadratic reciprocity" is then employed to transpose the numbers, and the new number on the left hand is depressed as before, and so on till the result is attained. Dr. McClintock has discovered simple relations affecting the right hand number so as to permit the depression of the two numbers alternately, without transposition. His paper treats of this class of functions in the most comprehensive manner, developing their various relations first from one definition and then from another, and closes with practical rules and illustrations. It has appeared in the January number of the *Transactions*.

Professor Jacoby's paper is in abstract as follows: Let there be given two series of observation equations as follows:

$$\begin{array}{c} a_1x + b_1y + c_1z + \cdots + n_1 = 0, \\ a_2n + b_2y + c_2z + \cdots + n_2 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \qquad (1)$$

$$\begin{array}{l} a_1x + b_1y + c_1z + \cdots + p_1w + \cdots + n_1 = 0, \\ a_2x + b_2y + c_2z + \cdots + p_2w + \cdots + n_2 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{2}$$

the equations being identical in the last two series except for the addition of one or more new unknowns $w, \cdots$ in (2). Let each series of equations be solved by the method of least squares, and let: $[vv]_1$, be the sum of the squares of the residuals resulting from the solution of equations (1), $[vv]_2$ be the sum of squares of the residuals resulting from the solution of equations (2); then, no matter what may be the law of the coefficients $p_1, p_2, \cdots$, and even if these coefficients are assigned at random, $[vv]_1$ is always larger than $[vv]_2$.

Corollary.—It is easy to extend the theorem to an additional case of equal importance. Instead of introducing the new unknowns $w, \cdots$, by adding them to those already occurring in equations (1), we may select out certain equations from the series (1), and simply substitute new unknowns, like $w$, for old ones, like $z$, leaving the coefficients unchanged.

Conclusion.—The method of least squares is used ordinarily to adjust series of observation equations so as to obtain the most probable values of the unknowns. But there is a subtler and perhaps more important use of the method: when it is employed to decide which of two hypothetical theories has the greater probability of really being a law of nature; or to decide between two methods of reducing observations. Cases abound in astronomy where the method of least squares is used for this purpose. It has been so employed, for instance, to decide whether stellar parallax observations should be reduced with equations involving terms depending on atmospheric dispersion, and terms depending on the hour angle; to ascertain whether portable transit observations should be reduced on the supposition of a change of azimuth on reversal of the instrument (an application of the Corollary); etc., etc.

In such cases, astronomers not infrequently give preference to that solution which brings out the smallest value of $[vv]$, the sum of the squared residuals. But in the light of the above theorem, it becomes clear that the mere diminution of $[vv]$ alone is insufficient to decide between two solutions, when one involves more unknowns than the other. To give preference to the second solution it is necessary that the diminution of $[vv]$ be quite large, and that the additional unknowns possess a decided a priori probability of having a real existence.

In Professor Hancock's paper the treatment is divided into two heads: 1° where no subsidiary conditions involving the variables are present; 2° where there are subsidiary conditions. The functions are supposed to be regular within the realms considered and may therefore be developed in convergent power series. The first necessary condition for the existence of maximal or minimal values is the vanishing of the first derivatives. The second condition is made to depend upon homogeneous quadratic forms. These forms must be definite and may vanish: 1° when all the variables vanish; 2° for other values of the variables. The discussion as regards the sign of these forms is then made to depend upon the quotients of determinants of different orders and upon the reality and signs of the roots of symmetrical determinants. The question of the existence of a maximum or minimum is considered, the asymptotic approach to an upper or lower limit, etc. The theory is illustrated with numerous examples and applications.

Professor Sir R. S. Ball's paper is in abstract as follows: The movement of a material system of the most general type can be represented with the help of a geometrical entity termed a screw chain, associated with a metric quantity possessing the dimensions of an angular velocity. Any small displacement of a system free or constrained in any way can be represented by a twist about a screw chain. Any set of forces acting on such a system can be represented by a wrench on a screw chain. The meaning of reciprocal screw chain systems is explained. The problem of impulsive forces was considered. It was shown that the number of principal screw chains of inertia equals the number of degrees of freedom of the system. A quiescent system receiving an impulsive wrench on a principal screw chain of inertia will commence to move by twisting about that screw chain. The differential equation which must be satisfied by the function expressing the kinetic energy of a system when screw chain coördinates are employed was obtained. The physical meaning of this equation was considered, and the meaning of permanent screw chains. A generalization of Euler's dynamical equation followed.

