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$\mathcal{A}$-topological triviality of map germs and Newton filtrations

Marcelo José Saia and Liane Mendes Feitosa Soares

Abstract.

We apply the method of constructing controlled vector fields to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x)+th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

For the case $n \geq p$, we follow the technique used by M. A. S. Ruas in her Ph.D. Thesis [7] and construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We use the control function of the target to give an estimate for the case $p \geq 2n$. Moreover, in this case we show that if the coordinates of the map germ satisfy a Newton non-degeneracy condition, deformations by terms of higher filtration are topologically trivial.

As an application we obtain for both cases, $n \geq p$ and $p \geq 2n$, the results of Damon in [3] for deformations of weighted homogeneous map germs.

§1. Introduction

The determinacy of topological triviality for families of map-germs is a fundamental subject in singularity theory. As we see in the articles of Damon, [4] and [3] for example, the method of constructing controlled vector fields is a very powerful tool to compute the topological triviality.

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M. A. S. Ruas in her PhD. Thesis gives an explicit order such that the $\mathcal{A}$-topological structure of a polynomial map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n \geq p$, is preserved after higher order perturbations.

In this paper we apply this method to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x) + th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

First we generalize the results of M. A. S. Ruas [7], by considering different Newton filtrations A_k for $\mathcal{O}_n$ and B_k for $\mathcal{O}_p$, these results are given for the case $n \geq p$. We construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We remark that in [7] these control functions are homogeneous, since they are associated to the usual filtration, given by the degree of monomials.

In the case $p \geq 2n$ we give an estimate in terms of the control function of the target. Moreover, if $p \geq 2n$, we apply the results of Gaffney in [6] to show that deformations by higher Newton filtration are $\mathcal{A}$-topologically trivial if the map germs satisfy a Newton non-degeneracy condition.

In both cases we also show that the results of Damon for the topological triviality of unfoldings of weighted homogenous map germs can be obtained from our results.

§2. Newton filtration and control functions

To construct controlled vector fields that guarantee the topological triviality we define a convenient *control function* in terms of a fixed Newton polyhedron. An analytic function $\rho : \mathbb{C}^n \to \mathbb{R}$ is a control if there exist constants C and α such that $\rho(x) \geq C|x|^\alpha$.

First we construct a control function in the target, denoted by ρ_{m} and a function in the source, denoted by ρ_{f}. When $n \geq p$, the control function ρ is defined from these, via a partition of the unity. For $p \geq 2n$, the control function is ρ_{m}.

Fix coordinate systems $\mathbf{x}$ in $(\mathbb{C}^n, 0)$, $\mathbf{y}$ in $(\mathbb{C}^p, 0)$ and denote by $\mathcal{O}_n$, $\mathcal{O}_p$, the sets of holomorphic germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$ and from $(\mathbb{C}^p, 0)$ to $(\mathbb{C}, 0)$. We identify these sets with the rings of convergent power series $\mathbb{C}[[x]]$ and $\mathbb{C}[[y]]$ respectively.

To fix the notation we follow [1] and say that a subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \dots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+, i = 1, \dots, r\}$

and Γ_+ intersects all the coordinate axis. Denote by Γ the union of the compact faces of Γ_+ and consider the *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}$, for all $q \in \mathbb{N}$, here ϕ_Γ is the Newton function of Γ.

We fix a Newton polyhedron Γ_+ in $\mathbb{R}^n_+$ with its associate Newton filtration, then for any germ of function $g = \sum_k a_k x^k$, denote $\mathrm{d}(g) = \max\{q : g \in \mathcal{A}_q\}$ and by $\mathrm{in}(g)$, the polynomial $\mathrm{in}(g) = \sum a_k x^k$ such that $\phi_\Gamma(k) = \mathrm{d}(g)$.

To define a Newton filtration in $\mathcal{O}_p$ we consider a fixed map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $g = (g_1, \ldots, g_p)$, call $D_i = \mathrm{d}(g_i)$ and say that $D_1 \leq D_2 \leq \ldots \leq D_p$. In this case we call $\mathrm{d}(g) = (D_1, D_2, \ldots, D_p)$.

Denote by M_I the determinant of the $p \times p$ minor of the matrix of the partial derivatives of g indexed by $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots n\}$, with $i_1 < \ldots < i_p$. We fix an order for these determinants calling $M_1, M_2, \ldots, M_r$ in such a way that $\mathrm{d}(M_j) \leq \mathrm{d}(M_{j+1})$ and call $L_j = \mathrm{d}(M_j)$.

Now, call $D = \text{m.c.m.}\{D_1, D_2, \ldots, D_p, L_1, \ldots, L_q\}$ and define the weighted homogeneous control function in the target, $\rho_{\mathrm{m}} : \mathbb{C}^p \to \mathbb{R}$ by

$$\rho_{\mathrm{m}}(y) = |y_1|^{2r_1} + |y_2|^{2r_2} + \ldots + |y_p|^{2r_p}, \text{ where } r_i = \frac{D}{D_i} \text{ for all } i = 1, \ldots, p.$$

The Newton filtration of $\mathcal{O}_p = \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \ldots$ is associate to the control function $\rho_{\mathrm{m}}(y)$. Therefore any ideal $\mathcal{B}_k$ has a Newton polyhedron which only one compact face with normal vector $w = (w_1, ..., w_p)$, where $w_i = \frac{R}{r_i}$ and $R = \text{m.c.m.}\{r_1, \ldots, r_p\}$, for all $i = 1, ..., p$.

