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Notes on the Topology of Hyperplane Arrangements and Braid Groups

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Introduction.

We will be concerned with the following problem. Let V be an $n-$dimensional vector space over $\mathbf{R}$. Denote its complexification $V_{\mathbf{C}} = V + iV$.

Consider furthermore a finite family $\mathcal{H} := \mathcal{H}_I := \{H_i\}_{i \in I}$ of real hyperplanes in V which for simplicity we assume all passing through the origin. The set of given hyperplanes and all their intersections form a finite set of subspaces of V partially ordered by inclusion.

We shall restrict to the case in which $\cap_i H_i = 0$ (such an arrangement is called *essential*) in fact this is not a serious restriction.

We shall denote by

$$L(\mathcal{H}) := \{\cap_{i \in T} H_i | T \subset I\}.$$

this finite set of subspaces (closed under intersection), which will be referred to as *the real arrangement.*

The complexification of all these subspaces is the corresponding complex arrangement in $V_{\mathbf{C}}$. Our main concern will be the study of the topology of the complement in $V_{\mathbf{C}}$ of the union $\cup_{i \in I}(H_i)_{\mathbf{C}}$.

Let us denote by $\mathcal{A} := V_{\mathbf{C}} - \cup_{i \in I}(H_i)_{\mathbf{C}}$ this open set.

Of particular interest is the case in which V is a Euclidean space and the H_i are the reflection hyperplanes of a finite reflection group [Bou].

These groups have been classified by Coxeter, the finite reflection group W acts freely on $\mathcal{A}$ and we can form the covering

$$\mathcal{A} \to \mathcal{A}/W.$$

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Among these reflection groups there is the type A_n wich is the group S_{n+1} of permutations of the coordinates of $\mathbf{R^{n+1}}$ (the reflection hyperplanes are the ones of equations $x_i - x_j = 0$). In this case $\mathcal{A}/S_{n+1}$ can be identified to the space of monic polinomials of degree n with distinct roots. The homotopy groups of $\mathcal{A}$, $\mathcal{A}/W$ have been determined by Brieskorn [Br] and in the case A_n we have the classical Artin braid group B_n. Moreover it has been proved by Deligne [De] that $\mathcal{A}$, $\mathcal{A}/W$ are both $K(\pi, 1)$ spaces.

Salvetti [S1] has described a very explicit finite CW complex homotopically equivalent to $\mathcal{A}$ resp. $\mathcal{A}/W$ and, with the use of this complex many cohomology computations for these groups can be performed (cf. also [B-Z]).

De Concini and Salvetti have used these methods also to compute the cohomology of finite reflection groups. In these notes we explain some of these topics.

These notes are a first draft of a project which may never see the light and I make them available in the hope that they may be useful. Nothing new is here just maybe some improvements in the notations and presentation.

A the moment, even if the Salvetti complex is very explicit there is no real simplification available in the proof of Deligne and this topic is not included. The main open problems are related to the genus of the fibration given by the action of the reflection group on the regular part and we refer to [DS2] for details.

Note added in proof. The following paper in fact is very relevant: C. C. Squier, The homological algebra of Artin groups, *Math. Scand.*, 75 (1995), 5–43.

§1. Real arrangements

We start our analysis from real arrangements, we give some basic definitions.

With the notations of the introduction we fix a finite family $\mathcal{H} := \mathcal{H}_I := \{H_i\}_{i \in I}$ of real hyperplanes in V and denote by $L(\mathcal{H}) := \{\cap_{i \in T} H_i | \, T \subset I\}$ the associated *real arrangement* (i.e. the set of all possible intersections of the H_i).

Definition. The connected components of $V - \cup_i H_i$ are called chambers of the arrangement.

Clearly the chambers are connected convex open sets of V.

Given any subspace $W \in L(\mathcal{H})$ of the arrangement the set of hyperplanes in the arrangement which do not contain W cuts on W a family $\mathcal{H}|_W$ of hyperplanes and the induced arrangent in W is a subset of $L(\mathcal{H})$. The chambers of all the induced arrangements in all the subspaces in $L(\mathcal{H})$ are called *faces*, [1] the set of all faces will be denoted by $F(\mathcal{H})$.

Lemma 1.1. *The faces form a partition of V.*

The proof is by easy induction.

Let us choose for each $i \in I$ an explicit linear equation $\alpha_i = 0$ for the hyperplane H_i.

Given a chamber F, by connectedness each α_i has a definite sign ($+$ or $-$) on the points of F and conversely if on 2 points p, q in $\mathcal{A} = V - \cup_i H_i$ the funcions α_i have the same sign then this happens on the entire segment $tp + (1-t)q,\ 0 \leq t \leq 1$ which connects p, q in $\mathcal{A}$.

Thus a chamber determines and is determined by a sequence of signs (of course not all sequences occur).

For a face in general some of the α_i are also 0 and thus we see that more generally a face determines and it is determined by a sequence of signs $+, -, 0$ indexed by I.