Dr. Stecker's paper considers the determination, in detail, of surfaces whose geodetic lines are represented by curves of the second degree in the plane. There are six different types of resulting differential equations, and six corresponding systems of partial differential equations. For three of these it is shown that no solution exists. For a fourth case the results are found to be

$$E = \frac{1}{[A(v)u^2 + B(v)u + C(v)]^2},$$

$$G = \frac{[D(v)u + H(v)]^2}{[A(v)u^2 + B(v)u + C(v)]^2} + \frac{\Psi(v)}{A(v)u^2 + B(v)u + C(v)},$$

$$F = \frac{D(v)u + H(v)}{[A(v)u^2 + B(v)u + C(v)]^2},$$

$$K = A(v)\,C(v) - \tfrac{1}{4}B^2(v),$$

where $K$ is the Gaussian curvature of the surface. For the fifth type

$$E = \frac{1}{\lambda c_1 v^2 + c_2}, \quad G = \frac{c_1 v^2}{[\lambda c_1 v^2 + c_2]^2}, \quad F = 0,$$

$$K = \tfrac{3}{2}\frac{(c_1 v - \lambda)}{v^2(\lambda c_1 v^2 + c_2)} + \lambda^2 c_1,$$

and $\lambda$ depends upon the eccentricity of the curve. The sixth case comes from the fifth by an interchange of $u$ and $v$ and a change of certain constants.

A recent memoir by Dr. F. S. Macaulay (*Proceedings of the London Mathematical Society*, volume 31) throws an interesting light on the nature of the conditions to which the coefficients in the equation of a curve must be subject in order that the equation may be expressible in the form $Pu + Qv = 0$. The conditions can be written as a single "prime equation" with all equations obtained from it by a particular process of derivation. The nature of the set of equations thus obtained is investigated by Miss Scott more minutely than in Dr. Macaulay's memoir, with the result that different and simpler proofs are obtained for the theorems; the fundamental theorem in these proofs is that if two equations have a common derivative, then they have a common source for any two curves for which the two equations are satisfied.

In a paper presented at the summer meeting Dr. Kasner considered the relation between the theories of double binary forms with respect to cogredient and digredient transformations. In the present paper the author extends the principle involved to: 1° multiple binary forms, *i. e.*, forms involving any number $k$ of binary variables, and 2°

double $n$-ary forms, *i. e.*, forms involving two sets of $n$-ary variables. The results are as follows: The cogredient theory of any system of forms 1° is, with certain restrictions, equivalent to the digredient theory of an enlarged system formed by adjoining $k$ binary bilinear forms. The contragredient theory of any system of forms 2° is, with certain restrictions, equivalent to the digredient theory of the same system after adjoining a certain bilinear form.

Using Lie's theory, Mr. Curtiss investigates those functions of the coefficients and their derivatives, for the equation

$$(y'')^2 + 4p_2(y')^2 + p_4y^2 + 4p_3yy' + 2q_2yy'' + 4p_1y'y'' = 0,$$

which remain invariant under the transformation

$$y = \lambda(x) \cdot \eta, \quad \xi = \mu(x),$$

$\lambda$ and $\mu$ being arbitrary functions of $x$. The dependent variable alone is first transformed and a complete system of seminvariants obtained; *i. e.*, a system whose members are functionally independent of each other and on which all other seminvariants must functionally depend. The invariants under the general transformation are then determined in terms of this system of seminvariants, a complete system of these also being obtained. The paper concludes with some applications connected with subgroups of the general transformation.

The notion of the length of any continuous line is obtained by the method of displacements contained in the theory of continuous groups. To measure the length of a limited portion of a line, or of a closed line, Euclid's algorithm, resulting in a continued fraction

$$a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+}\cdots,$$

is applied. In this manner the theory of irrational numbers as established by Cantor, Dedekind, and others is immediately applicable to measurable quantities. Studying the open polygon of an infinite number of finite equal sides inscribed in a circle of length $u$, Professor Emch is led to the theorem: Every real number $u > 1$ may be represented by a series of the form

$$u = \frac{1}{\frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \frac{1}{n_1 n_2 n_3 n_4} + \cdots},$$

where $n_1 < n_2 < n_3 < n_4 < \cdots$ are positive integers. For example,

$$\pi = \frac{1}{\frac{1}{3} - \frac{1}{3.22} + \frac{1}{3.22.118} - \cdots}.$$

For $u < 1$, we have

$$u = \frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \frac{1}{n_1 n_2 n_3 n_4} + \cdots.$$