For any monomial $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \ldots y_n^{\beta_n} \in \mathcal{O}_p$, denote $\mathrm{d}_w(y^\beta) = w_1\beta_1 + \ldots + w_p\beta_p$, and for any $g \in \mathcal{O}_p, \mathrm{d}_w(g) = \min . \mathrm{d}_w(y^\beta)$ for all y^β with nonzero coefficient in the Taylor series of g, then $B_k = \{g \in \mathcal{O}_p; \mathrm{d}_w(g) \geq k\}$.

Here we have $\mathrm{d}_w(\rho_m) = 2R$ and as $\rho_{\mathrm{m}}(g(x)) = |g_1|^{2r_1} + |g_2|^{2r_2} + \ldots + |g_p|^{2r_p}$, $\mathrm{d}(\rho_m \circ g) = \mathrm{d}\left(|g_1|^{2r_1} + |g_2|^{2r_2} + \ldots + |g_p|^{2r_p}\right) = 2D$.

Now define the control function in the source

$$\rho_v(x_1, \ldots, x_n) = \sum_{j=1}^{r} x_1^{2v_1^j} \ldots x_n^{2v_n^j},$$

with $v^j = (v_1^j, \ldots, v_n^j)$, $j = 1, \ldots, r$ being the vertices of the Newton polyhedron $\Gamma_+(A_D)$, therefore $\mathrm{d}(\rho_v) = 2D$.

We also define the function $\rho_{\mathfrak{f}}(g) : \mathbb{C}^n \to \mathbb{R}$, $\rho_{\mathfrak{f}}(g)(x) = \sum |M_j|^{2\alpha_j}$, where $\alpha_j = \frac{D}{L_j}$ for all $j = 1, \ldots, q$. We remember that $\rho_{\mathfrak{f}}(g)$ is not a control function, however, under some conditions, it is important in the construction of the controlled vector fields. We remark that all these constructions are done to obtain $\mathrm{d}(\rho_{\mathfrak{f}}(g)) = \mathrm{d}(\rho_m \circ g) = \mathrm{d}(\rho_v) = 2D$.

Example: Let $g(x, y) = (xy, x^4 + y^5 + xy^2)$ and fix the Newton polyhedron $\Gamma_+(g_2)$. Call Δ_1 the face with vertices $\{(0,5), (1,2)\}$ and Δ_2 the face with vertices $\{(4,0), (1,2)\}$, $C(\Delta_i)$ denotes the cone with vertex at 0 passing through Δ_i.

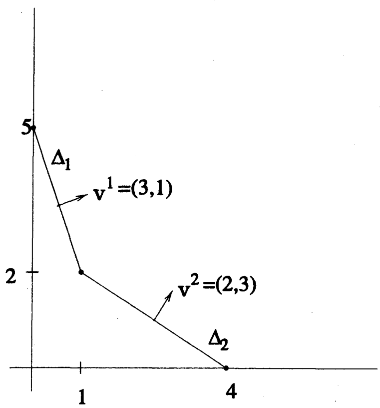

Fig. 1. The Newton polyhedron $\Gamma_+(g_2)$.

The Newton filtration $\varphi_{\Gamma_+(g_2)}$ is

$$\varphi(x^a y^b) = \begin{cases} 24a + 8b, & \text{if } (a,b) \in C(\Delta_1) \\ 10a + 15b, & \text{if } (a,b) \in C(\Delta_2). \end{cases}$$

Then $\mathrm{d}(g_1) = 25$ and $\mathrm{d}(g_2) = 40$, therefore $D = 200 = \mathrm{m.c.m}\{25, 40\}$ and the control function in the target is $\rho\mathrm{m}(y) = |y_1|^{16} + |y_2|^{10}$, and $\mathrm{d}_w(\rho\mathrm{m}(y)) = 80 = R$.

Now let $M(x, y) = 5y^5 - 4x^4 + xy^2$, be the determinant of the Jacobian matrix of g, then $\mathrm{d}(M) = 40$ and $\rho_{\mathfrak{f}}(g) = (5y^5 - 4x^4 + xy^2)^{10}$, with $\mathrm{d}(\rho_{\mathfrak{f}}(g)) = 2D = 400$.

Since $\mathrm{d}(g_1^{16}) = \mathrm{d}(g_2^{10}) = \mathrm{d}(\rho_{\mathfrak{f}}(g)) = 2D = 400$, the control function in the source ρ_v associate to $\Gamma_+(g_2^{10})$ is $\rho_v(x, y) = x^{40} + y^{32} + x^{20}y^{40}$ and $\mathrm{d}(\rho_v) = 2D = 400$.

