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Some Geometric Methods in Commutative Algebra

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In this series of lectures, I will discuss three examples of techniques of algebraic geometry which have applications to commutative algebra. The examples chosen are those which pertain most closely to my own research interests.

§1. Hilbert functions, regularity and Uniform Artin-Rees

In this Section we describe results giving explicit, effective bounds for the Hilbert functions and postulation number of a Cohen-Macaulay module M of given dimension d and multiplicity e over a Noetherian local ring $(A, \mathfrak{m})$, with respect to a given $\mathfrak{m}$-primary ideal I. We also discuss related results bounding the Castelnuovo-Mumford regularity of the associated graded module of M with respect to I in terms of Hilbert coefficients, assuming only that M has positive depth; this leads to a new proof of the Uniform Artin-Rees theorem of Duncan and O'Carroll, and other results. The geometric technique used here is the cohomological study of the blow up of the ideal I, using in particular Grothendieck's formal function theorem.

1.1. The finiteness theorem for Hilbert functions

Recall that if $(A, \mathfrak{m})$ is a Noetherian local ring, M a finite A-module and $I \subset \mathfrak{m}$ an ideal of A such that M/IM has finite length, the *Hilbert function* (or more properly, the Hilbert-Samuel function) of M with respect to I is the numerical function

$$H_I(M)(n) = \ell(M/I^n M), \ \ \forall\, n \geq 0,$$

where we use the symbol ℓ to denote the length of a module (which has a finite composition series). Then there exists a corresponding *Hilbert polynomial*

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$$P_I(M)(x) = e_0(I,M)\binom{x+d-1}{d} + e_1(I,M)\binom{x+d-2}{d-1} + \cdots + e_d(I,M),$$

where $e_0(I,M) > 0$ is the *multiplicity* of M with respect to I, $e_j(I,M) \in \mathbb{Z}$ for all $0 \leq i \leq d = \dim M$, and such that for some non-negative integer $n_0(I,M)$,

$$H_I(M)(n) = P_I(M)(n) \ \ \forall\, n \geq n_0(I,M).$$

The integers $e_j(I,M)$ are called the *Hilbert coefficients*, and $n_0(I,M)$ is called a *postulation number*, of M with respect to I.

The first result we state is the following *Finiteness Theorem*, taken from the paper [44] of V. Trivedi. The special case when $M = A$ is a Cohen-Macaulay local ring, and $I = \mathfrak{m}$ is the maximal ideal, was treated earlier in a paper of Trivedi and myself [39].

Theorem 1.1 (Finiteness Theorem). *Let $(A,\mathfrak{m})$, I, M be as above, where M is a Cohen-Macaulay module. Let $\mathbf{e} = e_0(I,M)$. Then*

(i) $|e_j(I,M)| \leq (9\mathbf{e}^5)^{j!}$ *for* $j \geq 1$, *and*
(ii) $n_0(I,M) = 3^{d!-1}\mathbf{e}^{3(d-1)!-1}$ *is a postulation number for M with respect to I.*

Corollary 1.2. *For fixed positive integers d, e there are only finitely many numerical functions which can arise as the Hilbert function $H_I(M)(n)$ of a Cohen-Macaulay module M of dimension d over a Noetherian local ring, which has multiplicity e with respect to some appropriate ideal I of that local ring. In particular, only finitely many numerical functions $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ can arise as the Hilbert function of a Cohen-Macaulay local ring of a given dimension and multiplicity.*

Remark 1.3. [44] contains some references to the earlier results in the direction of Theorem 1.1, which essentially pertain to very special situations, or to the coefficients $e_1(I,M)$ and $e_2(I,M)$. See [45] for an extension of Theorem 1.1 to the case of generalized Cohen-Macaulay modules.

Remark 1.4. Kleiman has shown (see [11, Exp.XIII]) that only finitely many polynomials can occur as the Hilbert polynomial of an integral projective scheme, of a given dimension, embedding dimension and degree, over a field. He gives an example to show that this is false for Cohen-Macaulay schemes. On the other hand, in [41], examples are given of infinitely many polynomials which occur as Hilbert polynomials of complete, local integral domains, of a fixed dimension and multiplicity, which are quotients of a fixed regular local ring. The examples given are of quotients of dimension 2 and multiplicity 4 of a power series ring in 6 variables over a field.

We now outline a proof of the Finiteness Theorem in the special case $M = A$, $I = \mathfrak{m}$, which contains most of the features of the general case. To simplify notation, we write $e_j(M)$, $n_0(M)$, etc. to mean $e_j(\mathfrak{m}, M)$, $n_0(\mathfrak{m}, M)$, etc., for any finite A-module M.

One input is a cohomological formula for the difference between the Hilbert function and Hilbert polynomial, obtained by Johnston and Verma (see [17], and [44], Theorem 1), generalizing "classical" results of Serre. This can be expressed in two equivalent ways, involving graded local cohomology of the Rees algebra, or coherent sheaf cohomology on a blow-up. We state their result in the latter form, since that is what is useful for us later.

Theorem 1.5. *Let $(A, \mathfrak{m})$ be Noetherian local of dimension d and depth > 0, and let*

$$R = \bigoplus_{n \geq 0} \mathfrak{m}^n, \quad R_+ = \bigoplus_{n > 0} \mathfrak{m}^n$$

be the $\mathfrak{m}$-adic (graded) Rees algebra of A and its irrelevant graded ideal, respectively. Let $\pi : X = \operatorname{Proj} R \to \operatorname{Spec} A$ be the blow-up of the maximal ideal of A.

(a) *If $d \geq 2$ and* $\operatorname{depth} A \geq 2$, *then*

$$P_{\mathfrak{m}}(A)(n) - H_{\mathfrak{m}}(A)(n) = \sum_{i=1}^{d-1} (-1)^{i-1} \ell(H^i(X, \mathcal{O}_X(n))).$$

In particular, taking $n = 0$, we get

$$e_d(A) = \sum_{i=1}^{d-1} (-1)^{i-1} \ell(H^i(X, \mathcal{O}_X)). \tag{1.1}$$

(b) (Northcott) *If $d = 1$, then*

$$e_1(A) = -\ell\left(\frac{H^0(X, \mathcal{O}_X)}{A}\right).$$

Remark 1.6. Note that if $\pi : X \to \operatorname{Spec} A$ is the blow-up of the maximal ideal, then π induces an isomorphism $X - \pi^{-1}(\mathfrak{m}) \to \operatorname{Spec} A - \{\mathfrak{m}\}$. Hence for any coherent sheaf $\mathcal{F}$ on X and any $j > 0$, the cohomology A-module $H^j(X, \mathcal{F})$, which is a finite A-module, has support contained in $\{\mathfrak{m}\}$; in particular, it has finite length. Hence the formulas in (a) of the above Theorem are meaningful. A similar comment applies to (b).

Remark 1.7. [17] contains a somewhat more general result, valid for any $\mathfrak{m}$-primary ideal, formulated in terms of graded local cohomology of the Rees algebra $R(I) = \oplus_{n\geq 0} I^n$ with respect to its irrelevant graded ideal $R_+(I)$, where the depth hypotheses on A are not needed. The above formulation results from the standard relation between sheaf cohomology on X and graded local cohomology of the Rees algebra. This is further generalized in [44] to the case of Hilbert functions of A-modules, and is also expressed cohomologically as above on the blow-up $X = \operatorname{Proj} R(I)$. The formula in [17] has a sign error, corrected in [44].

Next, we recall the notion of *Castelnuovo-Mumford regularity* of a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^N_A$ over a Noetherian ring A: we say that $\mathcal{F}$ is *m-regular* if $H^i(\mathbb{P}^N_A, \mathcal{F}(m-i)) = 0$ for all $i > 0$. This is closely related to the notion of Castelnuovo-Mumford regularity, in the sense of commutative algebra (see [7]), for the graded module $\oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^N_A, \mathcal{F}(n))$ over the polynomial algebra $A[X_0, \dots, X_N]$ (the homogeneous coordinate ring of $\mathbb{P}^N_A$). We now recall some of the standard properties of regularity (see [27]).

Proposition 1.8. (i) *Let A be a Noetherian ring, and let $\mathcal{F}$ be an m-regular coherent sheaf on $\mathbb{P}^N_A = \operatorname{Proj} A[X_0, \dots, X_N]$. Then:*

(a) *the graded $A[X_0, \dots, X_N]$-module*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^N_A, \mathcal{F}(n))$$

is generated by its homogeneous elements of degrees $\leq m$, and the sheaf $\mathcal{F}(n)$ is generated (as an $\mathcal{O}_{\mathbb{P}^N_A}$-module) by its global sections, for all $n \geq m$,

(b) *$H^i(\mathbb{P}^N_A, \mathcal{F}(j)) = 0$ for all $i > 0$ and $i + j \geq m$.*

(ii) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves on $\mathbb{P}^N_A$, for a Noetherian ring A, then:*

(a) *if $\mathcal{F}'$ and $\mathcal{F}''$ are m-regular, then $\mathcal{F}$ is m-regular,*

(b) *if $\mathcal{F}$ and $\mathcal{F}''$ are m-regular, then $\mathcal{F}'$ is also m-regular precisely when $H^0(\mathbb{P}^N_A, \mathcal{F}(m-1)) \to H^0(\mathbb{P}^N_A, \mathcal{F}''(m-1))$ is surjective.*

We also need a technical lemma of Mumford (see [27]). Recall that

$$\chi(\mathcal{F}) = \sum_{i\geq 0} (-1)^i \ell(H^i(X, \mathcal{F}))$$

denotes the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X, such that all the cohomology modules of $\mathcal{F}$ have finite length, and only finitely many are non-zero (for example, this holds for any coherent $\mathcal{F}$ if X is proper over an Artinian ring). Recall also that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of such coherent sheaves, then the long exact sequence of cohomology implies easily that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Hence, if $\mathcal{F}$ has a finite filtration $\{F^i\mathcal{F}\}$ by coherent subsheaves, then

$$\chi(\mathcal{F}) = \chi(\mathrm{gr}_F\mathcal{F}) = \sum_i \chi(F^i\mathcal{F}/F^{i+1}\mathcal{F}).$$

Lemma 1.9 (Mumford). *Let k be a field, and let $E \subset \mathbb{P}^N_k$ be a closed subscheme of dimension > 0, with Hilbert polynomial P_E (thus $P_E(n) = \chi(\mathcal{O}_E(n))$ for all $n \in \mathbb{Z}$). Let $F = E \cap H$ be the intersection with a hyperplane $H \subset \mathbb{P}^N_k$ such that the associated complex of sheaves*

$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E \to \mathcal{O}_F \to 0$$

is exact (i.e., the linear polynomial defining H is a non zero-divisor on $\mathcal{O}_E$). Suppose that the ideal sheaf $\mathcal{I}_F$ of F in $H = \mathbb{P}^{N-1}_k$ is m'-regular, where $m' \geq 1$. Then the following properties hold:

(i) $P_E(m'-1) = \chi(\mathcal{O}_E(m'-1)) \geq 0$,
(ii) *the ideal sheaf $\mathcal{I}_E$ of E in $\mathbb{P}^N_k$ is m-regular, with $m = m' + P_E(m'-1)$,*
(iii) *$\mathcal{O}_E$ is $(m'-1)$-regular.*

Now suppose $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 1$, with infinite residue field k (which we may assume without loss of generality). We outline the procedure which leads to an inductive proof of the Finitness Theorem.

Let

$$X = \mathrm{Proj}\, R = \mathrm{Proj} \bigoplus_{n \geq 0} \mathfrak{m}^n$$

be the blow-up of the maximal ideal, and

$$E = \mathrm{Proj} \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

the exceptional divisor. If $\ell(\mathfrak{m}/\mathfrak{m}^2) = N + 1$, then we can realize the Rees A-algebra R as a graded quotient of a polynomial algebra

$A[X_0, \dots, X_N]$. Hence X is realized as a closed subscheme of $\mathbb{P}^N_A$, such that we have a commutative diagram of closed immersions of A-schemes

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^N_A \\ \uparrow & & \uparrow \\ X \cap \mathbb{P}^N_k = E & \hookrightarrow & \mathbb{P}^N_k \end{array}$$

(here $\mathbb{P}^N_k$ is regarded as the fibre over the maximal ideal of $\pi : \mathbb{P}^N_A \to \operatorname{Spec} A$). The tautological line bundle (invertible sheaf) $\mathcal{O}_{\mathbb{P}^N_A}(1)$ restricts on X to $\mathcal{O}_X(1)$, which is naturally identified with the ideal sheaf $\mathcal{I}_{E,X}$ of E in X:

$$\mathcal{O}_{\mathbb{P}^N_A}(1) \otimes \mathcal{O}_X = \mathcal{O}_X(1) = \mathfrak{m}\mathcal{O}_X = \mathcal{I}_{E,X} \subset \mathcal{O}_X.$$

The Rees ring $R = \oplus \mathfrak{m}^n$ is identified with the homogeneous coordinate ring of $X \subset \mathbb{P}^N_A$, and the natural restriction map $H^0(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(n)) \to H^0(X, \mathcal{O}_X(n))$ induces an inclusion $\mathfrak{m}^n \hookrightarrow H^0(X, \mathcal{O}_X(n))$ for all $n \geq 0$, which is an isomorphism provided $H^1(\mathbb{P}^N_A, \mathcal{I}_X(n)) = 0$, where $\mathcal{I}_X$ is the ideal sheaf of X in $\mathbb{P}^N_A$.

Let $m = m(\mathcal{I}_E)$ be the smallest integer ≥ 1 such that the ideal sheaf $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ of E in $\mathbb{P}^N_A$ is m-regular. Our next goal will be to bound (in terms of $e_0(A)$ and d) the following quantities:

(i) $|e_j(A)|$ for $1 \leq j \leq d-1$,
(ii) m,
(iii) $\dim_k H^j(\mathcal{O}_E(r))$ for $j \geq 1$, $r \geq 0$,
(the bound sought is to be independent of r),
(iv) $\dim_k H^0(\mathcal{O}_E(r))$ for $r \geq 0$,
(with a bound allowed to depend on r),
(v) $|e_d(A)|$.

We have written these expressions in the above order because the proof of each bound will use the existence of the preceeding bounds. The bounds in (i) and (v) combine to give the finiteness theorem for Hilbert polynomials. We will later show (lemma 1.12 below) how the bound in (ii) yields a bound for the postulation number as well.

First consider the case $d = 1$. The bound (i) is vacuous. For (ii), one proves directly that $m \leq e - 1$, with $e = e_0(A)$. This is just the statement (applied to the scheme $E \subset \mathbb{P}^N_k$) that the ideal sheaf of a set of e points (or more generally, of a 0-dimensional subscheme of length e) in projective space over a field is e-regular. By Proposition 1.8(ii)(b), this is equivalent to the assertion that the natural restriction map

$$f : H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(e-1)) \to H^0(E, \mathcal{O}_E(e-1))$$

is surjective, which can be easily proved. Since $\dim E = 0$, the bound (iii) is also vacuous. For (iv), note that $\dim H^0(E, \mathcal{O}_E(r)) = e = e_0(A)$ for all $r \in \mathbb{Z}$, again since E is 0-dimensional. For (v), the above surjectivity of f, together with Nakayama's lemma, implies that there is a surjection (and hence an isomorphism) of A-modules $\mathfrak{m}^{e-1} \to H^0(X, \mathcal{O}_X(e-1)) = \mathfrak{m}^{e-1}H^0(X, \mathcal{O}_X)$. This bounds the length of the quotient A-module $H^0(X, \mathcal{O}_X)/A$, which bounds $|e_1(A)|$ by Theorem 1.5(b) (Northcott's formula). Note that, for the case $d = 1$, we have simultaneously bounded the postulation number by $e - 1$.