This remark has an immediate implication. If we consider the arrangement $L(H_J)$ associated to a subset $J \subset I$ of the given set of hyperplanes we have:

Proposition 1.2. *Each face of the arrangement $L(H_J)$ is a union of faces of the arrangement $L(H)$.*

Lemma 1.3. *The closure of a face F is a union of faces.*

Proof. We prove this statement by induction on the dimension of the face and thus we may assume that the face is a chamber.

If $p \in \overline{F}$ is a point then $\alpha_i(p)$ is either 0 or it has the same sign of $\alpha_i(q)$ for $q \in F$.

In particular we see that the half closed segment $tp + (1-t)q,\ 0 \leq t < 1$ is entirely contained in the chamber F.

Let F_1 be the face in which p is contained and $r \in F_1$ since the sequence of signs for r coincides with that of p we see that also the half closed segment $tr + (1-t)q,\ 0 \leq t < 1$ is entirely contained in the chamber F and thus $r \in \overline{F}$. □

[1] In the french literature one distinguishes between *faces* as the codimension 1 faces and *facettes* for the others.

We have thus defined a partial order on the set of faces and we shall denote by $\mathcal{F}$ the partially ordered set of faces, the usual convention is $F_1 \leq F_2$ if and only if $F_2 \subset \overline{F}_1$. Thus the chambers are the minimal faces.

§2. Fans

The fundamental combinatorial object is the *nerve* of the poset $\mathcal{F}$ i.e. the simplicial complex whose vertices are in 1-1 correspondence with the faces and whose simplices correspond to totally ordered subsets of faces.

Let us axiomatize this construction. Let us call *a cone* any subset $A \subset V$ such that $v \in A,\ a > 0,\ \Longrightarrow\ av \in A$.

Definition. A polyhedral fan[2] $\mathcal{F} := \{F_i\}_{i \in I}$ is a finite family of convex cones, called the *strata* such that:

1) 0 is a stratum.
2) The closure of a stratum is a union of strata.
3) $V = \cup_{i \in I} F_i$ is a decomposition (i.e. disjoint union) of V.

By definition then the set of strata is a poset by setting $F_1 \leq F_2$ if and only if $F_2 \subset \overline{F}_1$ (F_2 is contained in the closure $\overline{F}_1$ of F_1.)

Thus the set of faces of a hyperplane arrangement is a polyhedral fan, we will see another important example when we treat complex arrangements. Let us fix a polyhedral fan, before proceding let us remark some simple facts.

a) If we intersect a line l with the strata of a fan, it becomes decomposed as disjoint union of convex strata, such that the closure of a stratum is a union of strata. Then these strata are open segments (possibly infinite) and their extremal points.

b) If $W \subset V$ is a subspace the family $W \cap F_i$ of non empty intersections is a polyhedral fan in W.

c) A polyhedral fan in the line $\mathbf{R}$ is necessarily the decomposition $\mathbf{R}^-,\ \mathbf{0}, \mathbf{R}^+$.

d) A polyhedral fan in $\mathbf{R^2}$ is given by a finite set of half lines r_i and the connected components of their complement. Notice that such components are convex if and only if the angle between two successive lines il $\leq \pi$.

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[2]the definition we use is slightly more general that the one usually introduced in the theory of torus embeddings.

Part d) needs a proof. Consider a stratum S which is not a half line. S is a convex cone containing two linearly independent vectors a, b. Consider the intersection of S with the line through a, b it is a convex set A in this line which by the previous codiscussion is open, then it is easy to conclude the proof.

The main construction is a geometric realization of this poset in V but in fact this is a consequence of the construction of a *simplicial fan,* which is a pseudobaricentric subdivision of the given fan.

For this let us select in each stratum F, different from the stratum reduced to 0, a vector v_F.

There is a totally elementary but essential Lemma associated to this construction.

Lemma 2.1. *Given a vector $v \in F$ in a stratum F there exists, a unique vector $w \in \partial F$ in the boundary of F, and a unique positive numeber $a > 0$ such that:*

$$v = av_F + w.$$

Proof. If v is a multiple av_F of v_F then $a > 0$ and $w = 0$. Otherwise we work in the 2-dimensional plane π spanned by v, v_F in which the intersection $F \cap \pi$ appears as an open *convex angle* limited by two half lines which are in ∂F, then in this 2 dimensional picture the statement is clear. □

Theorem 2.2. 1) *Given a simplex $S := F_1 < F_2 < \cdots < F_k < 0$ the vectors $v_1 := v_{F_1}, v_2 := v_{F_2}, \ldots v_k := v_{F_k}$ are linearly independent.*

2) *Let $C_S := \{\sum a_i v_i,\ a_i > 0\}$ the corresponding open simplicial cone. Then $V - 0$ is the disjoint union on the cones C_S.*

3) *Each stratum F is the union of the cones C_S where the first element of S is F.*

proof. We claim that all these statements are immediatete consequences of the previous Lemma. In fact let us take a vector $v \in V - 0$ then $v \in F_1$ where F_1 is a non 0 stratum uniquely determined.

By the previous Lemma $v = a_1 v_{F_1} + w_1$. If $w_1 = 0$ we stop otherwise $w_1 \in F_2 \neq 0$, $w_1 = a_2 v_{F_2} + w_2$, $a_2 > 0$ with $F_1 < F_2$. Continuing in this way we see that each point has a unque expression of the form

$$v = \sum a_i v_{F_i},\ a_i > 0,\ F_1 < F_2 < \cdots < F_k < 0.$$

□

Let us now consider, for each combinatorial simplex $S := F_1 < F_2 < \cdots < F_k$ the geometrical simplex

$$|S| := v_{F_1} * v_{F_2} * \cdots * v_{F_k}$$

convex envelope (or join) of the (independent[3]) vertices v_{F_i} (we now allow also the stratum 0).

Corollary 2.3. *The simpleces $|S|$ form a simplicial subdivision on a combinatorial ball $B_{\mathcal{F}}$ with boundary Π the union of simpleces $|S|,\ S := F_1 < F_2 < \cdots < F_k < 0$ not containing the vertex 0.*

The map $j : \mathbf{R}^+ \times \Pi \to \mathbf{V} - \mathbf{0},\ \mathbf{j}(\mathbf{a}, \mathbf{v}) := \mathbf{av}$ is a homeomorphism.

Proof. We have seen that the cones C_S decompose $V - 0$ on the other hand clearly the closure $\overline{C}_S$ of the cone C_S is the union:

$$\overline{C}_S = \cup_{T \subset S} C_T$$

this implies that the simplices of Π form a simplicial complex.

For the second part it is clearly enough to show that j is bijective, for this we construct the inverse. Given a point $v \in V - 0$ we have that v is uniquely of the form:

$$v = \{\sum a_i v_i,\ a_i > 0\} \in C_S$$

we set $a = \sum_i a_i$ and $w := \frac{v}{a}$ then $w \in \Pi$ and $j^{-1}(v) = (a, w)$. □

§3. Subspace arrangements

Let us consider again a polyhedral fan $\mathcal{F} = \{F_i\}_{i \in I}$ and consider a closed subset $X \subset V$ with $X = \cup_{i \in J} F_i$ a union of strata. Let $\mathcal{A} := V - X = \cup_{i \notin J} F_i$ also a union of strata.

Denote ass before by Π the simplicial realization of the complex of non 0 strata in $\mathcal{F}$ and let $\Pi_X,\ \Pi_X^{\perp}$ be the two full subcomplexes of Π with the vertices in X and in $\mathcal{A}$ respectively.

From the last corollary it follows that the homeomorphism $j^{-1} : V - 0 \to \mathbf{R}^+ \times \Pi$ maps $X - 0,\ \mathcal{A}$ respectively to $\mathbf{R}^+ \times \Pi_{\mathbf{X}},\ \mathbf{R}^+ \times (\Pi - \Pi_{\mathbf{X}})$.

By standard facts $\Pi_X^{\perp}$ is a deformation retract of $\Pi - \Pi_X$ and thus we obtain:

Theorem 3.1. *The open set $\mathcal{A} = V - X$ has the same homotopy type as $\Pi_X^{\perp}$.*

[3] in the sense of affine geometry

Let us see the implication of this discussion to the topology of subspace arrangements. If we consider an arrangement of subspaces $W := \{W_j\}$ contained in $L(\mathcal{H})$ we have that:

(1) The union $V_W := \cup W_j$ of the subspaces W_j, is a union of faces.

(2) The intersection of V_W with Π is the full subcomplex Π_W with vertices the vertices v_F, $F \subset V_W$ or $v_F \in V_W$.

(3) Under the homeomorohism j the open set $V - V_W$ corresponds to

$$\mathbf{R}^+ \times (\Pi - \Pi_\mathbf{W}).$$

Thus consider the *orthogonal* subcomplex to Π_W i.e. the full subcomplex $\Pi_W^\perp$ having the vertices $v_F \notin V_W$.

We obtain:

Corollary 3.2. *The open set $V - V_W$ has the same homotopy type as $\Pi_W^\perp$.*

Since we will need it in a moment let us see what happens for non essential arrangements. Assume thus that the intersection $\cap H_i = A$ is a linear subspace of codimension m.

Fix a linear complement B to A so that $V = A \oplus B$ then the hyperplanes H_i intersect B in an essential arrangement $L_B(\mathcal{H})$. The faces of $L(\mathcal{H})$ can be identified with $A \times G$ with G face of $L_B(\mathcal{H})$.

Then the open set $V - \cup_i H_i$ is homeomorphic to $A \times (B - \cup_i(B \cap H_i)$.

Thus again $V - \cup_i H_i$ has the same homotopy type as the polyhedron Π associated to the induced arrangement on B.

Proposition 3.3. *If $A = \cap_i H_i$ is a subspace of codimension m the geometric realization of the poset of faces of the arrangement is a combinatorial $m-$ball.*

Before passing to complex arrangements it is useful to analyze a cellular structure of the polyhedrons $\Pi, \Pi_W, \Pi_W^\perp$.

For this we need a little more notations. Given a face F let us define by $\langle F \rangle$ the linear span of F (we know that $\langle F \rangle \in L(\mathcal{H})$ and that F is a chamber of $\langle F \rangle$).

Consider furthermore the set of indeces $J_F : \{i \in I | F \subset H_i\}$.

$$\mathcal{H}_{J_F} := \{H_i | F \subset H_i\}.$$

This is typically a non essential arrangement and $\langle F \rangle = \cap_{i \in J_F} H_i$.

We have seen that Π is a combinatorial sphere and its join with 0, $B_{\mathcal{F}} = \Pi * 0$ a ball. More generally if F is a face consider the poset $\mathcal{L}_F$ of all faces G such that $F \geq G$ i.e. such that $F \subset \overline{G}$.

We claim that:

Lemma 3.4. *As a poset $\mathcal{L}_F$ is isomorphic to the poset of faces of the configuration $\mathcal{H}_{J_F}$ of hyperplanes containing F.*

Proof. Take a face $G \in \mathcal{L}_F$, from Proposition 1 we know that it is contained in a unique face of the subarrangement $L(\mathcal{H}_{J_F})$.

Conversely take one such face G which we know (always by the same proposition) is a union of faces in $F(\mathcal{H})$.

These faces differ only for the signs of the equations α_i which do not vanish on F. Since $F \subset \overline{G}$ we must have that $F \subset \bar{F}'$ where $F' \subset G$ is a face in $F(\mathcal{H})$. This face is unique since on this face the signs of the equations α_i which do not vanish on F must have the same sign as on F. □

From the previous proposition we get:

Corollary 3.5. *The nerve of the poset $\mathcal{L}_F$ is a triangulation of a combinatorial ball B_F of dimension the codimension of F.*

This fact has an important implication:

Theorem 3.6. *The boundary of B_F is the union of the B_G with $G < F$.*

$$\partial B_F = \cup_{G<F} B_G.$$

The balls B_F as F varies on all faces of the hyperplane arrangement give a cellular decomposition of the ball $B_{\mathcal{H}}$.

For any given subspace arrangement W (of the hyperplane arrangement) the polyhedron $\Pi_W^{\perp}$ is a sub cell complex given by the balls B_F as F varies on the faces F of the arrangement which are not contained in the union of the subspaces.

We will refer to B_F as the *cell dual to F*.

Product of arrangements. Before we pass to complex arrangements let us treat briefly a simple general construction. Civen two vector spaces V_1, V_2 and in each an arrangement of hyperplanes $\mathcal{H}^1, \mathcal{H}^2$ we can define the product arrangement $\mathcal{H}^1 \times \mathcal{H}^2$ in $V_1 \times V_2$ in the obvious way.

$$\mathcal{H}^1 \times \mathcal{H}^2 := \{H \times V_2,\ V_1 \times K | H \in \mathcal{H}_1,\ K \in \mathcal{H}_2\}.$$

One easily sees that the faces of this arrangemens are just products:

$$F(\mathcal{H}^1 \times \mathcal{H}^2) = \{F_1 \times F_2 | F_1 \in F(\mathcal{H}^1),\ F_2 \in F(\mathcal{H}^2)\}$$

as poset we have that $F(\mathcal{H}^1 \times \mathcal{H}^2)$ is the product $F(\mathcal{H}^1), \times F(\mathcal{H}^2)$ of the two posets with the product order $(a, b) \le (c, d)$ if and only if $a \le c,\ b \le d$.

§4. Complex arrangements

It is now the time to look at complex arrangements, i.e. arrangements of hyperplanes given by real equations in complex space, or the complexification of a real arrangement $\mathcal{H}$ in V.

Of course the idea is to treat such arrangements as subspace arrangements in a real space. More precisely in $V_{\mathbf{C}} = V + iV = V \times V$ the complex hyperplane of equation $\alpha_k(v + iw) = 0$ is the real codimension 2 subspace $\tilde{H}_k := H_k + iH_k$ (where $H_k = \{v \in V | \alpha_k(v) = 0\}$).

Therefore the subspaces $\tilde{H}_k$ are part of the hyperplane arrangement associated to the real hyperplanes $H_k + iV,\ V + iH_j$, in the notations of the previous paragraph this is in fact $\mathcal{H} \times \mathcal{H}$. One can therefore apply the previous theory to this hyperplane arrangement. There is on the other hand a much more efficient way to procede due to Salvetti and we describe this.

Given a face A of the hyperplane arrangement $\mathcal{H}$ consider the hyperplane arrangement $\mathcal{H}_A$ generated by the hyperplanes containing A we consider the set

$$CF(\mathcal{H}) := \{(A, B) |\ A \in F(\mathcal{H}),\ B \in F(\mathcal{H}_A)\}$$

of pairs (A, B) where A is a face in the original hyperplane arrangement $\mathcal{H}$ while B is a face of the subarrangement $\mathcal{H}_A$.

Proposition 4.1. 1) *The sets* $A \times B = A + iB,\ (A, B) \in CF(\mathcal{H})$ *decompose* $V_{\mathbf{C}} = V + iV$.

2) *The closure* $\overline{A + iB}$ *is a union of strata* $A_k + iB_k,\ (A_k, B_k) \in CF(\mathcal{H})$.

Proof. 1) We have a decomposition $V + iV = \cup_{A \in F(\mathcal{H})} A + iV$ and then a decomposition $A + iV = \cup_{B \in F(\mathcal{H}_A)} A + iB$.

2) We have for the closure $\overline{A + iB} = \overline{A} + i\overline{B} = \overline{A} \times \overline{B}$ and $\overline{A} = \cup A_k$ is a union of faces in $F(\mathcal{H})$ while $\overline{B} = \cup B_h$ is a union of faces in $F(\mathcal{H}_A)$.

Thus $\overline{A} \times \overline{B} = \cup_{k,h} A_k \times B_h$ now the decomposition of V into faces for $F(\mathcal{H}_{A_k})$ is a refinement of the decomposition of V into faces for $F(\mathcal{H}_A)$, since A_k is in the closure of A and so the set of hyperplanes containing A_k contains the set of hyperplanes containing A. $\square$

Therefore in a natural way the set of pairs $CF(\mathcal{H})$ is also a partially ordered set and we are going (as in §1) to represent its nerve as a simplicial complex.

Remark that also the strata $A \times B$, $(A, B) \in CF(\mathcal{H})$ are convex cones (open in their closure). Thus

Theorem 4.2. *The set of strata $A \times B$, $(A, B) \in CF(\mathcal{H})$ is a polyhedral fan.*

We have to understand now how the open set $\mathcal{A}$ complement of the complex hyperplane arrangement, appears in this picture.

Proposition 4.3. *$\mathcal{A}$ is the union of the faces $A + iB$, $(A, B) \in CF(\mathcal{H})$ with B open.*

Proof. A vector $a + ib$ is in $\mathcal{A}$ if and only if b is not contained in any of the hyperplanes of $\mathcal{H}$ in which a is contained. This describes exactly the union of the strata in $CF(\mathcal{H})$ described by the proposition. $\square$

Let us thus set

$$\mathcal{F}_{\mathbf{C}} := \{A + iB \mid (A, B) \in CF(\mathcal{H}) \quad \text{with} \quad B \quad \text{open}\}.$$

This is a poset and, if we fix a vertex in each stratum of $\mathcal{F}_{\mathbf{C}}$ and construct the corresponding simplicial complex $\Pi_{\mathbf{C}}$ we have, by Theorem 3.1.

Theorem 4.4. *The complement $\mathcal{A}$ of the complex hyperplane arrangement has the same homotopy type as that of the simplicial complex $\Pi_{\mathbf{C}}$ geometric realization of $\mathcal{F}_{\mathbf{C}}$.*

We want to describe now the natural cellular structure of the poset $\mathcal{F}_{\mathbf{C}}$.

Fix a face $(A, B) \in \mathcal{F}_{\mathbf{C}}$. We want to consider the poset of all faces $(C, D) \leq (A, B)$.

By definition $(C, D) \leq (A, B)$ means $A \subset \overline{C}$, $B \subset \overline{D}$. Since B, D are open sets this condition is in fact equivalent to:

$$A \subset \overline{C}, \ B \subset D$$

thus D is the unique chamber of the configuration of hyperplanes through C which contains B. In other words, given $(A, B) \in \mathcal{F}_{\mathbf{C}}$, the subposet

$$\mathcal{F}(A, B) := \{(C, D) \in \mathcal{F}_{\mathbf{C}} | (C, D) \leq (A, B)\}$$

of $\mathcal{F}_{\mathbf{C}}$ formed by all faces $(C, D) \leq (A, B)$ is isomorphic to the poset L_A of all faces C of the hyperplane arrangement with $C \leq A$. By Corollary 3.2 the nerve of the poset $\mathcal{F}(A, B)$ is a triangulation of a combinatorial disk $\Delta(A, B)$ of dimension the codimension of A.

By construction the boundary of this ball is also a union of balls relative to pairs $(C, D) < (A, B)$ and thus:

Corollary 4.5. *We have a cell complex structure on the polyhedron Π in which the cells $\Delta(A, B)$ of dimension k are indexed by elements $(A, B) \in \mathcal{F}_{\mathbf{C}}$ with A of codimension k.*

The boundary of $\Delta(A, B)$ is

$$\partial(\Delta(A, B)) = \cup_{(A', B') < (A, B)} \Delta(A', B').$$

§5. Reflection arrangements

We consider now an $n-$dimensional Euclidean space V and the arrangement of reflection hyperplanes of a finite Coxeter group W. By this we mean that W is a finite group generated bty reflections with respect to some hyperplanes H_i and the arrangement is formed by these H_i and also all their transforms under the group W.

We plan to describe the various polyhedra considered, for real and complex arrangements, in this case and in a W equivariant way.

We start from the real polyhedron.

We assume that the only fixed vector is 0.

Fix for every H_i in the arrangement an orthogonal vector α_i so that $H_i := \{v \in V | (\alpha_i, v) = 0\}$.

The elements $\pm\alpha_i$ play the same role as the roots of a root system. Fixing a vector v outside all hyperplanes H_i determines *positive roots* and a *fundamental chamber.*

From the theory of these groups one can choose $n-$independent reflection hyperplanes $H_i, i = 1, \ldots, n$ which are the walls of a chamber C which conventionally we will call the *fundamental chamber.*

$H_i := \{v \in V | (\alpha_i, v) = 0\}$ the elements α_i correspond for root systemes to simple roots.

Thus the chamber C is a simplicial cone

$$C := \{\sum_{i=1}^{n} a_i u_i | a_i > 0\}$$

$(\alpha_i, u_j) = \delta_i^j$.

The wall H_i is spanned by the u_j, $j \neq i$, the group W is generated by the reflections s_i relative to the walls H_i.

The closure $\overline{C} := \{\sum_{i=1}^{n} a_i u_i | a_i \geq 0\}$ of C is a fundamental domain for the action of W.

The stabilizer of a face F of C acts trivially on the face and it is generated by the simple reflections s_i relative to the walls H_i with $F \subset H_i$.

F is determined by a subset $J \subset I := \{1, \dots, n\}$ we will denote it by F_J and we denote by W_J the subgroup generated by the s_i, $i \in J$. W_J is also a reflection group which may also be realized as a reflection group on the subspace $\langle F \rangle^\perp$ orthogonal to the span of the face F. The fixed vectors of W_J form the span $\langle F \rangle$ of the face F.

Consider now a vector $v_0 \in C$ in the open chamber. By what we have said the orbit $W v_0$ gives rise to a point in each chambers and it is in 1-1 correspondence with W.

Denote by $v_w := w v_0$, $w \in W$. Let Δ be the convex hull of the points v_w. Then it is also true that the v_w span V and hence Δ is a convex polyhedral ball of dimension n. Clearly Δ is stable under W and since its extremal points are among the points $W v_0$ it follows that all these points are extremal.

For any face F_J of C let us set

$$v_J := \frac{1}{|W_J|} \sum_{w \in W_J} w v_0$$

the baricenter of the orbit of v_0 under W_J.

Lemma 5.1. *We have that $v_J \in F$ and v_J is the orthogonal projection of v_0 to the span $\langle F \rangle$ of the face F.*

Proof. v_J is fixed by W_J hence it is in $\langle F \rangle$, if we decompose $v_0 = u + z$, $u \in \langle F \rangle$, $z \in \langle F \rangle^\perp$ we have that $\sum_{w \in W_J} wz = 0$ and hence the claim $u = v_J$.

We still have to prove that $v_J \in F$. By induction it is enough to do it when F is a codimension 1 face of C. If H_i is the wall through F and

s_i the corresponding simple reflection $v_F = 1/2(v_0 + s_i v_0)$ and H_i is the only wall separating the two chambers $C, s_i(C)$ thus the signs (α_j, w) for $j \neq i$ do not change crossing this wall and we see that $(\alpha_j, v_F) > 0$ for $j \neq i$ and so $v_F \in F$. □

Every other face F' is uniquely W equivalent to a face F_J and if $F' = wF_J$ the element w lies in a coset wW_J and so

$$v_{F'} := wv_J$$

is well defined.

We have thus defined, for all faces F of the refection arrangement a vector v_F characterized by the following properties:

1) If $F = wG$ then $v_F = wv_G$.

2) If $F \subset \overline{G}$ then v_F is the orthogonal projection of v_G to $\langle F \rangle$.

We can now consider the simplicial complex Π associated to the vertices v_F and simpleces induced from the poset structure of the faces. We have that:

Theorem 5.2. *Π is a triangulation of the ball Δ convex hull of the points v_w.*

Proof. By construction all the vertices of this polyhedron are contained in Δ and so Π triangulates some polyhedron contained in Δ but now the faces of Δ are balls of the same type for smaller reflection systems for which the coincidence is by induction and this proves the claim. □

Remark 5.3. With the notations of §3 notice that, the cell dual to a face F is the convex envelope of the orbit under the reflection group generated by the hyperplanes through F of a point v_w in a chamber of which F is a face. Let us pass now to the complexified picture and to the open set $\mathcal{A}$.

From §4 we know that this is stratified by the set

$$\mathcal{F}_{\mathbf{C}} := \{A + iB \mid (A, B) \in CF(\mathcal{H}) \quad \text{with} \quad B \quad \text{open}\}.$$

Here A is a face of the reflection arrangement while B by the description of §4 is a chamber of the reflection arrangement generated by the hyperplanes containg A.

Proposition 5.4. *There exists a unique $J \subset I$ and a unique $w \in W$ such that*

$$w(A, B) := (wA, wB) = (F_J, wB), \; C \subset wB.$$

Proof. Since $\overline{C}$ is a fundamental domain there exists a $J \subset I$ and a $w \in W$ such that $w(A) = F_J$, the set of elements $\{w' \in W | w'(A) = F_J\}$ is the coset $W_J w$.

The chamber wB is one of the chambers of the reflection arrangement generated by the hyperplanes containg F_J and W_J acts simply transitively on these chambers, of which one and only one contains C the statement follows. □

We have now to choose judiciously the points $v_{(A,B)}$, $(A, B) \in CF(\mathcal{H})$ so that the resulting polyhedron is W stable. Since there is a unique $w \in W$ with $wA = F_J$, $wB \supset C$ we define

$$v_{(A,B)} := w(v_J + iv_0).$$

We obtain that:

Theorem 5.5. *The simplicial complex $\Pi_{\mathbf{C}}$ with vertices $v_{(A,B)} := w(v_J + iv_0)$ and simplices induced by the poset structure of $CF(\mathcal{H})$ is W stable moreover the homotopy equivalence between A and $\Pi_{\mathbf{C}}$ is W equivariant.*

Proof. The homeomorphism j is clearly W equivariant, but if we have a polyhedro Π a full subpolyhedron Π_X and its orthogonal $\Pi_X^{\perp}$ the deformation from $\Pi - \Pi_X$ to $\Pi_X^{\perp}$ is canonical along the rays joining a point in Π_X and in $\Pi_X^{\perp}$ so if we have a simplicial action of a group preserving these two polyhedra also the deformation is equivariant. □

We can finally use all this to analyze the homotopy type of $\mathcal{A}/W$. From what we have seen this is homotopically equivalent to $\Pi_{\mathbf{C}}/W$.

We have seen (last corollary of §4) that $\Pi_{\mathbf{C}}$ has a cellular structure in which the cells $\Delta(A, B)$ of dimension k are indexed by elements $(A, B) \in \mathcal{F}_{\mathbf{C}}$ with A of codimension k.

Given a set $J \subset I$ with k elements we have in particular the k cell

$$C_J : \Delta(F_J, B),\ C \subset B.$$

By the previous Proposition each cell is W equivalent to one and only one of the cells C_J. Therefore we deduce that the space $\Pi_{\mathbf{C}}/W$ is obtained in some way attaching these cells.

The simplest way to describe these attachments is the following.

Consider the n cell $\Delta(0, C)$ which is the simplicial complex with vertices $v_F + iv_0$ as F runs through the faces of the real arrangement and is isomorphic (also as simplicial complex) to the ball Δ of the real picture by projection to the real part. The cells C_J are contained in $\Delta(0, C)$ and thus under projection

$$\pi : \Delta(0, C) \to \Pi_{\mathbf{C}}/W$$

is surjective. A face $\Delta(F, D)$, $C \subset D$ is identifyed to a unique face C_J by the element $w \in W$ with $wF = F_J$, $C \subset wD$.

$D_J := wD$ is the unique face of the arrangement generated by F_J and containg C. Since we have already that $C \subset D$ we must have also $wC \subset D_J$.

Lemma 5.6. *The unique element $w_0 \in W$ such that $w_0 F = F_J$, $w_0 D = D_J$ where D is the unique face of the arrangement generated by F and containg C is the shortest element in the coset $W_J w$.*

Proof. The set of elements $w | wF = F_J$ is the coset $W_J w_0$. We claim that the shortest element on the coset is characterized by the fact that $l(s_i w_0) = l(w_0) + 1$ for all $i \in J$ and this in turn is equivalent to $w_0^{-1}(\alpha_i) > 0$ for all the roots α_i associated to the hyprplanes H_i, $i \in J$. Now $C := \{v | (\alpha_i, v) = \alpha_i(v) > 0, \ \forall i \in I\}$ while $D_J := \{v | \alpha_i(v) > 0, \ \forall i \in J\}$ and thus since $w_0 C \subset D_J$ we have for $i \in J$ that:

$$v \in C, \ (w_0^{-1}\alpha_i, v) = (\alpha_i, w_0 v) > 0.$$

$\square$

So we have the

Theorem 5.7. *The space $\Pi_{\mathbf{C}}/W$ which is of homotopy type of $\mathcal{A}/W$ is obtained brom the ball Δ identifying each face F with the face C_J in its W orbit, using the shortest element w in the coset $W_J w$ for which $W_J wF = C_J$.*

Let us draw some interesting consequence of this.

First of all we deduce immediately Brieskorn presentation by generators and relations of the generalized braid group.

