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Numerical characterization for affine varieties be a cone over nonsingular projective varieties

Stephen S.-T. Yau[1]

§1. Introduction

Let V be an affine variety in $\boldsymbol{C}^{n+1}$. It is a natural question to ask when V is a cone over a nonsingular projective variety in $\boldsymbol{CP}^n$ after a biholomorphic change of coordinates in $\boldsymbol{C}^{n+1}$. This seems to be a very difficult problem even if V is a hypersurface in $\boldsymbol{C}^{n+1}$.

For example, let $f(x_1, x_2, x_3) = x_1^2 + x_2^3 + x_3^4$. Take a generic change of coordinates $x_i = \sum_{j=1}^{3} a_{ij} y_j$, $1 \leq i \leq 3$. Then we get a new polynomial

$$\begin{aligned} \widetilde{f}(y_1, y_2, y_3) &= (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)^2 \\ &\quad + c(a_{21}y_1 + a_{22}y_2 + a_{23}y_3)^3 \\ &\quad + (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)^4. \end{aligned}$$

On the other hand, consider $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. Take a generic change of coordinates $x_i = \sum_{j=1}^{3} b_{ij} y_j + q_i$, where q_i, $1 \leq i \leq 3$, are quadratic polynomials in y_1, y_2, y_3. Then we get a new polynomial

$$\begin{aligned} \widetilde{g}(y_1, y_2, y_3) &= (b_{11}y_1 + b_{12}y_2 + b_{13}y_3 + q_1)^2 \\ &\quad + (b_{21}y_1 + b_{22}y_2 + b_{23}y_3 + q_2)^2 \\ &\quad + (b_{31}y_1 + b_{32}y_2 + b_{33}y_3 + q_3)^2. \end{aligned}$$

Observe that both $\widetilde{f}$ and $\widetilde{g}$ are degree 4 polynomials in y_1, y_2 and y_3. The hypersurface defined by $\widetilde{g}$ is a cone over nonsingular projective curve in $\boldsymbol{CP}^2$ after biholomorphic change of coordinates while the hypersurface defined by $\widetilde{f}$ does not have this property.

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In this paper, we shall treat the simplest case when V is a hypersurface in $\boldsymbol{C}^{n+1}$. Obviously if V is a cone over a nonsingular projective variety in $\boldsymbol{CP}^n$, then V only has an isolated singularity at 0. Therefore we need to give a characterization when an analytic function $f(z_0, z_1, \ldots, z_n)$ with isolated critical point of 0 is a homogeneous polynomial after biholomorphic change of coordinate. In this paper we shall formulate a numerical criterion for this purpose.

§2. Geometric Genus and Milnor Number

Let $f(z_0, z_1, \ldots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose that f has an isolated critical point at the origin. f can be developed in a convergent Taylor series $f(z_0, z_1, \ldots, z_n) = \Sigma_\lambda a_\lambda z^\lambda$ where $z^\lambda = z_0^{\lambda_0} \ldots z_n^{\lambda_n}$. Recall that Newton boundary $\Gamma(f)$ is the union of the compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of the subsets $\{\lambda + (\boldsymbol{R}_+)^{n+1}\}$ for λ such that $a_\lambda \neq 0$. Finally, let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\boldsymbol{C}^*)^{n+1}$ for any $\Delta \in \Gamma(f)$ where $\boldsymbol{C}^* = \boldsymbol{C} \backslash \{0\}$.

Let $(V, 0)$ be an isolated hypersurface singularity defined by holomorphic function $f : (\boldsymbol{C}^{n+1}, 0) \longrightarrow (\boldsymbol{C}, 0)$. Let $\pi : M \longrightarrow V$ be a resolution of the singularity at 0. Define the geometric genus of the singularity $(V, 0)$ to be $p_g = \dim H^{n-1}(M, O)$. Let ω be a holomorphic n-form on $V - \{0\}$. ω is said to be L^2-integrable if $\int_{W-\{0\}} \omega \wedge \overline{\omega} < \infty$ for any sufficiently small relatively compact neighborhood W of 0 in V. Let $L^2(V - \{0\}, \Omega^n)$ be the set of all L^2-integrable holomorphic n-forms on $V - \{0\}$, which is a linear subspace of $\Gamma(V - \{0\}, \Omega^n)$. Then

$$p_g = \dim \Gamma(V - \{0\}, \Omega^n) / L^2(V - \{0\}, \Omega^n)$$

(See Laufer [1] for $n = 2$ and Yau [11] for $n > 2$).

We say that a point p of the integral lattice $\boldsymbol{Z}^{n+1}$ in $\boldsymbol{R}^{n+1}$ is positive if all the coordinates of p are positive. The following theorem is due to Merle-Teissier [3].