A group $G$ of order $p^m$ ($p$ being any prime) contains at least one invariant subgroup $H$ of order $p^{m-1}$. If $G$ is represented as a regular substitution group, then $H$ has $p$ systems of intransitivity and its $p$ transitive constituents ($H_1, H_2, \cdots, H_p$) are transformed cyclically by every substitution of $G$ which is not contained in $H$. The object of Professor Miller's paper is to find a practical method to determine the substitutions which must be added to $H$ in order to obtain all the groups of order $p^m$ that include $H$, even when $H$ is non-abelian. The special case when $H$ is abelian was considered by the author in the *Transactions*, volume 2, page 264. In the present paper the same general lines of thought are pursued, but the results are considerably more comprehensive. The paper will be offered for publication to the *American Journal of Mathematics.*

Dr. Slocum's paper is in abstract as follows: If we have given the finite equations defining a family of transformations $T_a$ (with continuous parameters $a_1, \cdots, a_r$) of a given $r$-parameter group $G$, then by the introduction of new parameters $\mu_1, \cdots, \mu_r$ these equations can be transformed into the canonical equations of the group by the method given by Lie in his Transformationsgruppen, volume 3, pp. 609–611. In the canonical form these equations will define a family of transformations $T_\mu$ of $G$. But to a definite transformation of the family $T_a$ with finite parameters $a_1, \cdots, a_r$ and generated by an infinitesimal transformation of $G$, there may correspond a transformation of the family $T_\mu$, which has at least one of its parameters $\mu_1, \cdots, \mu_r$ infinite in all branches, and which therefore can not be generated by an infinitesimal transformation of $G$. It is the purpose of the paper to illustrate this by an example, and also to point out the error in Lie's demonstration, whereby this exception arises.

The first paper of Dr. Hedrick deals with the characteristics of differential equations. The ordinary discussion of the subject is entirely laid aside, and characteristic curves are defined and developed (first for partial differential equations of the first order) as the curves for which the equation cannot be reduced to normal form, *i. e.*, for which the Cauchy-Kowalewski existence proofs cannot be carried out. Merely on this assumption the ordinary equations for characteristic curves and strips are produced; and it is shown that the curves and strips so defined possess the properties ordinarily attributed to characteristics. The further course of the theory would then be as ordinarily given, and is not dwelt upon in the paper. In this connection a proof is given that every partial differential equation of the first order has non-analytic solutions.

The possible families of characteristic curves are then considered, and it is shown that a given three-parameter family of curves is not, in general, the family of characteristic curves of any partial differential equation of the first order.

Finally, a connection is established between the theory of characteristic curves and the calculus of variations, it being shown that the characteristic curves of any equation are the extremals of a certain simple problem of the calculus of variations. The same general topics are then discussed with reference to equations of the second order, and the analogous results are obtained.

The paper, which follows to a large extent recent lectures by Professor Hilbert in Göttingen, will be offered for publication to the *Annals of Mathematics.*

The substance of Miss Scott's second paper is a proof that for any even degree there exist curves consisting of a single circuit without visible singularities of any kind. This proof is obtained by means of the fact that a curve $f=0$ cannot have a real inflexion within a region in which

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

is positive.

Clifford (Works, page 117) asserts that the general plane quartic $a_x^4$ possesses a contravariant curve $\alpha_\xi^4$ of class 4, and of degree 5 in the coefficients of $a$, such that taking any line conic $b$, if the polar of $b$ as to $a$ is $\beta$, then the polar of $\beta$ as to $a$ is $b$. This assertion is not true in general. The conditions under which it is true are obtained by Professor Morley and Mr. Coble in algebraic form and interpreted geometrically. There is, however, in general for two plane curves, one of order 4 and one of class 4, a "double-six" of conics say $b_i$ and $\beta_\kappa$ $(i, \kappa = 1, \cdots, 6)$ such that $b_i$ is apolar to $\beta_\kappa$ when $i \neq \kappa$; $b_i$ and $\beta_i$ have the polar property stated by Clifford. From this point of view (compare Hilbert, "Zur binären Invariantentheorie," *Mathematische Annalen*, volume 28, page 383) the quartic is attacked.

Professor Bôcher established for a system of two homogeneous linear differential equations of the first order theorems analogous to those given by Sturm in his famous paper for a single homogeneous linear differential equation of the second order. If we confine our attention to an interval in which the coefficients of the differential equations are continuous real functions of the real variable $x$, the case here considered is more general than that of Sturm; for in order to pass from the pair of equations of the first order to a single equation of the second order we must assume not only that one of the coefficients of the equations of the first order has a continuous first derivative, but also that this coefficient does not vanish in the interval in question. The advantage of the method used in the present paper as compared with the method of Sturm consists, however, not merely in its greater generality, but also in its greater simplicity and symmetry.