Henceforth we assume $d = \dim A > 1$, and establish bounds (i)-(v) by induction on d.

Choose a "general" element $x \in \mathfrak{m} - \mathfrak{m}^2$. Then x is a non zero-divisor on A, and its image $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$ is a homogeneous superficial element in the graded ring $\oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ (*i.e.*, $\overline{x}$ is a non zero-divisor in high enough degrees). Let $\overline{A} = A/xA$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{A}$, so that $\overline{A}$ is a Cohen-Macaulay local ring of dimension $d-1$. Let $Y = \operatorname{Proj} \overline{R}$ with $\overline{R} = \oplus \overline{\mathfrak{m}}^n$ the corresponding blow up, and with exceptional divisor $F \subset Y$. Then $F \subset E$ is a hyperplane intersection of $E \subset \mathbb{P}^N_k$ defined by the homogenous linear equation $\overline{x} = 0$.

One has the following easy lemma, proved via Artin-Rees, using that x is a non zero-divisor on A as well as a superficial element.

Lemma 1.10. *The Hilbert polynomial of $\overline{A}$ is $P_{\overline{A}}(t) = P_A(t) - P_A(t-1)$. In particular, $e_j(\overline{A}) = e_j(A)$ for $j < d$.*

By induction on $d = \dim A$, we may thus assume given bounds (in terms of $e = e_0(A)$ and d) on the following quantities: $|e_1(A)|, \dots,$ $|e_{d-1}(A)|$, $m' = m(\mathcal{I}_F)$ (the smallest integer $m' \geq 1$ such that $\mathcal{I}_F$ is m'-regular on $H = \mathbb{P}^{N-1}_k$), $\dim_k H^j(\mathcal{O}_F(r))$ for $j > 0$, $r \geq 0$ (bound independent of r), and $\dim_k H^0(\mathcal{O}_F(r))$ for $r \geq 0$ (bound depending on r). In particular, we are already given the desired bound (i) for the ring A.

We next observe that the Hilbert polynomial of $\mathcal{O}_E$ satisfies

$$P_E(n) = \chi(\mathcal{O}_E(n)) = P_A(n+1) - P_A(n) = \sum_{j=0}^{d-1} e_j(A)\binom{n+d-j-1}{d-j-1},$$

since we have the corresponding formula for the underlying Hilbert functions, at least for all sufficiently large n (the homogeneous coordinate ring of $E \subset \mathbb{P}^N_k$ coincides, in large enough degrees, with the graded ring $\oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$). Now Mumford's lemma (lemma 1.9), combined with the known bound (i) above, and the bound (ii) for the ring $\overline{A}$, implies that $\mathcal{I}_E$ is m-regular for some quantity $m \geq 1$ bounded in terms of $e_0(A)$ and d. This gives the desired bound (ii) for the ring A.

The exact sequences of sheaves for all $r \geq 0$

$$0 \to \mathcal{O}_E(r) \to \mathcal{O}_E(r+1) \to \mathcal{O}_F(r+1) \to 0 \tag{1.2}$$

yield exact sequences of finte dimensional k-vector spaces

$$\cdots \to H^{j-1}(F, \mathcal{O}_F(r+1)) \to H^j(E, \mathcal{O}_E(r)) \to H^j(E, \mathcal{O}_E(r+1)) \to \cdots$$

which imply, for $j \geq 1$, $r \geq 0$,

$$\begin{aligned}&\dim_k H^j(E, \mathcal{O}_E(r)) \\ &\leq \dim_k H^{j-1}(F, \mathcal{O}_F(1)) + \cdots + \dim_k H^j(F, \mathcal{O}_F(m'-j-1))\end{aligned}$$

since $\mathcal{O}_E$ is $m'-1$-regular, by lemma 1.9(iii) (as $\mathcal{I}_F$ is m'-regular), and since $H^j(E, \mathcal{O}_E(r)) = 0$ for all $j > 0$ for sufficiently large r, by Serre vanishing (see [13], III Thm. 5.2). This gives the desired bound (iii) for the ring A.

For the bound (iv), first note that by the definition of Euler characteristic of a sheaf, we have

$$\dim_k H^0(E, \mathcal{O}_E) \leq \chi(\mathcal{O}_E) + \sum_{0 \leq j \leq (d-2)/2} \dim_k H^{2j+1}(E, \mathcal{O}_E).$$

But $\chi(\mathcal{O}_E) = P_A(1) - P_A(0) = e_0(A) + \cdots + e_{d-1}(A)$, so the bound (iv) for $r = 0$ (*i.e.*, for $\dim_k H^0(E, \mathcal{O}_E)$) is deduced from the bound (iii) already obtained. For $r \geq 1$, we use the exact sequences (obtained from the sheaf exact sequence (1.2))

$$0 \to H^0(E, \mathcal{O}_E(r-1)) \to H^0(E, \mathcal{O}_E(r)) \to H^0(F, \mathcal{O}_F(r)) \to \cdots$$

which implies the inequality

$$\dim_k H^0(E, \mathcal{O}_E(r)) \leq \dim_k H^0(E, \mathcal{O}_E) + \sum_{j=1}^{r} \dim_k H^0(F, \mathcal{O}_F(j)).$$

The right side is bounded, by the bound (iv) for the ring $\overline{A}$, and the bound for $r = 0$ already obtained.

Finally, we are left with the bound (v), for the absolute value of the constant term $e_d(A)$ of the Hilbert polynomial. This is really the heart of the matter, in a way. The following lemma gives the desired bound.

Lemma 1.11. *We have inequalities*

$$-\sum_{j=0}^{m'-3} \chi(\mathcal{O}_E(j)) \leq e_d(A) \leq \sum_{j=0}^{m'-3} \left[\dim_k H^0(E, \mathcal{O}_E(j)) - \chi(\mathcal{O}_E(j))\right].$$

Proof. From Theorem 1.5(a), we have the formula (1.1)

$$e_d(A) = \sum_{i=1}^{d-1}(-1)^{i-1}\ell(H^i(X, \mathcal{O}_X)).$$

From the Formal Function Theorem (see [13], III, §11), we have isomorphisms for $i > 0$, and any $r \in \mathbb{Z}$,

$$H^i(X, \mathcal{O}_X(r)) \cong \varprojlim_n H^i(X^{(n)}, \mathcal{O}_{X^{(n)}}(r)), \tag{1.3}$$

where $X^{(n)} \subset X$ is the closed subscheme with ideal sheaf $\mathfrak{m}^n\mathcal{O}_X = \mathcal{O}_X(n)$ (in particular, $X^{(1)} = E$). We have exact sheaf sequences

$$0 \to \mathcal{O}_E(n) \to \mathcal{O}_{X^{(n+1)}} \to \mathcal{O}_{X^{(n)}} \to 0$$

which, since $\mathcal{O}_E$ is $m' - 1$-regular (by lemma 1.9), imply that

$$H^i(\mathcal{O}_{X^{(n+1)}}) \cong H^i(\mathcal{O}_{X^{(n)}}) \ \forall\, n \geq m' - i - 1.$$

Hence, taking $r = 0$ in the formula (1.3), we deduce from the formula (1.1) that

$$\begin{aligned} e_d &= \textstyle\sum_{i=1}^{d-1}(-1)^{i-1}\ell\left(H^i(X^{(n)}, \mathcal{O}_{X^{(n)}})\right) \ \ \forall\, n \geq m' - 2 \\ &= \ell(H^0(X^{(n)}, \mathcal{O}_{X^{(n)}})) - \chi(\mathcal{O}_{X^{(n)}}). \end{aligned}$$

But $\mathcal{O}_{X^{(n)}}$ is filtered by the sheaves $\mathcal{O}_E(j)$, $0 \leq j \leq n-1$, and so

$$\chi(\mathcal{O}_{X^{(n)}}) = \sum_{j=0}^{n-1}\chi(\mathcal{O}_E(j)),$$

and

$$0 \leq \ell(H^0(X^{(n)}, \mathcal{O}_{X^{(n)}})) \leq \sum_{j=0}^{n-1}\dim_k H^0(E, \mathcal{O}_E(j)).$$

This implies the inequalities stated in the lemma. Q.E.D.

We now discuss the bound on the postulation number.

Lemma 1.12. *With the above notation, if $m \geq 1$ is an integer such that $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ is m-regular and $\mathcal{O}_E$ is $(m-1)$-regular on $\mathbb{P}^N_k$, then $m - 1$ is a postulation number for the Hilbert function of A.*

Proof. Since $\mathcal{O}_E$ is $(m-1)$-regular on $\mathbb{P}^N_k$, the Formal Function Theorem (formula (1.3)) implies that $\mathcal{O}_X$ is $(m-1)$-regular on $\mathbb{P}^N_A$. Hence the graded R-module $\oplus_{n\geq 0}H^0(X,\mathcal{O}_X(n))$ is generated by its homogeneous elements of degrees $\leq m-1$. Thus $H^0(X,\mathcal{O}_X(n+1)) = \mathfrak{m}H^0(X,\mathcal{O}_X(n))$ if $n \geq m-1$. From the exact sheaf sequence

$$0 \to \mathcal{O}_X(n+1) \to \mathcal{O}_X(n) \to \mathcal{O}_E(n) \to 0,$$

we get an exact sequence (defining the map γ_n)

$$0 \to H^0(X,\mathcal{O}_X(n+1)) \to H^0(X,\mathcal{O}_X(n)) \xrightarrow{\gamma_n} H^0(E,\mathcal{O}_E(n))$$

so that

$$\text{(1.4)} \qquad \operatorname{image}\gamma_n = H^0(X,\mathcal{O}_X(n)) \otimes A/\mathfrak{m} \;\; \forall\, n \geq m-1.$$

Now in the diagram (defining maps $\alpha_n, \beta_n, \rho_n$)

$$\begin{array}{ccc} H^0(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(n)) & \xrightarrow{\alpha_n} & H^0(X,\mathcal{O}_X(n)) \\ \rho_n \downarrow & & \downarrow \gamma_n \\ H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(n)) & \xrightarrow{\beta_n} & H^0(E,\mathcal{O}_E(n)) \end{array}$$

we have that ρ_n is surjective for all $n \geq 0$, while β_n is surjective for $n \geq m-1$, since $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ is m-regular. Hence γ_n is surjective, and so by the formula (1.4) and Nakayama's lemma, α_n is surjective for $n \geq m-1$. But the image of α_n is $\mathfrak{m}^n \subset H^0(X,\mathcal{O}_X(n))$. Hence $m-1$ is a postulation number for the local ring A. Q.E.D.

1.2. The Regularity Number and applications

Next, we discuss the regularity number of a module M over $(A,\mathfrak{m})$ with respect to an ideal I such that M/IM has finite length. Let $d = \dim M$, and let $e_j(I,M)$, $0 \leq j \leq d$ denote the Hilbert coefficients of M with respect to I. Inductively define

$$\begin{aligned} m_1 = &\ e_0(I,M) \\ m_i = &\ m_{i-1} + e_0(I,m)\binom{m_{i-1}+i-2}{i-1} + e_1(I,M)\binom{m_{i-1}+i-3}{i-2} + \cdots \\ &+ e_{i-1}(I,M), \qquad \text{for } 2 \leq i \leq d. \end{aligned}$$

Note that m_d is a polynomial with rational coefficients in $e_j(I,M)$, $0 \leq j \leq d-1$. In particular, it is independent of $e_d(I,M)$, the constant coefficient of the Hilbert polynomial of M (this is a crucial point in the inductive proof of Theorem 1.1). The integer $\mathbf{m}(I,M) = m_d(I,M)$ is called the *regularity number* of M with respect to I.

This terminology is introduced in [44], because of its relation to Castelnuovo-Mumford regularity of a certain sheaf; we comment more on this later. Its relevance to us is due to the following result (see [44], Theorem 2 and Corollary 2).

Theorem 1.13. (a) *Let M be a finite module over a Noetherian local ring $(A, \mathfrak{m})$, with* depth $(M) > 0$, *and let $I \subset \mathfrak{m}$ be an ideal in A such that M/IM has finite length. Then $\mathbf{m}(I, M) - 1$ is a postulation number for M with repect to I, i.e., we have an equality between values of the Hilbert function and Hilbert polynomial*

$$H_I(M)(n) = P_I(M)(n)$$

for all $n \geq \mathbf{m}(I, M) - 1$.

(b) *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and depth > 0. Let $I \subset \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal, and $J \subset I$ a reduction ideal of I. Then the reduction number $r_J(I)$ is at most the regularity number $\mathbf{m}(I, A)$, i.e., we have the reduction formula*

$$J^n I^{\mathbf{m}} = I^{n+\mathbf{m}} \ \forall\, n \geq 0,$$

where $\mathbf{m} = \mathbf{m}(I, A)$.

Remark 1.14. We comment further on the regularity number $\mathbf{m}(I, M)$. Let $X = \operatorname{Proj} R(I)$ be the blow-up of the ideal I. Then the graded "Rees module" $M(I) = \oplus_{n \geq 0} I^n M$ yields a coherent sheaf $\mathcal{F}$ on X, and hence (by choosing a set of $N+1$ generators of I, and thus an embedding $X \hookrightarrow \mathbb{P}^N_A$) also on $\mathbb{P}^N_A$. This sheaf is generated by its global sections, since $M(I)$ is generated by its homogeneous elements of degree 0; hence we can find an exact sequence of coherent $\mathcal{O}_{\mathbb{P}^N_A}$-modules

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r}_{\mathbb{P}^N_A} \to \mathcal{F} \to 0.$$

It is shown in [44] that $\mathcal{K}$ is $\mathbf{m}(I, M)$ regular on $\mathbb{P}^N_A$; this in turn controls the regularity of various other related sheaves. Note that this first syzygy sheaf $\mathcal{K}$ is the analogue for A-modules M of the ideal sheaf of the projective scheme X.

Remark 1.15. The fact that there is a connection between regularity and the reduction number was noted earlier by Trung [46]. That this implies a connection also with Hilbert coefficients seems to be the new observation in [44].

The last related result we discuss in this Section is the following "explicit" form of the Artin-Rees lemma, due to Trivedi, taken from

[44]. If $N \subset M$ are finitely generated modules over a Noetherian ring R, and I any ideal of R, an *Artin-Rees number* of (M, N) with respect to I is a positive integer n_0 such that we have the Artin-Rees formula

$$I^{n+n_0}M \cap N = I^n(I^{n_0}M \cap N) \ \forall\, n \geq 0$$

(the choice of the terminology "Artin-Rees number", introduced in [44], is self-explanatory).