§3. $\mathcal{A}$-topological triviality

Denote by

$$F : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}, (0,0)), F(x,\lambda) = (f(x,\lambda),\lambda)$$

a one parameter unfolding of a finitely determined map germ

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$$

and call the family of map germs $f_\lambda(x) = f(x,\lambda)$ a deformation of the germ f.

An unfolding $F(x,\lambda)$ of f is $\mathcal{A}$-topologically trivial if, for small values of λ, there are germs of homeomorphisms $H : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to (\mathbb{C}^n \times \mathbb{C}, (0,0))$, of type $H(x,\lambda) = (h(x,\lambda),\lambda)$, with $h(0,\lambda) = 0$ and $K : (\mathbb{C}^p \times \mathbb{C}, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}, (0,0))$ of type $K(x,\lambda) = (k(x,\lambda),\lambda)$ with $k(0,\lambda) = 0$ such that $K \circ F \circ H^{-1} = (f_0(x),\lambda)$.

In this case we say that the deformation $f_\lambda(x)$ is $\mathcal{A}$-topologically trivial, since for small values of λ, the families of homeomorphisms $h_\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, with $h_\lambda(x) = h(x,\lambda)$ and $k_\lambda : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ with $k_\lambda(x) = k(x,\lambda)$ give

$$k_\lambda \circ f_\lambda \circ h_\lambda^{-1} = f_0.$$

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ satisfying

$$(1) \qquad A_{2D+D_1}\theta_g \subseteq tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p).$$

From the above constructions we have the following:

Proposition 3.1.

(1) *If $n \geq p$, suppose that in a neighborhood V of 0 in $\mathbb{C}^n$, there exist constants α e β such that $\rho_{\mathrm{f}}(g(x)) \geq \beta\rho_v(x)$, for all $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$.*

(2) *If $p \geq 2n$, suppose that $\rho_{m}(g(x)) \geq c\rho_v(x)$, $\forall x$ in a neighborhood V of 0.*

Then deformations $g_\lambda = g + \lambda h$ of g, with $\mathrm{d}(h_i) \geq D_i$, $\forall\, i = 1, \ldots, p$, are $\mathcal{A}$-topologically trivial for small values of λ.

In the next Lemma, essential in the proof of this Proposition, we show that it is possible to extend the filtration condition of the equation (1) to the tangent space of an unfolding of the germ g.

We call m_1 the maximal ideal in $\mathcal{O}_1$ and $\tilde{A}_{2D+D_1}$ the ideal in $\mathcal{O}_{n+1}$ generated by the monomial λ and the ideal A_{2D+D_1}.

Lemma 3.2. *Let $G(x,\lambda) = (g_\lambda(x), \lambda)$, with $g_\lambda(x) = (g_{1\lambda}(x), \dots g_{n\lambda}(x))$ be an unfolding of $g_0(x) = g(x)$, such that $g_{i\lambda} - g_{i0} \in m_1.A_{D_i}\theta_G$ for all $i = 1, \dots, n$ and $|\lambda| < \epsilon$ for small values of ϵ. If the equation (1) holds, then*

$$tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) \supseteq A_{2D+D_1}\theta_G.$$

Proof: Since

$$\begin{aligned} A_{2D+D_1}\theta_G &= A_{2D+D_1}\theta_g + \lambda A_{2D+D_1}\theta_G \\ &\subseteq tg(A_{2D}\theta_n) + wg(B_{2R+1}.\theta_p) + \lambda A_{2D+D_1}\theta_G, \end{aligned}$$

and

$$\begin{aligned} &tg(A_{2D}\theta_n) + wg(B_{2R+1}.\theta_p) \\ &\qquad \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G \end{aligned}$$

it follows that

$$(2) \quad A_{2D+D_1}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G.$$

Let E be the finitely generated $\mathcal{O}_{n+1}$-modulo defined as

$$E = \frac{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})}.$$

We remark that E is a $G^*(\mathcal{O}_{p+1})$-modulo and $(\lambda).E = E$ since

$$\begin{aligned} (\lambda).E &= \frac{tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})}{tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})} + \\ &\quad + \frac{(\lambda)[tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})+A_{2D+D_1}\theta_G]}{tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})} \\ &= \frac{tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})+A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1})+wG(B_{2R+1}.\theta_{p+1})} = E. \end{aligned}$$

Therefore if we show that E is finitely generated as $G^*(\mathcal{O}_{p+1})$-modulo we apply the Nakayama's Lemma to obtain $E = 0$, or

$$A_{2D+D_1}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}).$$

We show now that E is finitely generated as $G^*(\mathcal{O}_{p+1})$- module.

Let E' be the finitely generated $\mathcal{O}_{n+1}$-module

$$E' = \frac{tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1})}.$$

Then we need to show that E' is finitely generated as a $G^*(\mathcal{O}_{p+1})$-module.

From the Malgrange's Preparation Theorem, E' is a finitely generated $G^*(\mathcal{O}_{p+1})$- module if, and only if $\dim_{\mathbb{C}} \frac{E'}{G^*(m_{p+1})E'} < +\infty$.

Write

$$\frac{E'}{G^*(m_{p+1})E'} = \frac{\frac{tG(A_{2D}\theta_{n+1})+A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1})}}{\frac{G^*(m_{p+1})[tG(A_{2D}\theta_{n+1})+A_{2D+D_1}\theta_G]+tG(A_{2D}\theta_{n+1})}{tG(A_{2D}\theta_{n+1})}}$$

$$= \frac{tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})[tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G]},$$

denote $S = A_{2D+D_1}\theta_G$ and $T = tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G$.

Therefore by the isomorphism theorem we obtain $\dfrac{T+S}{T} \cong \dfrac{S}{T\cap S}$.

From

$$tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})\theta_G$$

and by the equation (2) we conclude that

$$tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})\theta_G \supseteq A_{2D+D_1}\theta_G.$$

Multiplying by A_{2D+D_1} we obtain

$$tG(A_{4D+D_1}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G \supseteq A_{4D+2D_1}\theta_G.$$

On the other hand,

$$tG(A_{4D+D_1}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G \supseteq \lambda A_{2D+D_1}\theta_G.$$

Hence, $\dim_{\mathbb{C}} \frac{S}{T\cap S} \leq \dim_{\mathbb{C}} \frac{A_{2D+D_1}\theta_G}{\tilde{A}_{2D+D_1}A_{2D+D_1}\theta_G} < +\infty$. ||

Proof of the Proposition 3.1: Let $G(x,\lambda) = (g_\lambda(x),\lambda)$ be an unfolding of g, with $g_\lambda(x) = g(x) + \lambda h(x)$ and $h = (h_1,\ldots,h_p)$ with each $h_i \in A_{D_i}$.

From the general hypotheses, since $A_{D_i} \subset A_{D_1}$ for all $i = 1,\ldots,n$ we obtain

$$h.\rho_{\mathrm{m}}(g) \in tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p).$$

From the Lemma 3.2. we conclude that there exist analytic vector fields $\xi \in A_{2D}\theta_{n+1}$ and $\eta \in B_{2R+1}\theta_{p+1}$ such that the above inclusion holds for deformations, i.e.

$$h.\rho_{\mathrm{m}}(g_\lambda) = tG(\xi) + \eta\circ G. \tag{3}$$

From the equation (3) we construct the vector field controlled by ρ_{m}. Define ω in $(\mathbb{C}^p\times\mathbb{C}, 0\times 0)$ as:

$$\omega(y,\lambda) = \begin{cases} \dfrac{\eta(y,\lambda)}{\rho_{\mathrm{m}}(y)}, & \text{if } y\neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

Since $\mathrm{d}_w(\eta) \geq 2R+1 > \mathrm{d}_w(\rho_{\mathrm{m}}) = 2R$ we apply Lemmas (1) e (2) of [9] to conclude that the vector field ω is integrable.

Proof of the case $n \geq p$. In order to define the vector field controlled by the function ρ_{f}, for each $I = \{i_1, i_2,\ldots,i_p\} \subset \{1,2,\ldots,n\}$ write $\dfrac{\partial g_\lambda}{\partial\lambda} M_{I_\lambda} = tG(\gamma_I)$, with $\gamma_I = \sum \gamma_i \frac{\partial}{\partial x_i}$ and each γ_i is defined as

$$\begin{cases} \gamma_i = 0, & \text{if } i\notin I \\ \gamma_{i_m} = \sum N_{ji_m}\left(\frac{\partial g_\lambda}{\partial\lambda}\right)_j, & \text{if } i_m \in I. \end{cases} \tag{4}$$

where N_{ji_m} is the $(p-1)\times(p-1)$ cofactor of $\frac{\partial g_j}{\partial x_{i_m}}$.

Since $\rho_{\mathrm{f}}(g_\lambda) = \sum |M_{I_\lambda}|^{2\alpha_I}$, we obtain

$$h.\rho_{\mathrm{f}}(g_\lambda) = tG\left(\sum \gamma_I M_{I_\lambda}^{\alpha_I - 1}\overline{M_{I_\lambda}}^{\alpha_I}\right),$$

therefore $h = tG(\psi)$, with $\psi = \frac{\sum \gamma_I M_{I_\lambda}^{\alpha_I - 1}\overline{M_{I_\lambda}}^{\alpha_I}}{\rho_{\mathfrak{f}}(g_\lambda)}$.

Denote $\gamma_{\mathcal{R}} = \sum \gamma_I M_{I_\lambda}^{\alpha_I - 1}\overline{M_{I_\lambda}}^{\alpha_I}$, then $\mathrm{d}(\gamma_{\mathcal{R}}) = \mathrm{d}(\rho_{\mathfrak{f}}(g_\lambda)) + r$, with $r = \min_{i,k}\left\{\frac{M}{\ell(v^k)}.v_i^k\right\}$.

The integrability of the vector field $\psi = \dfrac{\gamma_R}{\rho_{\mathfrak{f}}(g_\lambda)}$, follows from the hypotheses of the following:

Lemma 3.3. There exist positive constants α_1, β and a neighborhood V of the origin in $\mathbb{C}^n$ such that

$$\rho_{\mathfrak{f}}(g_\lambda(x)) \geq \alpha_1 \rho_v(x), \; \forall\, x \in \; V \cap \{\rho_{\mathrm{m}}(g_\lambda(x)) < \beta\rho_v(x)\}.$$

Proof: Since $g_\lambda = g + \lambda h$, and $\mathrm{d}(h_i) \geq \mathrm{d}(g_i)$ for all $i = 1, \ldots, p$ we obtain

$$\rho_{\mathfrak{f}}(g_\lambda) \geq \rho_{\mathfrak{f}}(g) - \lambda\theta(x,\lambda), \text{with} \quad \mathrm{d}(\theta) \geq \mathrm{d}(\rho_{\mathfrak{f}}(g)).$$

By hypotheses $\rho_{\mathfrak{f}}(g) \geq \alpha\rho_v(x)$ for $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \beta\rho_v(x)\}$ hence there exists a constant $c > 0$ such that $\lambda\theta(x,\lambda) \leq c\rho_v(x)$. Since $\rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x))$, for each $x \in V \cap \{x; \rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$, we obtain

$$\begin{aligned} \rho_{\mathfrak{f}}(g_\lambda(x)) &\geq \rho_{\mathfrak{f}}(g(x)) - \lambda\theta(x,\lambda) \\ &\geq (\alpha - c)\rho_v(x) \\ &= \alpha_1\rho_v(x). \end{aligned}$$

||

To finish the proof of the Proposition 3.1., consider the following partition of the unity.

Let $H = (V \times I) - (0 \times I)$, with $I = (-\epsilon, \epsilon)$ and the following sets

$F_1 = (\{(x,\lambda); g_\lambda(x) = 0\} - (0 \times \mathbb{C})) \cap H$, $F_2 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) \geq \alpha\rho_v(x)\} \cap H$,

$E_1 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_1\rho_v(x)\} \cap H$ and $E_2 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_2\rho_v(x)\} \cap H$,

with $\alpha_1 < \alpha < \alpha_2$.

We remark that F_1 and F_2 are closed and disjoint from H.

Define $\zeta(x) = \zeta_1(x) + \zeta_2(x)$, a partition of the unity related to $\{E_2, (\overline{E_1})^c\}$.

$$\zeta_1(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_1 \\ 0, & \text{if } (x,\lambda) \in (E_2)^c \end{cases}$$

and

$$\zeta_2(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_2 \\ 0, & \text{if } (x,\lambda) \in \overline{E_1}. \end{cases}$$

Call $\nu_2(x,\lambda) = \begin{cases} \frac{\xi(x,\lambda)}{\rho_{\mathrm{m}}(g_\lambda(x))}, & \text{if } (x,\lambda) \in {F_1}^c \\ 0, & \text{if } (x,\lambda) \in F_1, \end{cases}$

where $\xi(x,\lambda)$ is given in equation (3), and define

$$\nu_1(x,\lambda) = \begin{cases} \frac{\gamma_{\mathcal{R}}}{\rho_{\mathfrak{f}}(g_\lambda(x))}, & \text{if } (x,\lambda) \in {F_2}^c \\ 0, & \text{if } (x,\lambda) \in F_2. \end{cases}$$

Since all functions defined above can be extended in such a way that they are zero at $0 \times \lambda$, let ν be the vector field in $(\mathbb{C}^n \times \mathbb{C}, 0 \times 0)$ defined as

$$\nu(x,\lambda) = \begin{cases} \zeta_1(x,\lambda)\nu_1(x,\lambda) + \zeta_2(x,\lambda)\nu_2(x,\lambda), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then the vector field ν is continuous, integrable and $h = tG(\nu(x,\lambda)) + w(G(x,\lambda))$.

From the integral curve solutions of ν e ω we construct the germs of homeomorphisms

$H : (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n \times \mathbb{C}, 0 \times 0)$, $H(x,\lambda) = (h(x,\lambda),\lambda)$, $h(x,0) = x$,

and

$K : (\mathbb{C}^p \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$, $K(y,\lambda) = (k(y,\lambda),\lambda)$, $k(y,0) = y$

to obtain $K \circ G \circ H^{-1} = (g, id_{\mathbb{C}})$. ||

Proof of the case $p \geq 2n$: From the equation (3) we have

$$h = tG\left(\frac{\xi}{\rho_{\mathrm{m}}(g_\lambda)}\right) + \frac{\eta \circ G}{\rho_{\mathrm{m}}(g_\lambda)}.$$

By the general hypotheses $\xi \in A_{2D}$ and $\eta \in A_{2R+1}$, from (2) we see that

$\rho m(g_\lambda)(x) \geq c.\rho_v(x)$, therefore the vector field $\dfrac{\xi}{\rho m(g_\lambda)}$ is integrable, and also the vector field $\dfrac{\eta}{\rho m}$, then the homeomorphisms H and K are obtained as above. ||

§4. The non-degenerate case when $p \geq 2n$

In this section we are interested in the topological triviality of families $g_\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $g_\lambda = g + \lambda h$, with $p \geq 2n$ and $g = (g_1, g_2, \ldots, g_p)$ being an $\mathcal{A}$-finitely determined map germ.

We show the $\mathcal{A}$-topological triviality of the family g_λ in terms of the filtration of the map germ h, if the ideal I generated by the system $\{g_1, g_2, \ldots, g_p\}$ satisfies some non-degeneracy conditions with respect to its Newton polyhedron.

We recover the basic definitions needed for these non-degeneracy conditions.

Let $g = \sum_k a_k x^k$ in $\mathcal{O}_n$, denote $\operatorname{supp} g$ the set of points $k \in \mathbb{Z}^n$ with $a_k \neq 0$. If I is an ideal in $\mathcal{O}_n$, define $I = \cup_{g \in I} \operatorname{supp} g$.

Fix an ideal I, consider its Newton polyhedron $\Gamma_+(I)$, the convex hull in $\mathbb{R}^n_+$ of $\{k + v : v \in \mathbb{R}^n_+,\ k \in \operatorname{supp}(I)\}$ and its induced Newton filtration.

For each compact face Δ of $\Gamma(I)$, call $C(\Delta)$ the cone with vertex at the origin and passing through Δ and $\mathcal{A}_\Delta$ denotes the sub-ring with unity of $\mathcal{O}_n$, $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta)\}$. The Newton filtration of $\mathcal{O}_n$ induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

For any germ $g \in \mathcal{O}_n$, denote $g_\Delta = \sum a_k x^k$ with $k \in \operatorname{supp} g \cap \Delta$, and $\operatorname{in}_\Delta(g)$, the polynomial

$$\operatorname{in}_\Delta(g) = \sum \{a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \operatorname{d}(x^k) = \operatorname{d}(g)\}.$$

Definition 4.1. *The ideal I is Newton non-degenerate if there exists a system of generators $\{f_1, \ldots, f_s\}$ of I such that for each compact face $\Delta \subseteq \Gamma$, the ideal generated by the system $\{f_{1\Delta_1}, \ldots, f_{s\Delta_1}\}$ has finite colength in $\mathcal{A}_{\Delta_1}$, for all subfaces Δ_1 of Δ.*

Definition 4.2. *A system of generators* $\{f_1, \ldots, f_s\}$ *of an ideal* I *is non-degenerate on* $\Gamma_+(I)$ *if, for each compact face* $\Delta \subseteq \Gamma$, *the ideal of* $\mathcal{A}_\Delta$ *generated by* $\mathrm{in}_\Delta(f_1), \ldots, \mathrm{in}_\Delta(f_s)$ *has finite colength in* $\mathcal{A}_\Delta$.

Now we consider the ideal $I = \langle g_1, g_2, \ldots, g_p \rangle$, for each generator g_i of I, denote $\mathrm{d}(g_i) = D_i$ and consider $D_1 \leq D_2 \leq \ldots \leq D_p$.

In the case that the ideal I is non-degenerate on some Newton polyhedron Γ_+ we have the following:

Proposition 4.3. *Suppose that* I *is non-degenerate on some Newton polyhedron* Γ_+. *Then, deformations of* g *of type* $g_\lambda = g + \lambda h$, *with* $\mathrm{d}(h_i) \geq D_p$, *for all* $i = 1, \ldots, p$ *are* $\mathcal{A}$*-topologically trivial.*

When the ideal I is Newton non-degenerate we obtain the following:

Corollary 4.4. *Suppose that* I *is Newton non-degenerate. Then, deformations of* g *of type* $g_\lambda = g + \lambda h$, *with* $\mathrm{d}(h_i) \geq D_i$, *for all* $i = 1, \ldots, p$ *are* $\mathcal{A}$*-topologically trivial.*

Since $p \geq 2n$ any map germ $g = (g_1, \ldots, g_p)$ is $\mathcal{A}$-finitely determined if, and only if, g is $\mathcal{L}$- finitely determined, where $\mathcal{L}$ denotes the $\mathcal{L}$-group of Mather.

Let $G(x, \lambda) = (g_\lambda, \lambda)$ be the one parameter unfolding of g. Since g is $\mathcal{L}$-finitely determined we can choose an integer number s and a vector field $\eta \in m_p \theta_{p+1}$, such that

$$\frac{\partial g_\lambda}{\partial \lambda} \left(g_1^{2D/D_1} + g_2^{2D/D_2} + \ldots + g_p^{2D/D_p} \right)^s = \eta \circ G.$$

Consider the control function in the target $\rho : \mathbb{C}^p \to \mathbb{R}$, defined by

$$\rho(y) = \left(|y_1|^{2D/D_1} + |y_2|^{2D/D_2} + \ldots + |y_p|^{2D/D_p} \right)^{1/2}$$

To prove that the vector field $\dfrac{\eta \circ g}{(\rho(g))^{2s}}$ is integrable, it is sufficient to show that there exists a constant $C > 0$ such that $\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, y_1, \ldots, y_p) \right| \leq C\rho(y)$, (see Gaffney in [6] p.482 and Fukui-Paunescu in [5], p.87).

We can compose the terms of this inequality with G to get an equivalent inequality on $\mathbb{C}^n$, $\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \ldots, x_n) \right| \leq C\rho(g_\lambda)$.

Proof of the Proposition 4.3:

From the proof of the Theorem 3.6 of [1] we see that the ideal I is non-degenerate on Γ_+ if, and only if, the ideal

$$J = \left\langle g_1^{D/D_1}, g_2^{D/D_2}, \dots g_p^{D/D_p} \right\rangle$$

is Newton non-degenerate. In this case, if a germ h satisfies $\mathrm{d}(h) \geq D_p$, then $\Gamma_+(h^{D/D_p}) \in \Gamma_+(J)$, since J is Newton non-degenerate we obtain $h^{D/D_p} \in \overline{J}$. Now we can use the valuative criterion for the integral closure (see [12] p. 288), to obtain that

$$\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \dots, x_n) \right| \leq C\rho(g_\lambda)$$

and the result follows.

Proof of the Corollary 4.4:

If the ideal I is Newton non-degenerate, we obtain from the Theorem 3.4 of [10] that any germ h with $\Gamma_+(h) \subset \Gamma_+(I)$ is in the integral closure of I. ¿From the condition $\mathrm{d}(h_i) \geq \mathrm{d}(g_i)$, since $\Gamma_+(g_i) \subset \Gamma_+(I)$ we obtain

$\Gamma_+(h_i) \subset \Gamma_+(I)$. ||

4.1. An example in $\mathbb{C}^2 \to \mathbb{C}^4$

Let $g : \mathbb{C}^2 \to \mathbb{C}^4$ be the map germ $g = (g_1, g_2, g_3, g_4)$ with

$$\begin{aligned}
g_1(x,y) &= \alpha_1 x^5 + \alpha_2 y^5 + a_1 x^3 y + a_2 x y^3; \\
g_2(x,y) &= \beta_1 x^7 + \beta_2 y^7 + b_1 x^3 y^2 + b_2 x^2 y^3; \\
g_3(x,y) &= \theta_1 x^{11} + \theta_2 y^{11} + c_1 x^5 y^3 + c_2 x^3 y^5; \\
g_4(x,y) &= \gamma_1 x^{12} + \gamma_2 y^{12} + d x^4 y^4;
\end{aligned}$$

We see in the example 2.1 of [2] that g is $\mathcal{A}$-finitely determined for generic values of α_i, β_i, θ_i and γ_i, with a_i, b_i, c_i and d being all distinct prime numbers.

Here we fix the Newton polyhedron $\Gamma_+(g_4)$, with vertices $(12,0)$, $(0,12)$, $(4,4)$ to obtain that I is non-degenerate on $\Gamma_+(g_4)$, therefore any deformation of type $g_\lambda = g + \lambda h$ with $\mathrm{d}(h_j) \geq \mathrm{d}(g_4)$ for $j = 1,2,3,4$ is topologically trivial.

§5. The weighted homogeneous case

Damon in [3] investigates the topological triviality of unfoldings of $\mathcal{A}$-finitely determined map germs which are weighted homogenous. His theorem 1. shows that polynomial unfoldings of non negative weights of these map germs are topologically trivial.

From the results shown above we obtain similar results for one parameter linear unfoldings of any weighted homogenous $\mathcal{A}$-finitely determined map germ. We remark that in the weighted homogenous case, the results of Damon are for any pair of dimensions (n, p) while the results shown here are only for $n \geq p$ and $p \geq 2n$

Definition 5.1. *Given $(w_1, \ldots, w_n; d_1, \ldots, d_p)$ with $w_i,\ d_j \in \mathbb{Q}_+$, a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is weighted homogeneous of type $(w_1, \ldots, w_n; d_1, \ldots, d_p)$ if for all $\lambda \in K - \{0\}$*

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_n} x_n) = (\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), \ldots, \lambda^{d_p} f_p(x)).$$

For a fixed set of weights $w = (w_1, \ldots, w_n)$ consider the *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \mathrm{d}_w(g) \geq q\}$.

Proposition 5.2. *Let g be an $\mathcal{A}$-finitely determined map germ which is weighted homogenous of type $(w_1, \ldots, w_n; d_1, \ldots, d_p)$. Then deformations of $g_\lambda = g + \lambda h$ of g, with $\mathrm{d}_w(h_i) \geq d_i$, $\forall\ i = 1, \ldots, p$, are $\mathcal{A}$-topologically trivial for small values of λ.*

Proof: To show this result we follow the proof of the Proposition 3.2.

We should prove that the germ g satisfies the equation 1, however the main purpose of this equation is to guarantee that

$$h\rho_{\mathrm{m}}(g) \in tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p)$$

and them from the Lemma 3.2. we obtain that this condition also holds for deformations, i. e.,

$$h.\rho_{\mathrm{m}}(g_\lambda) \in tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}\theta_{p+1})$$

In the case of weighted homogenous map germs, we see in the item *ii* of the proposition 7.4 of [3] p.319, that it is possible to obtain vector fields η and ψ satisfying the condition

$$h.\rho_{\mathrm{m}}(g_\lambda) = tg_\lambda(\psi(x, \lambda)) + \eta(g(x, \lambda)).$$

and as g and ρ_m are weighted homogeneous, we may assume that the vector field ψ is in $A_{2D}\theta_{n+1}$ and η is in $B_{2R+1}\theta_{p+1}$.

In the case $p \geq 2n$ we use the fact that each germ g_j is weighted homogeneous of type $(w_1, \ldots, w_n; d_j)$, then we consider the ideal I generated by the system $\{g_1^{2r_1}, \ldots, g_p^{2r_p}\}$, where r_j are integers such that $r_j d_j = D$ for some D and each $g_j^{2r_j}$ is weighted homogenous of type $(w_1, \ldots, w_n; 2D)$.

Since g is $\mathcal{A}$-finitely determined it is also $\mathcal{L}$-finitely determined and the ideal I is Newton non-degenerate.

Therefore we obtain the inequality $\rho_{\mathrm{m}}(g(x)) \geq c\rho_v(x)$, $\forall x$ in a neighborhood V of 0.

In the case $n \geq p$ we need to show that there exist a neighborhood V of 0 in $\mathbb{C}^n$, and constants α e β such that $\rho_{\mathrm{f}}(g(x)) \geq \beta\rho_v(x)$, for all $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$, but in this case this condition follows from the lemma 7.7, p.319 of [3].

Therefore, we are ready to follow the final part of the proof of the Proposition 3.1. to obtain the result. ||

§6. Examples

Example 6.1. ([7], p. 102.) Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $f(x, y, z) = (x^2 + y^2 + x^3 + z^3, x^2 + y^3 + z^2)$.

We remark that it is not possible to apply the Proposition 5.2 for this case since the map germ f is not weighted homogenous.

The best filtrations to choose for $\mathcal{O}_n$ and $\mathcal{O}_p$ in this example are the usual filtrations given by the degree.

Here we have $tf(m_n^3\theta_n) + wf(m_p^2\theta_p) = m_n^4\theta_f$, moreover we see that $tf(m_n^3\theta_n) + wf(m_p^2\theta_p) = tf(m_n^3\theta_n) + f^*(m_p)m_n^{k-1}\theta_f = m_n^4\theta_f$, hence this germ is $(k-1) - C^0 - \mathcal{K}-$ determined. From this condition we show that this germ satisfies the conditions of the Proposition 3.2., therefore deformations by order higher than 2 are $\mathcal{A}$-topologically trivial.

Example 6.2. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (xy, g(x, y))$, with $g(x, y) = x^4 + xy^2 + y^5$.

This is a special case of a pre-weighted homogeneous map germ which is in the $\mathcal{K}$-orbit of the $\mathcal{A}$-finitely determined weighted homogenous map germ $k(x, y) = (xy, x^4 + y^5)$, therefore it is also $\mathcal{A}$-finitely determined. See [11] for more details about the $\mathcal{A}$-finite determinacy of pre-weighted homogenous map germs.

We fix the Newton polygon $\Gamma_+(g)$ and are interested in the topological triviality of families of type $g_\lambda(x,y) = (xy, x^4 + xy^2 + y^5 + \lambda h(x,y))$, with $\mathrm{d}(h) \geq \mathrm{d}(g)$.

The main difficulty with this type of example is to show that the equation 1 holds, or if we follow the proof for the weighted homogenous case, we need to show that it is possible to obtain vector fields η and ψ satisfying the condition

$$h.\rho_{\mathrm{m}}(g_\lambda) = tg_\lambda(\psi(x,\lambda)) + \eta(g(x,\lambda)).$$

in such a way that the vector field ψ is in $A_{2D}\theta_{n+1}$ and η is in $B_{2R+1}\theta_{p+1}$.

In fact, in this case we can show that for each germ h it is possible to find an specific vector field ψ, which depends of the cone $C(\Delta)$ that h belongs, such that ψ is not in A_{2D}, however it is in an appropriate level of filtration in such a way that we obtain the integrability of the vector field $\nu_1(x,\lambda)$ given in the proof of the Lemma 3.3.

Therefore we can follow the method of the proof of the Proposition 3.2. to show that any deformation of this type is topologically trivial for small values of λ.

This example is a particular case of the following:

Proposition 6.3. [8] Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x,y) = (xy, g(x,y))$, with $g(x,y) = x^a + x^r y^s + y^b$, be a pre-weighted homogeneous map germ in the $\mathcal{K}$-orbit of an $\mathcal{A}$-finitely determined map germ $k(x,y) = (xy, x^a + y^b)$. Then deformations of type $f(x,y) = (xy, x^a + x^r y^s + y^b + \lambda h(x,y))$ with $\Gamma_+(h) \subset \Gamma_+(g)$ are topologically trivial for small values of λ.

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Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail address: `mjsaia@icmc.usp.br`

Departamento de Matemática, Universidade Regional do Cariri, Crato, Ceará, Brazil
E-mail address: `liane@urca.br`