The homotopy group of $\Pi_{\mathbf{C}}/W$ is computed by just considering the 1 and 2 cells. the 1 cells give a bouquet of circles, corresponding to the 1 faces joining v_0 to $s_i v_0$, we denote by T_i the corresponding loop oriented from v_0 to $s_i v_0$. Thus the T_i are generators for the homotopy group. The 2 cells give the relations. Given 2 nodes i, j of the Dynkin diagram we deduce a relation between T_i, T_j and it easily seen to be:

$$T_iT_j = T_jT_i, \ T_iT_jT_i = T_jT_iT_j, \ T_iT_jT_iT_j = T_jT_iT_jT_i,$$
$$T_iT_jT_iT_jT_iT_j = T_jT_iT_jT_iT_jT_i,$$

according if the two nodes are joined by $0, 1, 2, 3$ edges.

First of all let us look at the 1-dimensional cells which are of the ones of vertices $ws_i v_0, wv_0$. If $l(ws_i) = l(w) + 1$ then w^{-1} is the element of shortest length identifying the 1-cell with $s_i v_0, v_0$. The generator T_i is by definition the loop associated to the oriented edge $v_0, s_i v_0$. Thus the lift of T_i from the point wv_0 goes to the point $ws_i v_0$ along this edge.

Next consider the universal covering space $\pi : \tilde{\Pi} \to \Pi \to \Pi/W$ of $\Pi_{\mathbf{C}}/W$ and of Π. Lifting the cellular structure of Π we have a paving of $\tilde{\Pi}$ by cells which are permuted by the group of deck transformations.

We fix a cell C of amaximal dimension mapping to $\Delta(0, C)$ and a base point p_0 in C mapping to v_0. Thus we identify the group of deck transformations with the generalized braid group B using this base point.

Under the homeomorphism of C to $\Delta(0, C)$ the vertices wv_0 are in the orbit of p_0 under the group of deck transformations

$$wv_0 = \pi(T_w p_0)$$

and this defines a canonical lift T_w of w.

If $w = s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced expression the we claim that

$$T_w = T_{i_1} T_{i_2} \dots T_{i_k}.$$

In fact there is a path from v_0 to wv_0 given by the edges $[s_{i_k} v_0, v_0]$, $[s_{i_1} s_{i_2} \dots s_{i_k} v_0, s_{i_2} \dots s_{i_k} v_0]$ which maps in Π/W to a path giving the element $T_{i_1} T_{i_2} \dots T_{i_k}$ of the homotopy group.

Next we identify in C the copies of the C_J which we denote by the same symbols.

We have to fix an orientation for the cells C_J this can be done by ordering the vertices and then orienting the cells C_J so that if $K \subset J$, $|K| = k - 1$ is obtained removing the h^{th} element of J the oriented cell C_K appears in the boundary of C_J with the sign $\epsilon_{K,J} := (-1)^h$.

We have thus:

Theorem 5.8. 1) *The cells in $\tilde{\Pi}$ are simply transitive orbits of the cells C_J.*

2) *Denoting by $C_k(\tilde{\Pi})$ the group of $k-$dimensional cells, under the action of B this is a free $\mathbf{Z}[\mathbf{B}]$ module with basis the cells C_J, $|J| = k$.*

3) *The boundary of the cell C_J is the sum*

$$\sum_{K \subset J,\ |K| = k-1} \epsilon_{K,J} \Big(\sum_{w \in W_J / W_K} (-1)^{l(w)} T_w \Big) C_K.$$

Where T_w denotes the canonical lift of the element of shortest length w in the coset.

Proof. The statements 1), 2) follow from the construction as for 0) and 3) we have to note that each cell F which in $\Delta(0, C)$ is in the orbit of C_J under W in $\tilde{\Pi}$ is exactly $F = T_w C_J$ (under the group of deck transformations) this is easily verified by considering the minimal path from wv_0 to v_0 followed by the two segments joining wv_0, v_0 to the centers of the respective cells. The sign $(-1)^{l(w)}$ depends of the fact that the reflections s_i reverse the orientation of the fundamental cell. □

§6. Reflection groups

In [DS2] the authors generalize the previous analysis as follows. Start from the real reflection representation V and consider instead of the complexification, the space V^m for all m. On V^m the reflection group W acts and it acts freely on the open subspace U^m obtained by removing the subspaces H^m for each reflection hyperplane.

One has naturally a set of inclusions $U^m \subset U^{m+1} \ldots$ and a space U^∞ which by a simple dimension argument is contractible and hence $B_W := U^\infty / W$ is a classifying space for W.

The same method used for the complexification allows to stratify in a W equivariant way the space V^m by products $F_1 \times F_2 \times \cdots \times F_m$ where inductively:

F_1 is a face of the reflection arrangement and $Fi + 1$ is a face of the subarrangement generated by the hyperplanes which contain F_i. In this way one has a fan and U^m is a union of the strata $F_1 \times F_2 \times \cdots \times F_m$ with F_m open. Then a similar analysis gives a cellular structure on B_W. We refer to the original paper for details.

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