Proposition 1.16. *Let $(A, \mathfrak{m})$ be Noetherian local, $N \subset M$ finite A-modules such that M/N has positive depth and dimension d. Then the regularity number $\mathbf{m}(\mathfrak{m}, M/N)$ is also an Artin-Rees number for (M, N) with respect to the maximal ideal $\mathfrak{m}$.*

Remark 1.17. We do not discuss the proof here; however, one of the points made by the latter part of [44] is that there is a relationship between the Artin-Rees number and Castelnuovo-Mumford regularity. The "standard" proof of the Artin-Rees lemma reduces to the statement that n_0 is an Artin-Rees number if a certain graded module is generated by its elements of degrees $\leq n_0$. On the other hand, Proposition 1.8 yields a statement that a certain graded module is generated by its homogeneous elements of bounded degree. In a sense, this is the explanation for the above mentioned relationship.

Using Proposition 1.16, and the technique of normally flat stratifications, a new proof (see [44], Theorem 4) is obtained of the following Uniform Artin-Rees Theorem of Duncan and O'Carroll (see [6] for the original proof).

Theorem 1.18 (Uniform Artin-Rees). *Let A be an excellent (or even $J2$) ring, and $N \subset M$ finitely generated A-modules. Then there is an $n_0 \geq 0$ such that for* any *maximal ideal $\mathfrak{m} \subset A$, we have*

$$\mathfrak{m}^{n+n_0}M \cap N = \mathfrak{m}^n(\mathfrak{m}^{n_0}M \cap N) \ \forall\, n \geq 0.$$

The idea of the new proof is the following: results from the theory of normal flatness imply that, since A is a J2 ring, there are only finitely many numerical functions occuring as Hilbert functions of localizations $(M/N)_\wp$ with respect to prime ideals $\wp$ of A. Hence there is a uniform bound on all the Hilbert coefficients of these localizations $(M/N)_\wp$, and hence also of the corresponding regularity numbers. Now Proposition 1.16 implies that there is a uniform bound on Artin-Rees numbers for the localizations $(M_\wp, N_\wp)$ with respect to the corresponding maximal ideals $\wp A_\wp$. For maximal ideals $\mathfrak{m}$, an Artin-Rees number for $(M_\mathfrak{m}, N_\mathfrak{m})$ with respect to $\mathfrak{m}A_\mathfrak{m}$ is automatically also an Artin-Rees number for (M, N) with respect to $\mathfrak{m}$.

If M/N is Cohen-Macaulay, then combining the Finiteness Theorem 1.1 with Proposition 1.16 yields the following result.

Corollary 1.19. *Let A be Noetherian, $N \subset M$ finitely generated modules such that $(M/N)_{\mathfrak{m}}$ is Cohen-Macaulay for each maximal ideal $\mathfrak{m} \subset A$. Then*

$$\sup_{\mathfrak{m}}\{3^{d!-1}e(\mathfrak{m}, (M/N)_{\mathfrak{m}})^{3(d-1)!-1}\}$$

is an Artin-Rees number for (M, N) with respect to all maximal ideals $\mathfrak{m}$.

This has the following application to graded rings (see [44], Corollary 5):

Corollary 1.20. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and $I \subset A$ an ideal generated by an A-sequence $f_1, \ldots, f_r$ of length r. Let $\mathbf{e} = e_0(\mathfrak{m}, A/I)$ be the $\mathfrak{m}$-adic multiplicity of A/I, and set*

$$N = \max\{3^{(d-r)!}\mathbf{e}^{3(d-r-1)!-1}, (d-r)!\mathbf{e} + 2\}.$$

Let J be the ideal generated by $f_1 + g_1, \ldots, f_r + g_r$, where $g_1, \ldots, g_r \in \mathfrak{m}^N$ are arbitrarily chosen elements. Then the $\mathfrak{m}$-adic associated graded rings (and hence the Hilbert functions) of A/I and A/J are canonically isomorphic.

§2. Chern Classes and Zero Cycles

In this Section, we discuss how the theory of Chern classes with values in the Chow ring, combined with results on 0-cycles, yields several counterexamples which are of interest in algebra. For simplicity, we will work only with varieties over an algebraically closed field k, and eventually restrict to the case $k = \mathbb{C}$, the complex number field. However, unlike the convention in [13], we will use the term "variety" even when referring to unions of irreducible varieties, and explicitly mention irreducibility when needed.

2.1. The Chow ring and Chern classes

First, we recall the definition of the graded *Chow ring* $CH^*(X) = \bigoplus_{p\geq 0} CH^p(X)$ of a non-singular variety X over an algebraically closed field k (see Fulton's book [8] for more details; see also Bloch [1]). The graded components $CH^p(X)$ generalize the more familiar notion of the *divisor class group*, which is just the group $CH^1(X)$.

If $Z \subset X$ is irreducible, let $\mathcal{O}_{Z,X}$ be the local ring of Z on X (*i.e.*, the local ring of the generic point of Z, in the terminology of Hartshorne's book [13]). The codimension of Z in X, denoted $\operatorname{codim}_X Z$, is the dimension of the local ring $\mathcal{O}_{Z,X}$. Now let

$Z^p(X) =$ Free abelian group on irreducible subvarieties of X of codimension p
$\qquad\quad =$ Group of algebraic cycles on X of codimension p.

For an irreducible subvariety $Z \subset X$, let $[Z]$ denote its class in $Z^p(X)$ (where $p = \operatorname{codim}_X Z$).

Let $Y \subset X$ be irreducible of codimension $p-1$, and let $k(Y)^*$ denote the multiplicative group of non-zero rational functions on Y ($k(Y)$, which is the field of rational functions on Y, is the residue field of $\mathcal{O}_{Y,X}$). For each irreducible divisor $Z \subset Y$, we have a homomorphism $\operatorname{ord}_Z : k(Y)^* \to \mathbb{Z}$, given by

$$\operatorname{ord}_Z(f) = \ell(\mathcal{O}_{Z,Y}/a\mathcal{O}_{Z,Y}) - \ell(\mathcal{O}_{Z,Y}/b\mathcal{O}_{Z,Y}),$$

for any expression of f as a ratio $f = a/b$ with $a, b \in \mathcal{O}_{Z,Y} \setminus \{0\}$. Here $\ell(M)$ denotes the length of an Artinian module M.

For $f \in k(Y)^*$, let $(f)_Y$ denote the divisor of f on Y, defined by

$$(f)_Y = \sum_{Z \subset Y} \operatorname{ord}_Z(f) \cdot [Z],$$

where Z runs over all irreducible divisors in Y; the sum has only finitely many non-zero terms, and is hence well-defined. Clearly we may also view $(f)_Y$ as an element of $Z^p(X)$.

Let $R^p(X) \subset Z^p(X)$ be the subgroup generated by cycles $(f)_Y$ as (Y, f) ranges over all irreducible subvarieties Y of X of codimension $p-1$, and all $f \in k(Y)^*$. We refer to elements of $R^p(X)$ as cycles *rationally equivalent to 0* on X. The p-th Chow group of X is defined to be

$$CH^p(X) = \frac{Z^p(X)}{R^p(X)}$$

= group of rational equivalence classes of codimension p-cycles on X.

We will abuse notation and also use $[Z]$ to denote the class of an irreducible subvariety Z in $CH^p(X)$.

The graded abelian group

$$CH^*(X) = \bigoplus_{0 \le p \le \dim X} CH^p(X)$$

can be given the structure of a commutative (graded) ring via the *intersection product.* This product is characterized by the following property — if $Y \subset X$, $Z \subset X$ are irreducible of codimensions p, q respectively, and $Y \cap Z = \cup_i W_i$, where each $W_i \subset X$ is irreducible of codimension $p+q$ (we then say Y and Z *intersect properly* in X), then the intersection product of the classes $[Y]$ and $[Z]$ is

$$[Y] \cdot [Z] = \sum_i I(Y, Z; W_i)[W_i]$$

where $I(Y, Z; W_i)$ is the *intersection multiplicity* of Y and Z along W_i, defined by Serre's formula

$$I(Y, Z; W_i) = \sum_{j \geq 0} (-1)^j \ell \left(\operatorname{Tor}_j^{\mathcal{O}_{W_j,X}} (\mathcal{O}_{W_j,Y}, \mathcal{O}_{W_j,Z}) \right).$$

One of the important results proved in the book [8] is that the above procedure does give rise to a well-defined ring structure on $CH^*(X)$.

The Chow ring is an algebraic analogue for the even cohomology ring

$$\bigoplus_{i=0}^{n} H^{2i}(X, \mathbb{Z})$$

defined in algebraic topology. To illustrate this, we note the following 'cohomology-like' properties, proved in Fulton's book [8]. Here, we follow the convention of [13], and use the term "vector bundle on X" to mean "(coherent) locally free sheaf of $\mathcal{O}_X$-modules", and use the term "geometric vector bundle on X", as in [13] II Ex. 5.18, to mean a Zariski locally trivial algebraic fiber bundle $V \to X$ whose fibres are affine spaces, with linear transition functions. With this convention, we can also identify vector bundles on an affine variety $X = \operatorname{Spec} A$ with finitely generated projective A-modules; as in [13], we use the notation $\widetilde{M}$ to denote the coherent sheaf corresponding to a finitely generated A-module M.

Theorem 2.1 (Properties of the Chow ring and Chern classes).

(1) $X \mapsto \bigoplus_p CH^p(X)$ *is a contravariant functor from the category of smooth varieties over* k *to graded rings. If* $X = \coprod_i X_i$, *where* X_i *are the irreducible* (= *connected*) *components, then* $CH^*(X) = \prod_i CH^*(X_i)$. *If* X *is irreducible, then* $CH^0(X) = \mathbb{Z}$ *generated by the class* $[X]$.

(2) *If* X *is irreducible and projective* (*or more generally, proper*) *over* k *and* $d = \dim X$, *there is a well defined* degree homomorphism $\deg : CH^d(X) \to \mathbb{Z}$ *given by* $\deg(\sum_i n_i[x_i]) = \sum_i n_i$. *This allows one to define* intersection numbers *of cycles of complementary dimension, in a purely algebraic way, which agree with those defined via topology when* $k = \mathbb{C}$ (*see* (7) *below*).

(3) *If* $f : X \to Y$ *is a proper morphism of smooth varieties, there are "Gysin" (or "push-forward") maps* $f_* : CH^p(X) \to CH^{p+r}(Y)$ *for all* p, *where* $r = \dim Y - \dim X$; *here if* $p+r < 0$, *we define* f_* *to be* 0; *the induced map* $CH^*(X) \to CH^*(Y)$ *is* $CH^*(Y)$*-linear* (projection formula), *where* $CH^*(X)$ *is regarded as a* $CH^*(Y)$*-module via the* (*contravariant*) *ring homomorphism* $f^* : CH^*(Y) \to CH^*(X)$. *If* $f : X \hookrightarrow Y$ *is the inclusion of a closed subvariety, then* f_* *is induced by the natural inclusions* $Z^p(X) \hookrightarrow Z^{p+r}(Y)$.

(4) $f^* : CH^*(X) \xrightarrow{\cong} CH^*(V)$ *for any geometric vector bundle* $f : V \to X$ (homotopy invariance). *In particular,* $CH^*(X \times \mathbb{A}^n) = CH^*(X)$, *and* $CH^*(\mathbb{A}^n) = \mathbb{Z}$.

(5) *If* V *is a vector bundle* (*i.e., locally free sheaf*) *of rank* r *on* X, *then there are* Chern classes $c_p(V) \in CH^p(X)$, *such that*
  (a) $c_0(V) = 1$,
  (b) $c_p(V) = 0$ *for* $p > r$, *and*
  (c) *for any exact sequence of vector bundles*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

  we have $c(V_2) = c(V_1)c(V_3)$, *where* $c(V_i) = \sum_p c_p(V_i)$ *are the corresponding* total Chern classes,
  (d) $c_p(V^\vee) = (-1)^p c_p(V)$, *where* $V^\vee$ *is the dual vector bundle.*

Moreover, we also have the following properties.

(6) *If* $f : \mathbb{P}(V) = \mathbf{Proj}\, S(V) \to X$ *is the projective bundle associated to a vector bundle of rank* r (*where* $S(V)$ *is the symmetric algebra of the sheaf* V *over* $\mathcal{O}_X$), *then* $CH^*(\mathbb{P}(V))$ *is a* $CH^*(X)$*-algebra generated by* $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, *the first Chern class of the tautological line bundle, which is subject to the relation*

$$\xi^r - c_1(V)\xi^{r-1} + \cdots + (-1)^r c_n(V) = 0;$$

in particular, $CH^*(\mathbb{P}(V))$ is a free $CH^*(X)$-module with basis $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

(7) *If $k = \mathbb{C}$, there are cycle class homomorphisms $CH^p(X) \to H^{2p}(X, \mathbb{Z})$ such that the intersection product corresponds to the cup product in cohomology, and for a vector bundle E, the cycle class of $c_p(E)$ is the topological p-th Chern class of E.*

(8) *The first Chern class determines an isomorphism $c_1 : \mathrm{Pic} X \to CH^1(X)$ from the Picard group of line bundles on X to the first Chow group (i.e., the divisor class group) of X, such that $c_1(\mathcal{O}_X(D)) = [D] \in CH^1(X)$ for any divisor D on X. For an arbitrary vector bundle V, of rank n, we have $c_1(V) = c_1(\det V)$, where $\det V = \bigwedge^n V$.*

(9) *If $f : X \to Y$ is a morphism between non-singular varieties, V a vector bundle on Y, then the Chern classes of the pull-back vector bundle f^*V on X are given by $c(f^*V) = f^*c(V)$, where on the right, f^* is the ring homomorphism $CH^*(Y) \to CH^*(X)$* (functoriality of Chern classes). *In particular, taking $Y =$ point, we see that $c(\mathcal{O}_X) = 1 \in CH^*(X)$.*

(10) *If $i : Y \hookrightarrow X$ is the inclusion of an irreducible smooth subvariety of codimension r in a smooth variety, with normal bundle $\mathcal{N} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$ (where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf of Y in X), then $\mathcal{N}$ is a vector bundle on Y of rank r with top Chern class*

$$c_r(\mathcal{N}) = i^* \circ i_*[Y],$$

where $[Y] \in CH^0(Y) = \mathbb{Z}$ is the generator (self-intersection formula).

Remark 2.2. If $X = \operatorname{Spec} A$ is affine, we will also sometimes write $CH^*(A)$ in place of $CH^*(X)$; similarly, by the Chern classes $c_i(P)$ of a finitely generated projective A-module P, we mean $c_i(\widetilde{P})$ where $\widetilde{P}$ is the associated vector bundle on X.

We remark that the total Chern class of a vector bundle on a smooth variety X is a *unit* in the Chow ring $CH^*(X)$, since it is of the form $1 +$ (nilpotent element). Thus the assignment $V \mapsto c(V)$ gives a homomorphism of groups from the Grothendieck group $K_0(X)$ of vector bundles (locally free sheaves) on X to the multiplicative group of those units in the graded ring $CH^*(X)$, which are expressible as $1 +$ (higher degree terms).

On a non-singular variety X, every coherent sheaf has a resolution by locally free sheaves (vector bundles) of finite rank, and the Grothendieck group $K_0(X)$ of vector bundles coincides with the Grothendieck group of coherent sheaves. There is a finite decreasing filtration $\{F^pK_0(X)\}_{p\geq 0}$

on $K_0(X)$, where $F^pK_0(X)$ is the subgroup generated by classes of sheaves supported in codimension $\geq p$. Further, $F^pK_0(X)/F^{p+1}K_0(X)$ is generated, as an abelian group, by the classes $\mathcal{O}_Z$ for irreducible subvarieties $Z \subset X$ of codimension p – for example, if $X = \operatorname{Spec} A$ is affine, we can see this using the fact that any finitely generated A-module M has a finite filtration whose quotients are of the form $A/\wp$ for prime ideals $\wp$, such that the minimal primes in $\operatorname{supp}(M)$ all occur, and their multiplicities in the filtration are independent of the choice of filtration. Thus, we have a natural surjection $Z^p(X) \to F^pK_0(X)/F^{p+1}K_0(X)$.

Now we can state the following result, sometimes called the *Riemann-Roch theorem without denominators* (see the book [8] for a proof).

Theorem 2.3. *Let X be a non-singular variety.*

(a) *If $x \in F^pK_0(X)$, then $c_i(x) = 0$ for $i < p$, and $c_p : F^pK_0(X) \to CH^p(X)$ is a group homomorphism.*
Let $\overline{c_p} : F^pK_0(X)/F^{p+1}K_0(X) \to CH^p(X)$ be the induced homomorphism.

(b) *The natural surjection $Z^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$ factors through rational equivalence, yielding a map $\psi_p : CH^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$.*

(c) *The compositions $\overline{c_p} \circ \psi_p$ and $\psi_p \circ \overline{c_p}$ both equal multiplication by the integer $(-1)^{p-1}(p-1)!$. In particular, both $\overline{c_p}$ and ψ_p are isomorphisms $\otimes\mathbb{Q}$.*

In particular, if $Z \subset X$ is an irreducible subvariety of codimension p, then $c_i([\mathcal{O}_Z]) = 0$ for $i < p$, and $c_p([\mathcal{O}_Z]) = (-1)^{p-1}(p-1)![Z] \in CH^p(X)$.

Remark 2.4. If $X = \operatorname{Spec} A$ is affine, any element $\alpha \in K_0(X)$ can be expressed as a difference $\alpha = [P] - [A^{\oplus m}]$ for some finitely generated projective A-module P and some positive integer m. Hence the total Chern class $c(\alpha)$ coincides with $c(P)$. The above theorem now implies that for any element $a \in CH^p(X)$, there is a finitely generated projective A module P with $c_p(P) = (p-1)!a$. By the Bass stability theorem, which implies that any projective A-module of rank $> d = \dim A$ has a free direct summand of positive rank, we can find a projective A-module P with $\operatorname{rank} P \leq d$ and $c_p(P) = (p-1)!a$.

Incidentally, this statement cannot be improved, in general: for any $p > 2$, there are examples of affine non-singular varieties X and elements $a \in CH^p(X)$ such that $ma \in \operatorname{image} c_p$ for some integer $m \iff (p-1)!|m$. For examples of Mohan Kumar and Nori, see [42], §17.

2.2. An example of a graded ring

We now discuss our first application of these constructions to commutative algebra, due to N. Mohan Kumar (unpublished). Let $k = \overline{k}$. We give an example of a 3-dimensional graded integral domain $A = \bigoplus_{n\geq 0} A_n$, with the following properties:

1. A is generated by A_1 as an A_0-algebra, where A_0 is a regular affine k-algebra of dimension 1,
2. the "irrelevant graded prime ideal" $P = \bigoplus_{n>0} A_n$ is the radical of an ideal generated by 2 elements,
3. P cannot be expressed as the radical of an ideal generated by 2 *homogeneous* elements.

For the example, take A_0 to be affine coordinate ring of a non-singular curve $C \subset \mathbb{A}^3_k$ such that the canonical module $\omega_{A_0} = \Omega_{A_0/k}$ is a non-torsion element of the divisor class group of A_0 (this implies k is not the algebraic closure of a finite field). In fact, if we choose A_0 to be a non-singular affine k-algebra of dimension 1 such that ω_{A_0} is non-torsion in the class group, then $C = \operatorname{Spec} A_0$ can be realized as a curve embedded in $\mathbb{A}^3_k$, by more or less standard arguments (see [13], IV, or [36], for example).

Let $R = k[x, y, z]$ denote the polynomial algebra, and let $\varphi : R \longrightarrow A_0$ be the surjection corresponding to $C \hookrightarrow \mathbb{A}^3_k$. Let $I = \ker\varphi$ be the ideal of C. Then I/I^2 is a projective A_0-module of rank 2; we let

$$A = S(I/I^2) = \bigoplus_{n\geq 0} S^n(I/I^2)$$

be its symmetric algebra over A_0. We claim this graded ring A has the properties stated above.

Consider the exact sequence of projective A_0-modules

$$0 \to I/I^2 \xrightarrow{\psi} \Omega_{R/k} \otimes A_0 \xrightarrow{\overline{\varphi}} \omega_{A_0} \to 0 \tag{2.1}$$

with $\overline{\varphi}$ induced by φ, and ψ by the derivation $d : R \to \Omega_{R/k}$. Let $h : \Omega_{R/k} \otimes A_0 \to I/I^2$ be a splitting of ψ. Use h to define a homomorphism of k-algebras

$$\Phi : R \to A,$$

by setting

$$\Phi(t) = \phi(t) + h(dt) \in A_0 \oplus A_1 = A_0 \oplus I/I^2$$

for $t = x, y, z$; this uniquely specifies a k-algebra homomorphism Φ defined on the polynomial algebra R.

Clearly $\Phi(I) \subset P = \bigoplus_{n>0} A_n$, the irrelevant graded ideal, and one verifies that Φ induces isomorphisms $R/I \to A/P$ and $I/I^2 \to P/P^2$, and in fact an isomorphism between the I-adic completion of R and the P-adic completion of A.

Since $C \subset \mathbb{A}^3_k$ is a non-singular curve, it is a set-theoretic complete intersection, from a result of Ferrand and Szpiro (see [43], for example). If $a, b \in I$ with $\sqrt{(a,b)} = I$, then clearly we have $\sqrt{(\Phi(a), \Phi(b))} = P \cap Q$, for some (radical) ideal Q with $P + Q = A$. We can correspondingly write $(\Phi(a), \Phi(b)) = J \cap J'$ with $\sqrt{J} = P$, $\sqrt{J'} = Q$. Then $J/J^2 \cong (A/J)^{\oplus 2}$. This implies (by an old argument of Serre) that $\operatorname{Ext}^1_A(J, A) \cong \operatorname{Ext}^2_A(A/J, A) \cong A/J$ is free of rank 1, and any generator determines an extension

$$0 \to A \to V \to J \to 0$$

where V is a projective A-module of rank 2, and such that the induced surjection $V \otimes A/J \to J/J^2 \cong (A/J)^{\oplus 2}$ is an isomorphism.

We claim the projective module V is necessarily of the form $V = V_0 \otimes_{A_0} A$; this implies $V_0 = V \otimes_A A/P \cong J/PJ \cong (A/P)^{\oplus 2}$ is free, so that V is a free A-module, and J is generated by 2 elements. To prove the claim, note that I/I^2 is a direct summand of a free $A/I = A_0$-module of finite rank; hence there is an affine A-algebra $A' \cong A_0[x_1, \ldots, x_n]$, which is a polynomial algebra over A_0, such that A is an algebra retract of A'. Now it suffices to observe that any finitely generated projective A'-module is of the form $M \otimes_{A_0} A'$, for some projective A_0-module M; this is the main result of [19] (see also [20]).

On the other hand, we claim that it is impossible to find two *homogeneous* elements $x, y \in P$ with $\sqrt{(x,y)} = P$. Indeed, let $X = \operatorname{Proj} A$, and $\pi : X \to C = \operatorname{Spec} A_0$ be the natural morphism. Then $X = \mathbb{P}(V)$ is the $\mathbb{P}^1$-bundle over C associated to the locally free sheaf $V = \widetilde{I/I^2}$ (the sheaf determined by the projective A_0-module I/I^2). Let $\xi = c_1(\mathcal{O}_X(1)) \in CH^1(X)$ be the 1st Chern class of the tautological line bundle $\mathcal{O}_X(1)$. Then by Theorem 2.1 and (2.1) above, $CH^*(X)$ is a free $CH^*(C)$-module with basis $1, \xi$, and ξ satisfies the monic relation

$$\xi^2 - c_1(V)\xi + c_2(V) = 0.$$

Since $\dim C = 1$, $CH^i(C) = 0$ for $i > 1$, and so this relation reduces to

$$\xi^2 = c_1(V)\xi.$$

From the exact sequence (2.1), we have a relation in $CH^*(C)$

$$1 = c(\mathcal{O}_C)^3 = c(\mathcal{O}_C^{\oplus 3}) = c(\Omega_{\mathbb{A}^3/k} \otimes \mathcal{O}_C) = c(V) \cdot c(\omega_C).$$

Hence $c_1(V) = -c_1(\omega_C)$, which by the choice of C is a non-torsion element of $CH^1(C)$ (which is the divisor class group of A_0). Thus $\xi^2 \in CH^2(X)$ is a non-torsion element of $CH^2(X)$.

If homogeneous elements $x, y \in P$ exist, say of degrees r and s respectively, such that $\sqrt{(x,y)} = P$, then we may regard x, y as determining global sections of the sheaves $\mathcal{O}_X(r)$ and $\mathcal{O}_X(s)$ respectively, *which have no common zeroes on* X. Let $D_x \subset X$, $D_y \subset X$ be the divisors of zeroes of $x \in \Gamma(X, \mathcal{O}_X(r))$ and $y \in \Gamma(X, \mathcal{O}_X(s))$ respectively. Then we have equations in $CH^1(X)$

$$[D_x] = c_1(\mathcal{O}_X(r)) = rc_1(\mathcal{O}_X(1)) = r\xi,$$
$$[D_y] = c_1(\mathcal{O}_X(s)) = sc_1(\mathcal{O}_X(1)) = s\xi.$$

But $D_x \cap D_y = \emptyset$. Hence in $CH^2(X)$, we have a relation

$$0 = [D_x] \cdot [D_y] = rs\xi^2,$$

contradicting that $\xi^2 \in CH^2(X)$ is a non-torsion element.

Remark 2.5. The construction of the homomorphism from the polynomial ring R to the graded ring A is an algebraic analogue of the *exponential map* in Riemannian geometry, which identifies a tubular neighbourhood of a smooth submanifold of a Riemannian manifold with the normal bundle of the submanifold (see [24, Theorem 11.1], for example). The exponential map is usually constructed using geodesics on the ambient manifold; here we use the global structure of affine space, where "geodesics" are lines, to make a similar construction algebraically. This idea appears in a paper[3] of Boratynski, who uses it to argue that a smooth subvariety of $\mathbb{A}^n$ is a set-theoretic complete intersection if and only if the zero section of its normal bundle is a set-theoretic complete intersection in the total space of the normal bundle.

2.3. Zero cycles on non-singular proper and affine varieties

In this section, we discuss results of Mumford and Roitman, which give criteria for the non-triviality of $CH^d(X)$ where X is a non-singular variety over $\mathbb{C}$ of dimension $d \geq 2$, which is either proper, or affine.

If X is non-singular and irreducible, and $\dim X = d$, then $Z^d(X)$ is just the free abelian group on the (closed) points of X. Elements of $Z^d(X)$ are called *zero cycles* on X (since they are linear combinations of irreducible subvarieties of dimension 0). In the presentation $CH^d(X) = Z^d(X)/R^d(X)$, the group $R^d(X)$ of relations is generated by divisors of rational functions on irreducible curves in X.

The main non-triviality result for zero cycles is the following result, called the *infinite dimensionality theorem for* 0*-cycles.* It was originally proved (without $\otimes\mathbb{Q}$) by Mumford [26], for surfaces, and extended to higher dimensions by Roitman [32]; the statement with $\otimes\mathbb{Q}$ follows from [33].

Theorem 2.6 (Mumford, Roitman). *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$. Suppose X supports a non-zero regular q-form (i.e., $\Gamma(X, \Omega^q_{X/\mathbb{C}}) \neq 0$), for some $q > 0$. Then for any closed algebraic subvariety $Y \subset X$ with* $\dim Y < q$, *we have $CH^d(X \setminus Y) \otimes \mathbb{Q} \neq 0$.*

Corollary 2.7. *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$, such that $\Gamma(X, \omega_X) \neq 0$. Then for any affine open subset $V \subset X$, we have $CH^d(V) \otimes \mathbb{Q} \neq 0$.*

The corollary results from the identification of ω_X with the sheaf $\Omega^d_{X/\mathbb{C}}$ of d-forms.

Bloch [1] gave another proof of the above result, using the action of algebraic correspondences on the étale cohomology, and generalized the result to arbitrary characteristics. In [37] and [38], Bloch's argument (for the case of characteristic 0) is recast in the language of differentials, extending it as well to certain singular varieties. One way of stating the infinite dimensionality results of [37] and [38], in the smooth case, is the following. The statement is technical, but it will be needed below when discussing M. Nori's construction of indecomposable projective modules.

We recall the notion of a k-generic point of an irreducible variety; we do this in a generality sufficient for our purposes. If X_0 is an irreducible k-variety, where $k \subset \mathbb{C}$ is a countable algebraically closed subfield, a point $x \in X = (X_0)_{\mathbb{C}}$ determines an irreducible subvariety $Z \subset X$, called the k-closure of X, which is the smallest subvariety of X which is defined over k (*i.e.*, of the form $(Z_0)_{\mathbb{C}}$ for some subvariety $Z_0 \subset X_0$) and contains the chosen point x. We call x a *k-generic point* if its k-closure is X itself.

In the case X_0 (and thus also X) is affine, say $X_0 = \operatorname{Spec} A$, and $X = \operatorname{Spec} A_{\mathbb{C}}$ with $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$, then a point $x \in X$ corresponds to a maximal ideal $\mathfrak{m}_x \subset A_{\mathbb{C}}$. Let $\wp_x = A \cap \mathfrak{m}_x$, which is a prime ideal of A, not necessarily maximal. Then, in the earlier notation, $\wp_x$ determines an irreducible subvariety $Z_0 \subset X_0$. The k-closure $Z \subset X$ of x is the subvariety determined by the prime ideal $\wp_x A_{\mathbb{C}}$ (since k is algebraically closed, $\wp_x A_{\mathbb{C}}$ is a prime ideal). In particular, x is a k-generic point $\iff \wp_x = 0$. In this case, x determines an inclusion $A \hookrightarrow A_{\mathbb{C}}/\mathfrak{m}_x = \mathbb{C}(x) \cong \mathbb{C}$. This in turn gives an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the quotient field K of A (*i.e.*, of the function field $k(X_0)$) into the complex numbers.

In general, even if X is not affine, if we are given a k-generic point $x \in X$, we can replace X by any affine open subset defined over k, which will (because x is k-generic) automatically contain x; one verifies easily that the corresponding inclusion $K \hookrightarrow \mathbb{C}$ does not depend on the choice of this open subset. Thus we obtain an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the function field $K = k(X_0)$ into $\mathbb{C}$, associated to any k-generic point of X.

It is easy to see that the procedure is reversible: any inclusion of k-algebras $i : K \hookrightarrow \mathbb{C}$ determines a unique k-generic point of X. Indeed, choose an affine open subset $\operatorname{Spec} A = U_0 \subset X_0$, so that K is the quotient field of A. The induced inclusion $A \hookrightarrow \mathbb{C}$ induces a *surjection* of $\mathbb{C}$-algebras $A_{\mathbb{C}} \to \mathbb{C}$, whose kernel is a maximal ideal, giving the desired k-generic point.

Suppose now that X_0 is proper over k, and so X is proper over $\mathbb{C}$ (e.g., X is projective). Let $\dim X_0 = \dim X = d$. Then by the Serre duality theorem, the sheaf cohomology group $H^d(X, \mathcal{O}_X)$ is the dual $\mathbb{C}$-vector space to

$$\Gamma(X, \Omega^d_{X/\mathbb{C}}) = \Gamma(X, \omega_X) = \Gamma(X_0, \omega_{X_0}) \otimes_k \mathbb{C}.$$

Hence we may identify $H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}$ with

$$\operatorname{Hom}_{\mathbb{C}}(\Gamma(X, \omega_X), \Omega^d_{\mathbb{C}/k}) = \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}).$$

Note that a k-generic point x determines, via the inclusion $i_x : K \hookrightarrow \mathbb{C}$, a k-linear inclusion $\Omega^n_{K/k} \hookrightarrow \Omega^n_{\mathbb{C}/k}$, and hence, via the obvious inclusion

$$\Gamma(X_0, \omega_{X_0}) = \Gamma(X_0, \Omega^n_{X_0/k}) \hookrightarrow \Omega^n_{K/k},$$

a canonical element

$$di_x \in \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}) = H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}.$$

Theorem 2.8. *Let $k \subset \mathbb{C}$ be a countable algebraically closed subfield, and X_0 an irreducible non-singular proper k-variety of dimension d, with $\Gamma(X_0, \omega_{X_0}) \neq 0$. Let $U_0 \subset X_0$ be any Zariski open subset. Let $X = (X_0)_{\mathbb{C}}$, $U = (U_0)_{\mathbb{C}}$ be the corresponding complex varieties. Then there is a homomorphism of graded rings*

$$CH^*(U) \to \bigoplus_{p \geq 0} H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k},$$

with the following properties.

(i) *If $x \in U$ is a point, which is not k-generic, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is zero.*

(ii) *If $x \in U$ is a k-generic point, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is (up to sign) the canonical element di_x described above.*

As stated earlier, the above more explicit form of the infinite dimensionality theorem follows from results proved in [37] and [38].

2.4. Some computations with Chern classes

We now study the following two problems, which turn out to have some similarities. We will show how, in each case, the problem reduces to finding an example for which the Chern classes of the cotangent bundle (*i.e.*, the sheaf of Kähler differentials) have appropriate properties. We will then see, in Example 2.12, how to construct examples with these properties. The discussion is based on the article [2] of Bloch, Murthy and Szpiro.

Problem 2.9. Find examples of n-dimensional, non-singular affine algebras A over (say) the complex number field $\mathbb{C}$, for each $n \geq 1$, such that A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, or such that the module of Kähler differentials cannot be generated by $2n - 1$ elements $da_1, \ldots, da_{2n-1}$ (in contrast, it is a "classical" result that such an algebra A can always be generated by $2n+1$ elements, and its Kähler differentials can always be generated by $2n$ exact 1-forms; see, for example, [36]).

Problem 2.10. Find examples of prime ideals I of height $< N$ in a polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$ such that $\mathbb{C}[x_1, \ldots, x_N]/I$ is regular, but I cannot be generated by $N - 1$ elements (the Eisenbud-Evans conjectures, proved by Sathaye [34] and Mohan Kumar [25], imply that such an ideal I can always be generated by N elements).

First we discuss Problem 2.9. Suppose A is an affine smooth $\mathbb{C}$-algebra which is an integral domain of dimension n. Assume $X = \operatorname{Spec} A$ can be generated by $2n$ elements, *i.e.*, that there is a surjection $f : \mathbb{C}[x_1, \ldots, x_{2n}] \to A$ from a polynomial ring. Let $I = \ker f$. If $i : X \hookrightarrow \mathbb{A}^{2n}_{\mathbb{C}}$ is the embedding corresponding to the surjection f, then the normal bundle to i is the sheaf $V^\vee$, where $V = \widetilde{I/I^2}$.

From the self-intersection formula, and the formula for the Chern class of the dual of a vector bundle, we see that

$$(-1)^n c_n(V) = c_n(V^\vee) = i^* i_*[X] = 0, \tag{2.2}$$

since $CH^n(\mathbb{A}^{2n}_{\mathbb{C}}) = 0$.

On the other hand, suppose $j : X \hookrightarrow Y$ is any embedding as a closed subvariety of a non-singular affine variety Y whose cotangent bundle (*i.e.*, sheaf of Kähler differentials) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle. For example, we could take $Y = \mathbb{A}^{2n}_{\mathbb{C}}$, and $j = i$, but below we will consider a different example as well.

Let W be the conormal bundle of X in Y (if $Y = \operatorname{Spec} B$, and $J = \ker j^* : B \twoheadrightarrow A$, then $W = \widetilde{J/J^2}$). We then have an exact sequence of vector bundles on X

$$0 \to W \to j^*\Omega_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to 0.$$

Since $\Omega_{Y/\mathbb{C}}$ is a trivial vector bundle, we get that

$$c(W) = c(\Omega_{X/\mathbb{C}})^{-1} \in CH^*(X). \tag{2.3}$$

Note that this expression for $c(W)$, and hence the resulting formula for $c_n(W)$ as a polynomial in the Chern classes of $\Omega_{X/\mathbb{C}}$, is in fact *independent* of the embedding j. In particular, from (2.2), we see that $c_n(W) = 0$ for *any* such embedding $j : X \hookrightarrow Y$.

Remark 2.11. In fact, the stability and cancellation theorems of Bass imply that in the above situation, the vector bundle W itself is, up to isomorphism, independent of j, and is thus an invariant of the variety X. We call it the *stable normal bundle* of X; this is similar to the case of embeddings of smooth manifolds into Euclidean spaces. We will not need this fact in our computations below.

Returning to our discussion, we see that to find a $\mathbb{C}$-algebra A with $\dim A = n$, and which cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, it suffices to produce an embedding $j : X \hookrightarrow Y$ of $X = \operatorname{Spec} A$ into a smooth variety Y of dimension $2n$, such that

(i) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle, and
(ii) if W is the conormal bundle of j, then $c_n(W) \neq 0$; in fact it suffices to produce such an embedding such that $j_*c_n(W) \in CH^{2n}(Y)$ is non-zero.

We see easily that the same example $X = \operatorname{Spec} A$ will have the property that $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements; in fact if $P = \ker(f : A^{\oplus 2n-1} \twoheadrightarrow \Omega_{A/\mathbb{C}})$ for some surjection f, then $\widetilde{P}$ is a vector bundle of rank $n-1$, so that $c_n(\widetilde{P}) = 0$, while on the other hand, the exact sequence

$$0 \to P \to A^{\oplus 2n-1} \xrightarrow{f} \Omega_{A/\mathbb{C}} \to 0$$

implies that

$$c(\widetilde{P}) = c(\Omega_{X/\mathbb{C}})^{-1},$$

so that we would have

$$0 = c_n(\widetilde{P}) = c_n(W) \neq 0,$$

a contradiction.

Next we discuss the Problem 2.10 of finding an example of a "non-trivial" prime ideal $I \subset \mathbb{C}[x_1, \dots, x_N]$ in a polynomial ring such that the quotient ring $A = \mathbb{C}[x_1, \dots, x_N]/I$ is smooth of dimension > 0, while I cannot be generated by $N-1$ elements (by the Eisenbud-Evans conjectures, proved by Sathaye and Mohan Kumar, I can always be generated by N elements).

Suppose I can be generated by $N-1$ elements, and $\dim A/I = n > 0$. Then $I/I^2 \oplus Q = A^{N-1}$ for some projective A-module Q of rank $n-1$; hence

$$(I/I^2 \oplus Q \oplus A) \cong A^{\oplus N} \cong (I/I^2 \oplus \Omega_{A/\mathbb{C}}).$$

Hence we have an equality between total Chern classes

$$c(\Omega_{X/\mathbb{C}}) = c(\widetilde{Q}),$$

and in particular, $c_n(\Omega_{X/\mathbb{C}}) = 0$.

So if $X = \operatorname{Spec} A$ is such that $c_n(\Omega_{X/\mathbb{C}}) \in CH^n(X)$ is non-zero, then for any embedding $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}}$, the corresponding prime ideal I cannot be generated by $N-1$ elements.

Example 2.12. We now show how to construct an example of an n-dimensional affine variety $X = \operatorname{Spec} A$ over $\mathbb{C}$, for any $n \geq 1$, such that, for some embedding $X \hookrightarrow Y = \operatorname{Spec} B$ with $\dim Y = 2n$, and ideal $I \subset B$, the projective module $P = I/I^2$ has the following properties:

(i) $c_n(P) \neq 0$ in $CH^n(X) \otimes \mathbb{Q}$,

(ii) if $c(P) \in CH^*(X)$ is the total Chern class, then $c(P)^{-1}$ has a non-torsion component in $CH^n(X) \otimes \mathbb{Q}$.

Then, by the discussion earlier, the affine ring A will have the properties that

(a) A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra,

(b) $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements,

(c) for any way of writing $A = \mathbb{C}[x_1, \dots, x_N]/J$ as a quotient of a polynomial ring (with N necessarily at least $2n+1$), the ideal J requires N generators (use the formula (2.3)).

The technique is that given in [2]. Let E be an elliptic curve (*i.e.*, a non-singular, projective plane cubic curve over $\mathbb{C}$), for example,

$$E = \operatorname{Proj} \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3).$$

Let $E^{2n} = E \times \cdots \times E$, the product of $2n$ copies of E. Let $Y = \operatorname{Spec} B \subset E^{2n}$ be any affine open subset. By the Mumford-Roitman infinite dimensionality theorem (Theorem 2.6 above), $CH^{2n}(Y) \otimes \mathbb{Q} \neq 0$. Also, since $Y \subset E^{2n}$, clearly the $2n$-fold intersection product

$$CH^1(Y)^{\otimes 2n} \to CH^{2n}(Y)$$

is surjective. Hence we can find an element $\alpha \in CH^1(Y)$ with $\alpha^{2n} \neq 0$ in $CH^{2n}(Y) \otimes \mathbb{Q}$. Let P be the projective B-module of rank 1 corresponding to α. Since Y is affine, by Bertini's theorem, we can find elements $a_1, \dots, a_n \in P$ such that the corresponding divisors $H_i = \{a_i = 0\} \subset Y$ are non-singular, and intersect transversally; take $X = H_1 \cap \cdots \cap H_n$. Then $X = \operatorname{Spec} A$ is non-singular of dimension n, and the ideal $I \subset B$ of $X \subset Y$ is such that $I/I^2 \cong (P \otimes_B A)^{\oplus n}$. Thus, if $j : X \hookrightarrow Y$ is the inclusion, then we have a formula between total Chern classes

$$c(I/I^2) = j^*c(P)^n = (1 + j^*c_1(P))^n = (1 + j^*\alpha)^n.$$

Hence

$$c_n(I/I^2) = j^*(\alpha)^n,$$

and so by the projection formula,

$$j_*c_n(I/I^2) = j_*(1)\alpha^n = \alpha^{2n},$$

since

$$j_*(1) = [V] = [H_1] \cdot [H_2] \cdot \cdots \cdot [H_n] = \alpha^n \in CH^n(Y),$$

as X is the complete intersection of divisors H_i, each corresponding to the class $\alpha \in CH^1(Y)$. By construction, $j_*c_n(I/I^2) \neq 0$ in $CH^{2n}(Y) \otimes \mathbb{Q}$, and so we have that $c_n(I/I^2) \neq 0$ in $CH^n(X) \otimes \mathbb{Q}$, as desired.

Similarly

$$c(I/I^2)^{-1} = (1 + j^*\alpha)^{-n}$$

has a non-zero component of degree n, which is a non-zero integral multiple of $j^*\alpha^n$.

Remark 2.13. The existence of n-dimensional non-singular affine varieties X which do not admit closed embeddings into affine $2n$-space is in contrast to the situation of differentiable manifolds — the "hard embedding theorem" of Whitney states that any smooth n-manifold has a smooth embedding in the Euclidean space $\mathbb{R}^{2n}$.

2.5. Indecomposable projective modules

Now we discuss M. Nori's (unpublished) construction of indecomposable projective modules of rank d over any affine $\mathbb{C}$-algebra $A_{\mathbb{C}}$ of dimension d, such that $U = \operatorname{Spec} A_{\mathbb{C}}$ is an open subset of a non-singular projective (or proper) $\mathbb{C}$-variety X with $H^0(X, \omega_X) = H^0(X, \Omega^d_{X/\mathbb{C}}) \neq 0$.

The idea is as follows. Fix a countable, algebraically closed subfield $k \subset \mathbb{C}$ such that X and U are defined over k; in particular, we are given an affine k-subalgebra $A \subset A_{\mathbb{C}}$ such that $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$. We also have a k-variety X_0 containing $U_0 = \operatorname{Spec} A$ as an affine open subset, such that $X = (X_0)_{\mathbb{C}}$.

Let K_n be the function field of $X_0^n = X_0 \times_k \cdots \times_k X_0$ (equivalently, K_n is the quotient field of $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$). We have n induced embeddings $\varphi_i : K \hookrightarrow K_n$, where $K = K_1$ is the quotient field of A, given by $\varphi_i(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$ with a in the i-th position.

Choose an embedding $K_n \hookrightarrow \mathbb{C}$ as a k-subalgebra. The inclusions φ_i then determine n inclusions $K \hookrightarrow \mathbb{C}$, or equivalently, k-generic points $x_1, \ldots, x_n \in X$ (in algebraic geometry, these are called "n independent generic points of X"). Let $\mathfrak{m}_i$ be the maximal ideal of $A_{\mathbb{C}}$ determined by x_i, and let $I = \cap_{i=1}^n \mathfrak{m}_i$. Clearly I is a local complete intersection ideal of height d in the d-dimensional regular ring $A_{\mathbb{C}}$. Thus we can find a projective resolution of I

$$0 \to P \to F_{d-1} \to \cdots \to F_1 \to I \to 0,$$

where F_i are free. By construction, $c(P) = c(A/I)^{(-1)^d}$. By theorem 2.3, we have

$$c(A/I) = 1 + (-1)^{d-1}(d-1)!(\sum_{i=1}^{n}[x_i]) \in CH^*(X).$$

Hence $c_i(P) = 0$ for $i < d$, while $c_d(P)$ is a non-zero integral multiple of the class $\sum_i [x_i] \in CH^d(U)$. This class is non-zero, from theorem 2.8 (we will get a stronger conclusion below). Hence $\operatorname{rank} P \geq d$.

By Bass' stability theorem, if $\operatorname{rank} P = d + r$, we may write $P = Q \oplus A^{\oplus r}$, where Q is projective of rank d. Then P and Q have the same Chern classes. So we can find a projective module Q of rank d with $c(Q) = 1 + m(\sum_i [x_i]) \in CH^*(U)$, for some non-zero integer m.

Suppose $Q = Q_1 \oplus Q_2$ with $\operatorname{rank} Q_1 = p$, $\operatorname{rank} Q_2 = d - p$, and $1 \leq p < d$ (thus $d > 1$). Then in $CH^*(U) \otimes \mathbb{Q}$, the class $\sum_i [x_i]$ is expressible as

$$\sum_i [x_i] = \alpha \cdot \beta, \quad \alpha \in CH^p(U) \otimes \mathbb{Q}, \ \beta \in CH^{d-p}(U) \otimes \mathbb{Q}.$$

Using the homomorphism of graded rings of Theorem 2.8,

$$CH^*(U) \otimes \mathbb{Q} \to \bigoplus_{j \geq 0} H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k},$$

we see that the element

$$\xi = \sum_{i=1}^{n} di_{x_i} \in H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$$

is expressible as a product

$$\begin{gathered} \xi = \textstyle\sum_{i=1}^n di_{x_i} = \alpha \cdot \beta, \quad \alpha \in H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k}, \\ \beta \in H^{d-p}(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^{d-p}_{\mathbb{C}/k}. \end{gathered}$$

Let L be the algebraic closure of K_n in $\mathbb{C}$. The graded ring

$$\bigoplus_{j=0}^{d} H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k} = \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{\mathbb{C}/k}$$

has a graded subring

$$\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$$

which contains the above element ξ. We claim that ξ is then expressible as a product $\alpha \cdot \beta$ of homogeneous elements of degrees $p, d-p$ with α, β lying in this subring. Indeed, since $\mathbb{C}$ is the direct limit of its subrings B which are finitely generated L-subalgebras, we can find such a subring B, and homogeneous elements $\widetilde{\alpha}$, $\widetilde{\beta}$ of degrees p, $d-p$ in $\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k}$ such that $\xi = \widetilde{\alpha} \cdot \widetilde{\beta}$. Choosing a maximal ideal in B, we can find an L-algebra homomorphism $B \twoheadrightarrow L$, giving rise to a graded ring homomorphism

$$f : \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k} \to \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}.$$

Then $\xi = f(\widetilde{\alpha}) \cdot f(\widetilde{\beta})$ holds in $\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$ itself.

Now

$$\Omega^1_{L/k} = \Omega^1_{K_n/k} \otimes_{K_n} L = \bigoplus_{j=1}^{n} \Omega^1_{K/k} \otimes_K L,$$

where the j-th summand corresponds to the j-th inclusion $K \hookrightarrow K_n$. We may write this as

$$\Omega^1_{L/k} = \Omega^1_{K/k} \otimes_K W,$$

where $W \cong L^{\oplus n}$ is an n-dimensional L-vector space with a distinguished basis. Then there are natural surjections

$$\Omega^r_{L/k} = \bigwedge_L^r (\Omega^1_{K/k} \otimes_K W) \twoheadrightarrow \Omega^r_{K/k} \otimes_K S^r(W),$$

where $S^r(W)$ is the r-th symmetric power of W as an L-vector space. In particular, since $\Omega^d_{K/k}$ is 1-dimensional over K, we get a surjection $\Omega^d_{L/k} \twoheadrightarrow S^d(W)$. This determines the component of degree d of a graded ring homomorphism

$$\Phi : \bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k} \to \bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W).$$

As in the discussion preceeding Theorem 2.8, by Serre duality on X_0, the natural inclusion $H^0(X_0, \Omega^d_{X_0/k}) \hookrightarrow \Omega^d_{K/k}$ determines a canonical element $\theta \in H^d(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^d_{K/k}$. Identifying the symmetric algebra $S^\bullet(W) = S^\bullet(L^{\oplus n})$ with the polynomial algebra $L[t_1, \ldots, t_n]$, we have that $\Phi(\xi) = \theta \cdot (t_1^d + \cdots + t_n^d)$. Hence, in the graded ring

$$\bigoplus_{j=0}^d H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W),$$

the element $\theta \cdot (t_1^d + \cdots + t_n^d)$ is expressible as a product of homogeneous elements α, β of degrees p and $d-p$. Hence, by expressing

$$\alpha \in H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k} \otimes_K S^p(W),$$
$$\beta \in H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k} \otimes_K S^{d-p}(W)$$

in terms of K-bases of $H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k}$ and $H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k}$, we deduce that in the polynomial ring $S^\bullet(W) = L[t_1, \ldots, t_n]$, the "Fermat polynomial" $t_1^d + \cdots + t_n^d$ is expressible as a sum of pairwise products of homogeneous polynomials

$$t_1^d + \cdots + t_n^d = \sum_{m=1}^N a_m(t_1, \ldots, t_n) b_m(t_1, \ldots, t_n)$$

with

$$N = \binom{d}{p}\binom{d}{d-p}(\dim_k H^p(X_0, \mathcal{O}_{X_0}))(\dim_k H^{d-p}(X_0, \mathcal{O}_{X_0})).$$

If $n > 2N$, the system of homogeneous polynomial equations $a_1 = b_1 = \cdots = a_N = b_N = 0$ defines a non-empty subset of the projective variety $t_1^d + \cdots + t_n^d = 0$ in $\mathbb{P}_L^{n-1}$, along which this Fermat hypersurface is clearly singular — and this is a contradiction!

§3. Variants of the Noether-Lefschetz Theorem

We discuss techniques from topology and Hodge theory, namely the monodromy theory of Lefschetz pencils, and Deligne's mixed Hodge structures, which lead to the construction of unique factorization domains, and to the construction of algebraic local rings over $\mathbb{C}$ of dimension 2 with a prescribed normal singularity, and minimal divisor class group (a cyclic group, generated by the class of the canonical module). As an example, we mention the following result: any ring of the form $\mathbb{C}[x, y, z]/(z^2 + xy + f(x, y, z))$, with f a "general" polynomial of degree ≥ 5 and vanishing at $(0, 0, 0)$ to order 4, is a UFD.

3.1. Background and results

If A is a (Noetherian) normal local ring, $\widehat{A}$ its completion, then the map on divisor class groups $C\ell(A) \to C\ell(\widehat{A})$ is injective. This leads to a natural question.

Question 3.1. *Given $\widehat{A}$, what are the possibilities for the subgroup $C\ell(A) \hookrightarrow C\ell(\widehat{A})$?*

We will restrict attention here to the case when A has the coefficient field $\mathbb{C}$, the complex numbers; *we will assume henceforth that all local rings under consideration have coefficient field* $\mathbb{C}$, unless explicitly noted otherwise.

There is one case when the question is trivially answered: when $C\ell(\widehat{A}) = 0$, or equivalently, $\widehat{A}$ is a UFD. We recall some "classical" results along these lines (see [4], [22], [10], [14]).

Theorem 3.2 (Brieskorn, Lipman). *Let $\widehat{A}$ be a complete non-regular UFD with* $\dim \widehat{A} = 2$. *Then* $\widehat{A} \cong \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^5)$.

Theorem 3.3 (Grothendieck). *Let $\widehat{A} = R/(x_1, \ldots, x_n)$ where R is a power series ring over $\mathbb{C}$, such that* (i) $\dim A \geq 4$, (ii) $x_1, \ldots, x_n$ *is a regular sequence in R,* (iii) $\widehat{A}_\wp$ *is a UFD for all primes $\wp$ of $\widehat{A}$ of height ≤ 3. Then $\widehat{A}$ is a UFD.*

Theorem 3.4 (Hartshorne, Ogus). *Let $(\widehat{A}, \mathfrak{m})$ be a complete local ring with* $\dim \widehat{A} = d \geq 3$, *such that* (i) $\widehat{A}$ *has an isolated singularity,* (ii) $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \leq 2d-3$, (iii) $\operatorname{depth} \widehat{A} \geq 3$. *Then $\widehat{A}$ is a UFD.*

In a related vein, one can ask the following.

Question 3.5. *Given $\widehat{A}$, when does there exist a Noetherian local ring A with completion $\widehat{A}$, such that A is a UFD?*

One has the following result in this direction [15] (see also [16]).

Theorem 3.6 (R. C. Heitmann). *Let R be a complete local ring over $\mathbb{C}$ of depth ≥ 2. Then there exists a local UFD A with completion $\widehat{A} = R$.*

However, the UFD constructed by Heitmann is very far from being "geometric". For example, suppose $\dim R = 2$, and R is normal but not Gorenstein. Then the corresponding ring A cannot have a dualizing module, from an old result of Murthy [28], which states that a Cohen-Macaulay UFD with a dualizing module is Gorenstein. Thus Heitmann's ring A is not a quotient of a regular local ring, for example.

So we will restrict attention, in Question 3.1, to local rings A which are essentially of finite type over $\mathbb{C}$ (*i.e.*, are localizations of finitely generated $\mathbb{C}$-algebras). We will refer to such local rings A as *geometric.* By Murthy's theorem [28] mentioned above, Question 3.5 can have a positive answer for geometric A, in dimension 2, only if $\widehat{A}$ is Gorenstein. So we are finally led to the following modification of Question 3.5.

Question 3.7. *Let R be a normal local ring. Does there exist a geometric local ring A with completion $\widehat{A} \cong R$, such that $C\ell(A)$ is the cyclic group generated by ω_A?*

Remark 3.8. Here ω_A is the dualizing module for A in the sense of [13], III,§7 — if X is any irreducible projective variety with a point $x \in X$ such that $A = \mathcal{O}_{X,x}$, then ω_A is the stalk at x of the dualizing sheaf ω_X, as defined in [13]; ω_A is in fact independent of the choice of X. As in the definition of ω_X in [13], one can characterize ω_A by a suitable dualizing property (phrased in terms of local cohomology), or as a suitable cohomology module of a dualizing complex for A.

The discussion above has been centered around making the divisor class group as small as possible. We give an example addressing the question as to how large the class group can be. This suggests that Question 3.7 is probably the only reasonable "general" question one can ask in the direction of Question 3.1.

Example 3.9. If $\widehat{A} = \mathbb{C}[[x,y,z]]/(x^2+y^3+z^7)$, then image $(C\ell\,(A) \hookrightarrow C\ell\,(\widehat{A}))$ is always a finitely generated subgroup, while $C\ell\,(\widehat{A}) = \mathbb{C}$. Without going into details, the reason for this is as follows. The *local Picard variety* of the singularity defined by $\operatorname{Spec}\widehat{A}$ (*i.e.*, the Picard variety of a minimal resolution of singularities of $\operatorname{Spec}\widehat{A}$) has connected component of the identity equal to the additive group $\mathbb{C}$ ($= \mathbb{G}_a$), which is affine; on the other hand, if $A = \mathcal{O}_{X,x}$ is the local ring of a point $x \in X$ on a projective complex algebraic variety X, and if $Y \to X$ is a resolution of singularities, then $C\ell\,(A)$ can be realized as a quotient of the Picard group $\operatorname{Pic}(Y)$ of Y. The subgroup $\operatorname{Pic}^0(Y) \subset \operatorname{Pic}(Y)$ is an abelian variety, with finitely generated quotient group $NS(Y) = \operatorname{Pic}(Y)/\operatorname{Pic}^0(Y)$ (the *Neron-Severi group* of the projective non-singular surface Y). Now we note that any homomorphism from an abelian variety to $\mathbb{G}_a$ is zero. Presumably the finitely generated subgroup $C\ell\,(A) \subset C\ell\,(\widehat{A})$ can be of arbitrary rank, as we vary the geometric subrings A. Incidentally, $\mathbb{C}[x,y,z]/(x^2+y^3+z^7)$ is a UFD, from results of Samuel.

We now state two results in the direction of Question 3.7 which are the main focus of our discussion. These are taken from the papers [29] and [30], respectively.

Theorem 3.10 (Parameswaran + Srinivas). *Question* 3.5 (*and hence also Quesion* 3.7) *has a positive answer for isolated complete intersection singularities.*

Theorem 3.11 (Parameswaran + van Straten). *Question* 3.7 *has a positive answer for an arbitrary normal surface singularity.*

The proofs of Theorems 3.10 (in [29]) and 3.11 (in [30]) are motivated by the "classical" proof, essentially due to Lefschetz, of the Noether-Lefschetz Theorem. We next recall this statement, in two equivalent forms.

Theorem 3.12 (Noether-Lefschetz Theorem).

(a) (Algebraic Version): *Let* $F(x,y,z,w) \in \mathbb{C}[x,y,z,w]$ *be a "general" homogeneous polynomial of degree* ≥ 4. *Then* $\mathbb{C}[x,y,z,w]/(F)$ *is a UFD.*

(b) (Geometric Version): *Let* $F(x,y,z,w) \in \mathbb{C}[x,y,z,w]$ *be a "general" homogeneous polynomial of degree* ≥ 4, *and let* $X \subset \mathbb{P}^3_{\mathbb{C}}$ *be the corresponding surface. Then* $\operatorname{Pic}(X) = \mathbb{Z}$ *generated by the class of the tautological line bundle* $\mathcal{O}_X(1)$. *Equivalently, the restriction map* $\operatorname{Pic}(\mathbb{P}^3_{\mathbb{C}}) \to \operatorname{Pic}(X)$ *is an isomorphism.*

The equivalence of the two versions follows from the projective normality of X, and the formula for the divisor class group of the affine cone over a projectively normal variety (see [13] II Ex. 6.3).

3.2. Outline of the proof of the Noether-Lefschetz Theorem

The idea of the proof of the Noether-Lefschetz Theorem is to view the surface X as a general member in a 1-parameter family of such hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. Now one uses the *monodromy theory of Lefschetz pencils* to show that $\mathrm{Pic}(\mathbb{P}^3_{\mathbb{C}}) \to \mathrm{Pic}(X)$ is an isomorphism, when $d = \deg X \geq 4$.

We outline this below, suppressing many technical details, but trying to indicate the main points of the argument (see [18], [21] and [12] for more details).

Remark 3.13. Later, we will comment on how this proof can be modified to prove Theorem 3.10 for hypersurface singularities of dimension 2. The proof of the general case of Theorem 3.10 is similar in dimension 2, but with some additional technical difficulties in dealing with complete intersections instead of hypersurfaces; the higher dimensional case of Theorem 3.10 in fact turns out to be simpler than the 2-dimensional case (for example, in dimensions ≥ 4, it follows at once from Theorem 3.3). The proof of Theorem 3.11 has the ingredients of the proof for hypersurface case, together with additional inputs from singularity theory, like a finite determinacy theorem of Pellikaan [31] and classification results of Siersma [35] on "line singularities" (singularities with 1-dimensional singular locus).

First, since the polynomial F is "general", Bertini's theorem implies that $X \subset \mathbb{P}^3_{\mathbb{C}}$ is a non-singular hypersurface. From the exact sequence of cohomology associated to the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0.$$

we get $H^1(X, \mathcal{O}_X) = 0$ and (since $d \geq 4$) $H^2(X, \mathcal{O}_X) \neq 0$.

Now on an arbitrary proper $\mathbb{C}$-variety T, with associated analytic space T_{an} (which is a compact complex analytic space), the exponential sheaf sequence (with $\exp(f) = e^{2\pi i f}$)

$$0 \to \mathbb{Z}_{T_{an}} \to \mathcal{O}_{T_{an}} \xrightarrow{\exp} \mathcal{O}^*_{T_{an}} \to 0$$

gives an exact sequence in cohomology

$$0 \to \frac{H^1(T_{an}, \mathcal{O}_{T_{an}})}{H^1(T_{an}, \mathbb{Z})} \to \mathrm{Pic}(T_{an}) \to H^2(T_{an}, \mathbb{Z}) \to H^2(T_{an}, \mathcal{O}_{T_{an}})$$

where $\mathrm{Pic}(T_{an})$ is the group of isomorphism classes of analytic line bundles on T_{an}. By Serre's GAGA, the canonical map $\mathrm{Pic}(T) \to \mathrm{Pic}(T_{an})$, obtained by regarding algebraic line bundles as analytic ones, is an isomorphism. GAGA also implies that for any coherent algebraic sheaf $\mathcal{F}$ on T, with associated analytic sheaf $\mathcal{F}_{an}$, the canonical maps $H^i(T, \mathcal{F}) \to H^i(T_{an}, \mathcal{F}_{an})$ are isomorphisms of $\mathbb{C}$-vector spaces. Hence, if $H^1(T, \mathcal{O}_T) = 0$, we get an induced exact sequence

$$0 \to \mathrm{Pic}(T) \to H^2(T_{an}, \mathbb{Z}) \to H^2(T, \mathcal{O}_T).$$

This sequence is clearly also functorial in T.

Applying these remarks to the inclusion $i : X \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$, since $H^i(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}) = 0$, $i = 1, 2$ we obtain a commutative diagram with exact bottom row

$$\begin{array}{ccccc}
 & \mathrm{Pic}(\mathbb{P}^3_{\mathbb{C}}) & \overset{\cong}{\to} & H^2((\mathbb{P}^3_{\mathbb{C}})_{an}, \mathbb{Z}) & \\
 & \downarrow & & \downarrow & \\
0 \to & \mathrm{Pic}(X) & \to & H^2(X_{an}, \mathbb{Z}) & \to H^2(X, \mathcal{O}_X).
\end{array}$$

So we are reduced to proving that

$$H^2((\mathbb{P}^3_{\mathbb{C}})_{an}, \mathbb{Z}) \to H^2(X_{an}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

is exact.

Since X is non-singular, Hodge Theory (in fact, the *Hodge decomposition* $H^2(X_{an}, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$) implies that the natural map

$$H^2(X_{an}, \mathbb{Z}) \otimes \mathbb{C} = H^2(X_{an}, \mathbb{C}) \to H^2(X_{an}, \mathcal{O}_{X_{an}}) = H^2(X, \mathcal{O}_X)$$

is surjective (it is the projection onto the summand $H^{0,2}$). Hence

$$\mathrm{Pic}(X) = \ker\left(H^2(X_{an}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)$$

is a proper subgroup of $H^2(X_{an}, \mathbb{Z})$, with torsion-free quotient.

To simplify notation, we now omit the subscript *an*. We can consider a general 1-parameter family of such hypersurfaces $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$, corresponding to a 2-dimensional $\mathbb{C}$-vector subspace V_0 of the vector space V_d of homogeneous polynomials of degree d. Let $B \subset \mathbb{P}^3_{\mathbb{C}}$ be the non-singular complete intersection curve defined by $F_1 = F_2 = 0$, for any basis $\{F_1, F_2\}$ of V_0 (by Bertini's theorem, B is a non-singular complete intersection, since $V_0 \subset V_d$ is general). (The family of hypersurfaces $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$ is usually called a *pencil*, and $B = \bigcap\limits_{t \in \mathbb{P}^1_{\mathbb{C}}} X_t$ is called the *base locus*).

One shows that, since the subspace $V_0 \subset V_d$ is general, $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$ forms a *Lefschetz pencil*, which means the following.

(a) If $\widetilde{\mathbb{P}} = \{(x,t) \in \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \mid x \in X_t\}$, then $\widetilde{\mathbb{P}} \to \mathbb{P}^3_{\mathbb{C}}$ is a birational proper morphism between non-singular projective varieties, which is the blow-up of $\mathbb{P}^3_{\mathbb{C}}$ along the base locus (the smooth curve B).

(b) Let $f : \widetilde{\mathbb{P}} \to \mathbb{P}^1_{\mathbb{C}}$ be induced by the projection $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Then all fibers of f are irreducible, and for a finite set of (closed) points $\Delta \subset \mathbb{P}^1_{\mathbb{C}}$, we have that
   (i) X_t is non-singular for $t \notin \Delta$, and
   (ii) for any $t \in \Delta$, there is a unique singular point $x_t \in X_t$, at which X_t has an ordinary double point singularity (*i.e.*, the complete local ring $\widehat{\mathcal{O}}_{X_t,x_t}$ is isomorphic to $\mathbb{C}[[x,y,z]]/(z^2-xy)$).

Remark 3.14. A Lefschetz pencil is the complex algebraic analogue of a *Morse function* in the theory of compact differentiable manifolds.

Now $f : \widetilde{\mathbb{P}} \setminus f^{-1}(\Delta) \to \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ is a smooth proper morphism. This implies the underlying map of C^∞ manifolds is a locally trivial C^∞ fiber bundle (Ehresmann fibration theorem; see, for example, [18] for further discussion of this point). This fiber bundle structure implies that all fibers of f over $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ are diffeomorphic, and have isomorphic singular cohomology groups. In fact, if t_1, t_2 are any two points of $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$, the choice of a path (continuous image of the unit interval $[0,1] \subset \mathbb{R}$) joining t_1 and t_2 determines an isomorphism $H^*(X_{t_1},\mathbb{Z}) \to H^*(X_{t_2},\mathbb{Z})$ between singular cohomologies; further, this isomorphism in fact depends only on the homotopy class of this path (keeping the end points fixed). In particular, taking $t_2 = t_1 = t_0$, for a chosen base point $t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$, we obtain the *monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0) \to \mathrm{Aut}\,(H^*(X_{t_0},\mathbb{Z}))$$

of the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ (based at t_0) into the group of graded ring automorphisms of the cohomology of the fiber.

The Leray spectral sequence for $f : \widetilde{\mathbb{P}} \to \mathbb{P}^1$, together with the fact that the fibers X_t over points $t \in \Delta$ have only ordinary double points, is used to show that

$$\text{image}\left(H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z})\right) = \{\text{elements of } H^2(X_{t_0},\mathbb{Z}) \text{ fixed under the monodromy action}\}$$

(this is the "easy" part of what is often called *local Lefschetz theory*). A similar spectral sequence argument, applied to cohomology with finite coefficients, also implies that $\operatorname{coker}(H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z}))$ is torsion-free — in fact one shows that image $H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \hookrightarrow H^2(X_{t_0},\mathbb{Z}))$ is a direct summand of $H^2(X_{t_0},\mathbb{Z})$; but the universal coefficient theorem in topology implies that $H^2(X_{t_0},\mathbb{Z})$ is a torsion-free abelian group, since X_{t_0} is a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$. hence simply connected.

On the other hand, one has

$$\operatorname{Pic}(\widetilde{\mathbb{P}}) \cong \operatorname{Pic}(\mathbb{P}^3_{\mathbb{C}}) \oplus \mathbb{Z}[E] \cong H^2(\widetilde{\mathbb{P}},\mathbb{Z}),$$

where E is the exceptional divisor, and $E \cap X_{t_0} = B$, with $\mathcal{O}_X(B) = \mathcal{O}_X(d)$. Hence

$$\begin{aligned}\text{image}\left(H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z})\right) &= \text{image}\left(H^2(\mathbb{P}^3_{\mathbb{C}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z})\right)\\ &= \mathbb{Z}[\mathcal{O}_{X_{t_0}}(1)],\end{aligned}$$

and we are reduced to proving that

$$\begin{aligned}&\ker\left(H^2(X_{t_0},\mathbb{Z}) \to H^2(X_{t_0},\mathcal{O}_{X_{t_0}})\right) =\\ &\qquad\{\text{elements of } H^2(X_{t_0},\mathbb{Z}) \text{ fixed under the monodromy action}\}.\end{aligned}$$

Since $\operatorname{coker}(H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z}))$ is torsion-free, we further reduce to proving that

$$\begin{aligned}&\ker\left(H^2(X_{t_0},\mathbb{Q}) \to H^2(X_{t_0},\mathcal{O}_{X_{t_0}})\right) =\\ &\qquad\{\text{elements of } H^2(X_{t_0},\mathbb{Q}) \text{ fixed under the monodromy action}\}.\end{aligned}$$

Suppose we allow the base point $t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ to vary, and consider the corresponding variations in $H^2(X_{t_0},\mathbb{C})$ and $H^2(X_{t_0},\mathcal{O}_{X_{t_0}})$. In other words, we consider the sheaves $R^2f_*\mathbb{Z}$ and $R^2f_*\mathcal{O}_{\widetilde{\mathbb{P}}}$ restricted to $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$. These sheaves are respectively a local system, and a holomorphic vector bundle. Hence, if we fix an element of $H^2(X_{t_0},\mathbb{Z})$, *i.e.*, an element of the stalk $(R^2f_*\mathbb{Z})_{t_0}$, it determines a well-defined section of $R^2f_*\mathbb{Z}$ in any open disc D around t_0 in the Riemann surface $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$; the image of this section in $R^2f_*\mathcal{O}_{\widetilde{\mathbb{P}}} \mid_D$ is a section of a holomorphic vector bundle on D, *i.e.*, after choosing a local trivialization of this vector bundle, the section becomes a vector-valued holomorphic function on D. Using the fact that a holomorphic function on a domain in $\mathbb{C}$ has a discrete set of zeroes, one sees that the kernel of the sheaf map $R^2f_*\mathbb{Z} \to R^2f_*\mathcal{O}_{\widetilde{\mathbb{P}}}$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ is a *local sub-system* of $R^2f_*\mathbb{Z} \mid_{\mathbb{P}^1_{\mathbb{C}}\setminus\Delta}$.

In more concrete language, this means that for a sufficiently general choice of base point t_0, the subspace

$$\ker\left(H^2(X_{t_0},\mathbb{Q}) \to H^2(X_{t_0},\mathcal{O}_{X_{t_0}})\right)$$

is at least a $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$-submodule (here "general" means "in the complement of a countable subset"). This submodule clearly contains the image of $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})$. Hence it suffices to show that the quotient

$$\frac{H^2(X_{t_0}, \mathbb{Q})}{\text{image } H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})}$$

is a *simple* $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$-module. This is a purely topological statement, which is proved by more carefully analyzing the monodromy representation.

Since $(\mathbb{P}^1_{\mathbb{C}})_{an}$ is the Riemann sphere S^2, it is simply connected. We may assume after reindexing that $\Delta \subset \mathbb{C} = \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. The fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$ is then generated by the classes of suitably chosen loops γ_t based at t_0, indexed by elements $t \in \Delta$, which are pairwise non-intersecting (except at t_0). Here γ_t is a simple closed loop in $\mathbb{C}$, going once around t, and with winding number 0 with respect to the other points of Δ. To simplify notation, we will henceforth denote $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$ by just π_1.

Recall that the topological *intersection number* (a, b) of two elements $a, b \in H^2(X_{t_0}, \mathbb{Z})$ is defined using the non-degenerate, symmetric bilinear *intersection pairing* (Poincaré duality pairing)

$$H^2(X_{t_0}, \mathbb{Z}) \otimes H^2(X_{t_0}, \mathbb{Z}) \to H^4(X_{t_0}, \mathbb{Z}) = \mathbb{Z}.$$

We thus have intersection quadratic forms on $H^2(X_{t_0}, \mathbb{Q})$ and $H^2(X_{t_0}, \mathbb{R})$. The *Hodge index theorem* implies that, since X_t is a non-singular projective surface, the intersection form on $H^2(X_{t_0}, \mathbb{R})$ has signature $(1, -1, -1, \ldots, -1)$. This means, for example, that on the orthogonal complement of $\mathbb{R}[\mathcal{O}_{X_{t_0}}(1)]$ for the intersection product, the intersection form is *negative definite*, since $[\mathcal{O}_{X_{t_0}}(1)]$ has positive self-intersection d (equal to the degree of X_{t_0} in $\mathbb{P}^3_{\mathbb{C}}$). In fact the Hodge theoretic proof of the index theorem amounts to directly proving this negative definiteness statement (a particular case of the *Hodge-Riemann bilinear relations*; see [9]).

Using the condition that the singular fibers X_t of f are irreducible with 1 ordinary double point, Lefschetz associates to each γ_t an element $\delta_t \in H^2(X_{t_0}, \mathbb{Z})$, called a *vanishing cycle*, and describes the monodromy action of the corresponding element $\rho(\gamma_t)$ on $H^2(X_{t_0}, \mathbb{Q})$ via the *Picard-Lefschetz formula*

$$\rho(\gamma_t)(a) = a + (a, \delta_t)\delta_t \ \ \forall\ a \in H^2(X_{t_0}, \mathbb{Q}),$$

where $(a, \delta_t) \in \mathbb{Z}$ is the intersection number of a with δ_t (see [12], [18], [21] for details). Further, one has a self-intersection formula

$$(\delta_t, \delta_t) = -2.$$

Another step in the proof of the Theorem is a lemma ("conjugacy of the vanishing cycles") that all the classes $\rho(\gamma_t)$, $t \in \Delta$, are contained in the same conjugacy class in the monodromy image group $\rho(\pi_1) \subset \mathrm{Aut}(H^2(X_{t_0}, \mathbb{Z}))$. This is deduced using standard homotopy arguments from a "global" geometric fact, that in the (dual) projective space parametrizing the set of all hypersurfaces of degree d in $\mathbb{P}^3_{\mathbb{C}}$, the subvariety parametrizing singular hypersurfaces (called the *discriminant locus*) is irreducible (see [18] for more details of this argument).

Next, one observes that the subspace of $H^2(X_{t_0}, \mathbb{Q})$ of elements invariant under $\rho(\pi_1)$ is the orthogonal complement of the span of the vanishing cycles δ_t, for the non-degenerate intersection pairing. This is clear, because any element of $H^2(X_{t_0}, \mathbb{Q})$ which is orthogonal to all the δ_t must, by the Picard-Lefschetz formula, be invariant under all the $\rho(\gamma_t)$, and hence under all of π_1.

Note that since $[\mathcal{O}_{X_{t_0}}(1)] \in$ image $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})$ is π_1-invariant, it is orthogonal to all δ_t. Hence the intersection form is *negative definite* on the span of the δ_t. This implies that $H^2(X_{t_0}, \mathbb{Q})$ is the orthogonal direct sum of its π_1-submodules

$$\text{image } H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}) = H^2(X_{t_0}, \mathbb{Q})^{\pi_1}$$

(the subspace of π_1-invariants) and

$$\mathbf{V} = \sum_{t \in \Delta} \mathbb{Q}\delta_t$$

(this direct sum decomposition is the only "easy" case of the so-called "Hard Lefschetz theorem"). Since image $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}) \neq 0$ is a proper subspace of $H^2(X_{t_0}, \mathbb{Q})$ as noted earlier, $\mathbf{V} \neq 0$.

In the light of this, the proof has been reduced to the following assertion.

Claim 3.15. $\mathbf{V} \subset H^2(X_{t_0}, \mathbb{Q})$ *is a non-trivial* simple π_1*-submodule of* $H^2(X_{t_0}, \mathbb{Q})$.

Indeed, since any two elements $\rho(\gamma_{t_1}), \rho(\gamma_{t_2}) \in \rho(\pi_1)$ are conjugate, we deduce (using the Picard-Lefschetz formula) that the corresponding vanishing cycles δ_{t_1}, δ_{t_2} are in the same $\rho(\pi_1)$-orbit. Hence any π_1-submodule of $\mathbf{V}$ containing some δ_t must be all of $\mathbf{V}$. Now if $a \in \mathbf{V}$ is a non-zero element, then negative definiteness of the intersection form yields $(a, a) < 0$, which implies $(a, \delta_t) \neq 0$ for some $t \in \Delta$. This in turn implies $a - \rho(\gamma_t)(a)$ is a non-zero multiple of δ_t, *i.e.*, the π_1-submodule generated by $\mathbb{Q}a$ contains δ_t.

3.3. Some elements of the proofs of Theorem 3.10 and Theorem 3.11

We now discuss how the Lefschetz pencil technique can be used to obtain Theorem 3.10 (in dimension 2) and 3.11.

Theorem 3.10, in dimension 2, is proved in the following stronger form (there is a similar strengthening in dimension 3, and in dimensions$\geq$ 4, it follows from Grothendieck's Theorem 3.3).

Theorem 3.16. *Let $f_i(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$, $1 \leq i \leq n-2$ be polynomials vanishing at the origin $P = (0, 0, \dots, 0)$ such that*

$$\mathbb{C}[[x_1, \dots, x_n]]/(f_1, \dots, f_{n-2})$$

has an isolated complete intersection singularity. Then there exist integers $d_0 > r_0 > 0$ such that if

$$r \geq r_0, \quad d \geq \sup\{d_0, r+1, \deg f_j \ \forall\, j\}$$

and

$$V_{r,d} = \{\textit{polynomials of degree} \leq d \textit{ vanishing to order} \geq r \textit{ at } P\},$$

then for "general" $g_1, \dots, g_{n-2} \in V_{r,d}$, we have:

(i) *$A = \mathbb{C}[x_1, \dots, x_n]/(f_1 + g_1, \dots, f_{n-2} + g_{n-2})$ is a UFD, with*

$$\widehat{A} \cong \mathbb{C}[[x_1, \dots, x_n]]/(f_1, \dots, f_{n-2}),$$

(ii) *if F_i, G_i are homogenous of degree d in $X_0, \dots, X_n$ such that*

$$F_i(1, x_1, \dots, x_n) = f_i, \quad G_i(1, x_1, \dots, x_n) = g_i, \quad 1 \leq i \leq n-2,$$

then

$$\mathbb{C}[X_0, \dots, X_n]/(F_1 + G_1, \dots, F_{n-2} + G_{n-2})$$

is a 3-dimensional graded UFD,

(iii) *with notation as in* (ii)*, the projective variety*

$$X = \operatorname{Proj} \mathbb{C}[X_0, \dots, X_n]/(F_1 + G_1, \dots, F_{n-2} + G_{n-2})$$

satisfies a Noether-Lefschetz theorem: the natural map $C\ell\,(\mathbb{P}^n_{\mathbb{C}}) \to C\ell\,(X)$ is an isomorphism.

Remark 3.17. In the above result, "general" means "in the complement of a countable union of hypersurfaces" (in the affine space $(V_{r,d})^{n-2}$).

Remark 3.18. To summarize the theorem in a less technical way, if we perturb the polynomials f_i defining an isolated complete intersection singularity, by adding "general" polynomials g_i vanishing to high enough order (depending on the singularity), then we obtain an algebraic local UFD from the perturbed equations $f_1+g_1 = \cdots = f_{n-2}+g_{n-2} = 0$, which has the same completion, *i.e.*, has the same singularity.

In order to prove this, one considers $S = (V_{r,d})^{n-2} = V_{r,d}\times\cdots\times V_{r,d}$, the affine space parametrizing ordered $(n-2)$-tuples $(g_1,\dots,g_{n-2})$. For $s \in S$, let

$$X_s = \{F_1 + G_1 = \cdots = F_{n-2} + G_{n-2} = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$$

be the corresponding projective subscheme. Let $L \subset S$ be a "general" line (*i.e.*, $L \cong \mathbb{A}^1_{\mathbb{C}}$ is an affine linear 1-dimensional subspace of S), and set

$$X_L = \{(x,s) \in \mathbb{P}^n_{\mathbb{C}} \times L \mid x \in X_s\}.$$

We would like to "do Lefschetz theory" for the "pencil" $f : X_L \to L$.

This requires several modifications of the earlier argument. First of all, the general subscheme X_s, $s \in L$, will be an irreducible, complete intersection surface, which has $\{P\}$ as its singular locus; further, one can ensure that every X_s is irreducible, and has at most one other singular point which is an ordinary double point. This resembles the conditions of a Lefschetz pencil, but $\{P\} \times L \subset X_L$ is still part of the singular locus.

Next, we need to construct a "simultaneous resolution of singularities" for the family $X_L \to L$ along the above curve $\{P\} \times L$ of singularities. Using a Hilbert scheme argument, this is reduced in [29] to the following problem on Hilbert functions in local algebra:

> given a complete intersection quotient $A = \mathbb{C}[[x_1,\dots,x_n]]/(f_1,\dots,f_{n-2})$ of a power series ring $R = \mathbb{C}[[x_1,\dots,x_n]]$, with an isolated singularity, and an $\mathfrak{m}$-primary ideal $J \subset \mathfrak{m} = (x_1,\dots,x_n) \subset R$, show that there exists $r > 0$ such that is an equality between Hilbert functions
>
> $$\ell\,(R/J^m + (f_1,\dots,f_{n-2})) = \ell\,(R/J^m + (f_1+g_1,\dots,f_{n-2}+g_{n-2})) \qquad \forall\, m \geq 0,$$
>
> for an arbitrary choice of $g_1,\dots,g_{n-2} \in \mathfrak{m}^r$.

In fact there is a result of Mather [23] which implies that, for some $r > 0$, and for each choice of $g_j \in \mathfrak{m}^r$, $1 \leq j \leq n-2$, there is an automorphism σ of the power series ring R such that $\sigma \equiv \text{identity} \pmod{J}$, and there is an equality between ideals $\sigma(f_1,\dots,f_r)=(f_1+g_1,\dots,f_r+g_r)$. This implies a positve answer to above problem.

Remark 3.19. The above question on Hilbert functions led naturally to the papers [40], [39], and thus ultimately to Theorem 1.1!

Let $Y_L\to X_L$ be the resulting "simultaneous resolution of singularities", so that $Y_L\to L$ does look more like a Lefschetz pencil (all fibers are irreducible, all but finitely many non-singular, and singular fibers have only 1 ordinary double point).

Now much of the argument is similar to the case of the Noether-Lefschetz theorem. One first uses the fact that if $s\in L$ is general, so that X_s is a surface with a unique singularity P, then

$$C\ell\,(X_s)=\mathrm{Pic}(X_s\setminus\{P\})=\mathrm{Pic}(Y_s\setminus E),$$

where $Y_s\to X_s$ is the above resolution of singularities (obtained from $Y_L\to X_L$), with exceptional set E. Using Mather's result cited above, one can arrange that the exceptional set $E\subset Y_s$ is in fact independent of $s\in L$, in the following sense — Y_L is obtained as a closed subvariety of $\widetilde{\mathbb{P}^n_{\mathbb{C}}}\times L$ for some blow-up $\widetilde{\mathbb{P}^n_{\mathbb{C}}}\to\mathbb{P}^n_{\mathbb{C}}$, such that the exceptional set for $Y_L\to X_L$ has the form $E\times L$ for some subvariety $E\subset\widetilde{\mathbb{P}^n_{\mathbb{C}}}$.

If $E_1,\dots,E_r$ are the irreducible components of E, then since Y_s is a non-singular surface, one computes that

$$\mathrm{Pic}(Y_s\setminus E)=\mathrm{Pic}(Y_s)/\{\text{subgroup generated by the classes } [\mathcal{O}_{Y_s}(E_i)]\}.$$

Thus the Theorem is equivalent to that statement that $\mathrm{Pic}(Y_s)$ is generated by the classes of $\mathcal{O}_{Y_s}(E_i)$ and the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Now suppose, for simplicity, that $n=3$, *i.e.*, we are still dealing with hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. One can then realize Y_L as an open subset of the blow-up $\overline{Y_L}\to\mathbb{P}^3_{\mathbb{C}}$ along the union of a suitable complete intersection curve B, and a subscheme of finite length supported at P (Y_L itself is not compact, since it is proper over $L=\mathbb{A}^1_{\mathbb{C}}$). Hodge theory and the exponential sheaf sequence reduce one to proving, after some further analysis of the geometry, that

$$\text{image}\left(H^2(\overline{Y_L},\mathbb{Q})\to H^2(Y_s,\mathbb{Q})\right)=\ker\left(H^2(Y_s,\mathbb{Q})\to H^2(Y_s,\mathcal{O}_{Y_s})\right).$$

Using the theory of vanishing cycles, etc. one ends up showing that

$$\text{image}\left(H^2(Y_L,\mathbb{Q})\to H^2(Y_s,\mathbb{Q})\right)=\ker\left(H^2(Y_s,\mathbb{Q})\to H^2(Y_s,\mathcal{O}_{Y_s})\right).$$

To conclude, one appeals to the following result of Deligne, proved in [5] using his theory of *mixed Hodge structures*.

Theorem 3.20 (Deligne). *Let Z be a non-singular proper $\mathbb{C}$-variety, $Y \subset Z$ a non-singular closed subvariety, and $U \subset Z$ a Zariski open subset containing Y. Then*

$$\text{image}\,(H^*(Z,\mathbb{Q}) \to H^*(Y,\mathbb{Q})) = \text{image}\,(H^*(U,\mathbb{Q}) \to H^*(Y,\mathbb{Q})).$$

Remark 3.21. The application of Deligne's result, in "monodromy" situations as above, is usually called the *theorem of the fixed part* (see [5] for more discussion) – it says that if $f : U \to L$ is a smooth proper morphism between non-singular varieties, Z a non-singular complete variety containing U as a Zariski open set, and Y a fiber of f, then the subspace of $\pi_1(L)$-invariant elements of $H^*(Y,\mathbb{Q})$ is the image of the composite restriction map $H^*(Z,\mathbb{Q}) \to H^*(U,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$. The special case of this result, when $Z = U$ and L are proper, is due to Griffiths.

We now make some further remarks on the proofs of the general case of Theorem 3.16, and of Theorem 3.11.

For Theorem 3.16, if $n \geq 4$, then in our above set-up, the variety Y_L, which has dimension 3, cannot be realized as an open subset of a blow-up of the ambient projective space $\mathbb{P}^n_{\mathbb{C}}$. So, in addition to the "pencil" $Y_L \to L$, one needs to also consider the "total family" of varieties parametrized by S itself, and (for example) compare the Leray spectral sequences for these two, etc. There is a "bad subvariety" $\Delta(S) \subset S$, parametrizing the fibers with additional singularities (apart from the chosen one P), such that the singular fibers of $Y_L \to L$ lie over the points of $\Delta = L \cap \Delta(S)$. Additional results needed at this stage are that $\Delta(S)$ is an *irreducible divisor*, and that (since $L \subset S$ is general) the natural map on fundamental groups $\pi_1(L \setminus \Delta) \to \pi_1(S \setminus \Delta(S))$ is *surjective* (a result of Zariski).

Finally, as before, one will end up proving that the subspace of $\pi_1(L \setminus \Delta)$-invariant elements in $H^2(Y_t,\mathbb{Z})$ coincides with the subgroup generated by the cohomology classes of the exceptional divisors E_i, and the pull-back of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$. This will give the desired conclusion.

The strategy in proving Theorem 3.11 is a bit different. One first chooses some algebraic "model" for the surface singularity, *i.e.*, one finds an irreducible normal projective surface $X \subset \mathbb{P}^n_{\mathbb{C}}$ together with a point $x \in X$ such that $\mathcal{O}_{X,x}$ has the given completion, and $X \setminus \{x\}$ is non-singular. Then choose a generic linear projection $\mathbb{P}^n_{\mathbb{C}} \setminus H \to \mathbb{P}^3_{\mathbb{C}}$, which restricts to a finite, birational morphism $X \to Y$ onto a non-normal surface $Y \in \mathbb{P}^3_{\mathbb{C}}$, with X as its normalization. Finally, one analyzes the (usually 1-dimensional) singular locus $Z = Y_{sing}$ of Y; for example, one shows that at a "general" point of Z, the surface Y has complete local rings isomorphic to $\mathbb{C}[[x,y,z]]/(xy)$ (*i.e.*, Y has an "ordinary double curve singularity" at such points).

Applying the singularity theory results of Pellikaan and Siersma cited earlier, one considers deformations $\{(Y_t, Z_t)\}_t$ in $\mathbb{P}^3_{\mathbb{C}}$ of the *pair* (Y, Z). Taking "simultaneous normalizations" gives rise to corresponding deformations $\{(X_t, x_t)\}_t$ of (X, x), such that each of the complete local rings $\widehat{\mathcal{O}}_{X_t,x_t}$ is isomorphic to $\widehat{\mathcal{O}}_{X,x}$ (this is a consequence of the singularity theory inputs). Again one arrives at a sort of Noether-Lefschetz situation, with one difference: the inverse image in X_t under the normalization map $X_t \to Y_t$ of the singular locus $Z_t \subset Y_t$ (which is only a Weil divisor on Y_t) is a "new" divisor class, which does not come from a line bundle on $\mathbb{P}^3_{\mathbb{C}}$. In fact, *adjunction theory* (*i.e.*, "Grothendieck duality for the finite morphism $X_t \to Y_t$"; see for example [13]III Ex. 6.10 and Ex. 7.2) implies that this divisor represents the canonical (Weil) divisor class of X_t, upto a twist by some $\mathcal{O}_{X_t}(n)$. This gives the generator of the cyclic class group $C\ell\,(\mathcal{O}_{X_t,x_t})$ for a sufficiently general choice of the parameter t.

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