In his "Grundlagen der Geometrie" (1899), Hilbert exhibited a body of axioms of three-dimensional euclidean geometry which, according to his statements, and the subsequent statements of others, involved no redundancy. Pasch, (1882) developed this geometry on the basis of the point, the linear-segment, and the planar-segment as elements, defining by means of suitable axiomatic relations, imposed upon these elements, the line and the plane Peano (1889 and 1894) retained the point and the linear-segment as elements, but defined the line and the plane in terms of these elements. Hilbert uses as elements the point, the line, and the plane. Schur, in his paper in a recent number of the *Mathematische Annalen* (1901), points out that in view of the possibility of a definition of a plane in terms of the other elements, Hilbert's system of axioms contains redundancy. This statement of Schur implies that the plane postulated by Hilbert is to be directly identified with the plane defined by Peano; this identification, it is proved, can in the Hilbert system I–II of axioms of connection and order be made only by use of two of the three axioms held by Schur to be redundant.

However, Hilbert's system does contain two plain redundancies. His axioms I 4: "Any three non-collinear points of a plane determine it," and II 4: "To four distinct collinear points a notation, $A$, $B$, $C$, $D$, may be given such that $B$ lies between $A$ and $C$ and between $A$ and $D$; and $C$ lies between $A$ and $D$ and between $B$ and $D$," are consequences of the remaining axioms, I, II.

The paper of Professor Moore exhibits these redundancies, and in that connection a new system of axioms of connection and order closely related to, but in various respects simpler than, those of Hilbert.

In his second paper, Dr. Hedrick makes an attempt, based on recent lectures by Professor Hilbert, to simplify the discussion of the subject. Beginning with Hilbert's new proof of Weierstrass's sufficient condition,* the author shows that Jacobi's criteria even with a slight extension can be obtained directly from the Weierstrass-Hilbert theory, by the aid of the latter's invariant integral; and Jacobi's equation itself is finally produced. Hilbert's notion of limited variation having been introduced, Legendre's necessary condition is deduced in a simple manner from the above; and the conditions known as Lagrange's, Legendre's, Jacobi's, and Weierstrass's are grouped to form necessary conditions and sufficient conditions in the cases of weak and strong minima, limited and unlimited. The discussion closes with a remark on the purpose of Hilbert's existence proof.

In his first note Dr. Eisenhart considers isotropic congruences as they arise in the study of the infinitesimal deformation of the sphere. The discussion leads to the following theorem: The middle envelope of an isotropic congruence is the adjoint of the minimal surface which is the associate surface in the infinitesimal deformation of the sphere, which is the directrix of the congruence.

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* See Osgood, *Annals of Math.*, 2d Series, vol. 2, No. 3, p. 121.

Dr. Eisenhart's second note deals with the double system of imaginary lines of length zero on real surfaces. After recalling that these lines can never be the lines of curvature for the surface, that when they form a conjugate system the surface is minimal and that when they are asymptotic the surface is a sphere, he shows that the sphere and minimal surfaces are the only real surfaces whose lines of length zero have for spherical representation the rectilinear generatrices of the sphere. Further, in order that the asymptotic lines or a conjugate system on a surface may be represented upon a sphere by these generatrices, they must be lines of length zero on the surface.

Dr. Fite showed that if a group of order $p^m$, where $p$ is an odd prime, contains an operator of order $p^{m-2}$, the group is of class $k$, where $k \leqq 3$; if it contains an operator of order $p^{m-3}$, where $p$ is a prime greater than 3, it is of class $k$, where $k \leqq 4$.

The curves considered in Dr. Kasner's second paper are defined by the form to which their cartesian equation may be reduced

$$y = A_0 x^n + A_1 x^{n-1} + \cdots + A_n.$$

It is quite evident that the polar conic of any point with respect to such a parabolic curve of $n$th order is a parabola. The paper proves that this property is characteristic, *i. e.*, any curve whose polar conics are all parabolas can be expressed in the above form. The proof depends upon the determination of the rational integral functions $\varphi(x, y)$ which satisfy the partial differential equation

$$\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy} = 0.$$

The result is applied to a certain class of developable surfaces.