Theorem 2.1. (Merle-Teissier) *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function f : $(\boldsymbol{C}^{n+1}, 0) \longrightarrow (\boldsymbol{C}, 0)$. Then the geometric genus $p_g = \#\{p \in \boldsymbol{Z}^{n+1} \cap \Gamma_-(f) : p$ is positive$\}$.*

Notice that in the above formula, positive lattice points on $\Gamma(f)$ are counted.

Let $f : (\boldsymbol{C}^{n+1}, 0) \longrightarrow (\boldsymbol{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. For $\epsilon > 0$ suitably small and δ yet smaller, the space $V' = f^{-1}(\delta) \cap D_\epsilon$ (where D_ϵ denotes the closed disk of radius ϵ about 0) is a real $2n$-manifold with boundary whose diffeomorphism type depends only on f. Milnor [4] proved that V' has the homotopy type of a wedge of n-spheres. The number of these n-spheres is called Milnor number μ. In fact, $\mu = \dim \boldsymbol{C}\{z_0, z_1, \dots, z_n\}/(f_{z_0}, f_{z_0}, \dots, f_{z_n})$. Recall also that $\tau := \dim \boldsymbol{C}\{z_0, \dots, z_n\}/(f, f_{z_0}, \dots, f_{z_n})$ is an analytic invariant.

A polynomial $f(z_0, z_1, \dots, z_n)$ is a weighted homogeneous of type $(w_0, w_1, \dots, w_n)$, where $w_0, w_1, \dots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$ for which $i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1$.

Theorem 2.2. (Milnor and Orlik) [5] *Let $f(z_0, z_1, \dots, z_n)$ be a weighted homogeneous polynomial of type $(w_0, w_1, \dots, w_n)$ with isolated singularity at the origin. Then the Milnor number is $\mu = (w_0 - 1)(w_1 - 1) \dots (w_n - 1)$.*

The following deep theorem which gives a numerical characterization of weighted homogeneous polynomial is due to Saito [7].

Theorem 2.3. (Saito) *Let $f : (\boldsymbol{C}^{n+1}, 0) \longrightarrow (\boldsymbol{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let μ and τ be defined as above. Then $\mu = \tau$ if and only if f is a weighted homogeneous polynomial after biholomorphic change of coordinates.*

§3. Numerical Characterization of Homogeneous Polynomial

In view of Theorem 2.3 above, we only need to give intrinsic numerical characterization when a weighted homogeneous polynomial is actually an homogeneous polynomial. The first theorem is due to Xu and Yau [9] in 1993.

Theorem 3.1. (Xu-Yau) *Let $(V, 0)$ be a two-dimensional isolated singularity defined by a weighted homogeneous polynomial $f(z_0, z_1, z_2) = 0$. Let μ be the Milnor number, p_g be the geometric genus and ν be the multiplicity of the singularity. Then*

$$\mu - \nu + 1 \geq 6p_g$$

with equality if and only if $(V, 0)$ is defined by the homogeneous polynomial.

Theorem 3.1 implies the Durfee conjecture $\mu \geq 3!p_g$ in this case.

Theorem 3.2. (Xu-Yau) [9]. *Let $(V,0)$ be a two-dimensional isolated hypersurface singularity defined by $f(x,y,z) = 0$. Let μ be the Milnor number, p_g be the geometric genus, ν be the multiplicity of the singularity and τ = dimension of the semi-universal deformation space of $(V,0) = \dim \boldsymbol{C}\{x,y,z\}/Cf, f_x, f_y, f_z)$. Then after a biholomorphic change of coordinate f is a homogeneous polynomial if and only if $\mu - \nu + 1 = 6p_g$ and $\mu = \tau$.*

In view of Theorem 3.2, we have made the following conjecture in 1995.

Conjecture *Let $f : (\boldsymbol{C}^{n+1}, 0) \longrightarrow (\boldsymbol{C}, 0)$ be a weighted homogeneous polynomial with an isolated critical point at the origin. Then*

$$\mu - h(\nu) \geq (n+1)! p_g$$

with equality if and only if f is a homogeneous polynomial, where $h(\nu)$ is a polynomial function on multiplicity with the properties $h(\nu) \geq 0$ and $h(\nu) = 0$ if and only if $\nu = 1$. Note that h is a polynomial function from $\boldsymbol{Z}_+$ to $\boldsymbol{Z}_+ \cup \{0\}$.

For two-dimensional isolated singularity, Theorem 3.2 asserts that Yau conjecture is true. In fact $h(\nu) = \nu - 1$. For 3-dimensional singularity, the conjecture is very challenging because we need to find $h(\nu)$ explicity. After several years of hard work, we have proved the conjecture for a 3-dimensional case with K.-P. Lin [2].

Theorem 3.3. (Lin-Yau) *Let $(V,0)$ be a three dimensional isolated singularity defined by a weighted homogeneous polynomial $f(x_0, x_1, x_2, x_3) = 0$. Let p_g be the geometric genue, ν be the multiplicity and μ be the Milnor number of the singularity. Then we have*

$$\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geq 4! p_g$$

with equality if and only if $(V,0)$ is defined by a homogeneous polynomial.

Theorem 3.2 implies the Durfee conjecture $\mu \geq 4! p_g$ in this case.

As a corollary of Theorem 2.3 and Theorem 3.3, we have the following theorem.

Theorem 3.4. (Lin-Yau) *Let $(V,0)$ be a three dimensional isolated hypersurface singularity defined by $f(x_0, x_1, x_2, x_3) = 0$. Let μ be the Milnor number, p_g be the geometric genus, ν be the multiplicity of the singularity and τ = dimension of the semi-universal deformation space of $(V,0) = \dim \boldsymbol{C}\{x_0, x_1, x_2, x_3\}/(f, f_{x_0}, f_{x_1}, f_{x_2}, f_{x_3})$. Then after a biholomorphic change of coordinates f is a homogeneous polynomial if and only if $\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 24p_g$ and $\mu = \tau$.*

Theorem 3.3 is related to the following theorem of Xu and Yau [10].

Theorem 3.5. (Xu-Yau) *Let* $a \geq b \geq c \geq d \geq 2$, *and* P_4 *be the number of positive integral solutions of* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$, *i.e.* $P_4 = \#\{(x, y, z, w) \in \boldsymbol{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. *If* $P_4 > 0$, *then*

$$24P_4 \leq abcd - \frac{3}{2}(abc + abd + acd + bcd) + \frac{11}{3}(ab + ac + bc) - 2(a + b + c)$$

and equality is attained if and only if $a = b = c = d =$ *integer.*

However Theorem 3.3 does not follow from Theorem 3.5 because the minimal weight of the variables x_i may not be an integer and we also need to analyze the case when the geometric genus vanishes. It is quite easy to see that the multiplicity ν is given by $\inf\{n \in \boldsymbol{Z}_+ : n \geq \inf\{w_0, w_1, w_2, w_3\}$ where w_i is the weight of $x_i\}$, see for example Saeki [6]. We observe that if $w_0 \geq w_1 \geq w_2 \geq w_3$ and w_3 is not an integer, then $w_3 = [w_3] + \beta$, $0 < \beta < 1$ and β is either $\frac{w_3}{w_0}$, or $\frac{w_3}{w_2}$. We then get an even sharper estimate in these three particular cases in the following Theorem 3.6 and Theorem 3.7 then those obtained in Theorem 3.5 of Xu and Yau [10].

Theorem 3.6. (Lin-Yau) *Let* $a \geq b \geq c \geq d \geq 3$ *be real numbers. Consider* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1$. *Let* P_4 *be the number of positive integral solutions of the above equation, i.e.,* $P_4 = \#\{(x, y, z, w) \in \boldsymbol{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. *Suppose* d *is not an integer and* $d = [d] + \beta$ *where* β *is either* $\frac{d}{c}$ *or* $\frac{d}{b}$ *or* $\frac{d}{a}$. *Define* $\mu = (a-1)(b-1)(c-1)(d-1)$. *Then*

$$\begin{aligned} 24P_4 &< \mu - (2\nu^3 - 5\nu^2 + 2\nu + 1)\Big|_{\nu = d - \beta + 1} \\ &= abcd - (abc + abd + acd + bcd) \\ &\quad + (ab + ac + ad + bc + bd + cd) \\ &\quad - (a + b + c) - (2d^3 + d^2 - d - 1) \\ &\quad + 2\beta^3 - \beta^2(6d + 1) \\ &\quad + \beta(6d^2 + 2d - 2). \end{aligned}$$

Theorem 3.7. (Lin-Yau) *Let* $a \geq b \geq c \geq d \geq 2$ *be real numbers. Consider* $\frac{x}{a} + \frac{y}{b} + \frac{z}{d} \leq 1$. *Let* P_4 *be the number of positive integral solutions of the above equation, i.e.,* $P_4 = \#\{(x, y, z, w) \in \boldsymbol{Z}_+^4 : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} \leq 1\}$. *Suppose* $P_4 > 0$ *and* d *is not an integer and* $d = [d] + \beta$ *where* β *is either* $\frac{d}{c}$, *or* $\frac{d}{b}$, *or* $\frac{d}{a}$. *Then the same assertion of Theorem 3.6 holds.*

Unlike the surface singularities treated in Xu and Yau [9], we still need to handle the case when the geometric genus is equal to zero. Thus Theorem 3.3 is substantially harder to prove than Theorem 3.1.

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Department of Mathematics, Statistics
and Computer Science, M/C 249
851 S. Morgan Street
Chicago, IL 60607-7045
USA