Mr. Gundersen's paper deals with different methods of measuring assemblages of points, particularly the methods of Cantor, Peano, Jordan, and Borel, and with properties of assemblages suggested by these methods. Among the results reached may be mentioned that if $F_{(1)}$ is the frontier assemblage of any linear assemblage, and $F_{(2)}$ the frontier of $F_{(1)}$ and so on, then $F_{(n)}$ is identical with $F_{(2)} \cdot (n \geqq 3)$. The relation between Borel's system of measure and that of Peano and Jordan is given by the equation $s = a + \theta\,(A - a)$, where $s$ is the Borel measure, $a$ the internal and $A$ the external measure as defined by Peano and Jordan, and $0 \gtreqless \theta \gtreqless 1$. If the given assemblage contains its derived, the above formula takes the form $s = A$. The necessary and sufficient conditions of measurability in the different systems are discussed.

Mr. Young's paper is in abstract as follows: We denote by $\alpha$-isomorphism any isomorphism of a group with itself (or with one of its subgroups), which may be obtained by putting each operator into correspondence with its $\alpha$th power, and by $\alpha$-holomorphism any $\alpha$-isomorphism which is simple. Prof. Miller has shown that $\alpha$ holomorphisms exist for any abelian group $A$ ($\alpha$ being prime to the order of every operator in $A$) and further that, regarded as operators in the group of isomorphisms of $A$, the $\alpha$-holomorphisms constitute the totality of invariant operators in the group of isomorphisms of $A$. It is shown for any group $G$ that if there is a number $\delta$ such that for every pair of operators $s$, $t$ of $G$, $s^\delta t = ts^\delta$ and $s^\delta t^\delta = (st)^\delta$, and if $\delta + 1$ is prime to the order of every operator of $G$, then and only then does $G$ admit an $\alpha$-holomorphism; and $\alpha$ may have any value of the form $\varkappa\delta + 1$ which is prime to the order of every operator of $G$. It is then proved that the $\alpha$-holomorphisms of any group $G$ are invariant in the group of isomorphisms of $G$, and certain related theorems are derived.

Professor Smith's paper treats the subject of orthogonal transformations by means of a result first obtained by Voss, that such a transformation in $n$ variables may be resolved into $n$ or less point-plane reflections, point and plane being in the polar relation with respect to the fundamental quadric. This theorem having been established, Cayley's formulas are derived and the generality of the types derived by Loewy proven. The method followed brings out more clearly than heretofore the completeness of the results. The question of the resolution of orthogonal transformations into involutory transformations is next discussed, the possibility of resolution into two such transformations established, and some applications given.

The object of Mr. Ford's paper is to determine an upper limit to the absolute value of the remainder after $n$ terms of a Fourier's series representing the function $f(x)$, this upper limit to hold good for all values of $x$ within a fixed interval $(a, b)$ lying within the interval $(-\pi, \pi)$. The result, besides being of service in the study of the uniform convergence of Fourier's series, furnishes a means for determining an upper limit to the number of terms necessary to take in a given Fourier development in order that the error committed by stopping with the last term taken shall be less than some assigned quantity, this being true when the series is considered for *all* values of $x$ lying within the interval $(a, b)$ ($a$ and $b$ included), it being assumed throughout that $f(x)$ satisfies certain preliminary conditions (such as Dirichlet's conditions) at every point of this interval.

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# THE JANUARY MEETING OF THE CHICAGO SECTION.

THE tenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Illinois, on Thursday and Friday, January 2–3, 1902. Two sessions were held each day, opening at 10 o'clock A. M. and 2.30 o'clock P. M. In the absence of Professor E. H. Moore, President of the Society, during the morning sessions, Professor Ziwet occupied the chair on the first day and Professor White on the second day. The attendance at the various sessions was unusually large, including a number of prominent educators in the vicinity of Chicago, and the following twenty-five members of the Society:

Professor R. J. Aley, Dr. G. A. Bliss, Professor Oskar Bolza, Dr. C. L. Bouton, Professor D. F. Campbell, Mr. A. Crathorne, Professor E. W. Davis, Professor L. E. Dickson, Professor L. W. Dowling, Dr. J. W. Glover, Dr. T. P. Hall, Professor Thomas F. Holgate, Dr. H. G. Keppel, Dr. Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Miss I. M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Professor E. J. Townsend, Professor C. A. Waldo, Dr. Jacob Westlund, Professor H. S. White, Professor Alexander Ziwet.

At the business session on Friday morning the following officers of the section were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; Additional members of the programme committee, Professor E. J. Townsend and Professor L. W. Dowling.

The following papers were presented and read: