Advanced Studies in Pure Mathematics 33, 2002
Computational Commutative Algebra and Combinatorics
pp. 203–212

The Zarankiewicz Problem via Chow Forms

Marko Petkovšek, James Pommersheim and Irena Swanson

The well-known *Zarankiewicz problem* [Za] is to determine the least positive integer $Z(m,n,r,s)$ such that each $m \times n$ 0-1 matrix containing $Z(m,n,r,s)$ ones has an $r \times s$ submatrix consisting entirely of ones. In graph-theoretic language, this is equivalent to finding the least positive integer $Z(m,n,r,s)$ such that each bipartite graph on m black vertices and n white vertices with $Z(m,n,r,s)$ edges has a complete bipartite subgraph on r black vertices and s white vertices.

A complete solution of the Zarankiewicz problem has not been given. While exact values of $Z(m,n,r,s)$ are known for certain infinite subsets of m,n,r and s, only asymptotic bounds are known in the general case; for example, see Čulík [Č], Füredi [F], Guy [G], Hartmann, Mycielski and Ryll-Nardzewski [HMR], Hyltén-Cavallius [HC], Irving [I], Kövári, Sós and Turán [KST], Mörs [M], Reiman [Re], Roman [Ro], Znám [Zn]. Even the case $r = s = 2$ has not been answered in general. Here we quote some known facts about this case: Hartmann, Mycielski and Ryll-Nardzewski [HMR] proved the asymptotic bounds

$$c_1 n^{4/3} < Z(n,n,2,2) < c_2 n^{3/2}$$

for some constants $c_1 \cong \frac{3}{4}$ and $c_2 \cong 2$. Kövári, Sós and Turán [KST] proved that

$$Z(n,n,2,2) \le 2n + n^{3/2}, \quad \lim_{n\to\infty} n^{-3/2} Z(m,n,2,2) = 1.$$

Moreover, when p is a prime integer, [KST] proved that $Z(p^2+p, p^2, 2, 2) = p^2(p+1)+1$. Hyltén-Cavallius [HC] proved that $Z(m,n,2,2) \le \frac{n}{2} + \sqrt{nm(m-1) + \frac{n^2}{4}} + 1$. Čulík [Č] proved that for $n \ge \binom{m}{2}$,

$$Z(m,n,2,2) = \binom{m}{2} + n + 1.$$

Received May 16, 2000.
2000 AMS Subject Classification: 05C65, 13A15, 15A15.

Reiman [Re] showed the same equality for infinitely many other m and n, and he also established a connection between finding $Z(m,n,2,2)$ and the existence of projective planes of given orders. This last existence question is still wide open, and hence Reiman's work provides convincing evidence that finding $Z(m,n,2,2)$ for all m and n is a highly non-trivial problem. Guy [G] calculated $Z(m,n,2,2)$ for many small values of m and n. Further asymptotic and exact values were established in [Ro], [F].

This paper is an analysis of the $r = s = 2$ case of Zarankiewicz problem from the point of view of commutative algebra. Our motivation came from the complexity theory of permanental ideals of generic matrices. This brought forth a new connection between combinatorics, computational algebra, commutative algebra, and algebraic geometry involving not only permanental ideals, but also complexity of parameters and Chow forms. We describe these connections in the first two sections of this paper. In the final section, we also exhibit a connection with hypergraphs and three dimensional matrices. However, with these new connections we have not been able to shed any new light on the Zarankiewicz problem; we have simply found several reformulations.

§1. Permanental ideals and balanced matrices

We begin by introducing permanental ideals, parameters, and complexity of parameters via Chow forms. We present the Chow form for the permanental ideals, and rephrase the question of computing $Z(m,n,2,2)$ in terms of the complexity of parameter ideals and their Chow forms.

Let F be a field, and let X_{ij} be indeterminates over F, where $i = 1,\ldots,m$ and $j = 1,\ldots,n$, with $m,n \geq 2$. Let X be the $m \times n$ matrix whose ij-th entry is X_{ij}. The matrix X is the so-called generic $m \times n$ matrix. Let P be the ideal in the polynomial ring $F[X_{ij}|i,j]$ generated by all 2×2 subpermanents of X. Specifically,

$$P = (X_{ij}X_{i'j'} + X_{i'j}X_{ij'} | i < i' \leq m, j < j' \leq n).$$

Note that the permanent is like the determinant but with all minus signs replaced by plus signs. The ideal P is called the 2×2 permanental ideal of X.

Permanental ideals have not been studied a great deal. This is partly because that they do not seem to describe geometric properties, and partly because permanents are very difficult to compute. One can calculate the determinant of an $n \times n$ matrix in $O(n^3)$ steps, but for a permanent, many more steps are needed. Calculating the permanent

is in fact a #P-complete problem (see for example [V, Valiant], [B, Bürgisser]).

One measure of the complexity of an ideal is the sparsity or non-sparsity of a system of parameters modulo it. Not surprisingly, the parameters modulo the 2×2 permanental ideal are not sparse. The goal is to determine this complexity more precisely.

Definition 1. For an element $\sum_{ij} c_{ij} X_{ij}$ to be a *parameter modulo an ideal* I, it is necessary and sufficient that it avoids all the minimal primes of I. A *system of parameters modulo* I is a sequence of elements $a_1, \ldots, a_d$, where d is the dimension of the ring modulo I, such that for all $i = 1, \ldots, d$, a_i is a parameter modulo the ideal $I + (a_1, \ldots, a_{i-1})$.

A parameter is said to be *sparse* if most of the c_{ij} are zero. The *complexity of a parameter* is defined to be the number of nonzero c_{ij}. The *complexity of* I is then defined to be the smallest possible sum of all the complexities of the parameters in a system of parameters, as we vary the systems.

When $m = n = 2$, the permanental 2×2 ideal P is a prime ideal, so that any one of the four X_{ij} variables is a parameter. In this case the complexity of a single parameter is 1.

When $2 = m < n$, Laubenbacher and Swanson [LS] showed that an element $\sum_{ij} c_{ij} X_{ij}$ is a parameter modulo the permanental ideal P exactly when for each row i, at least one c_{ij} is nonzero, and for each 2×2 submatrix of X, at least one of the corresponding c_{ij} is nonzero. Thus, one can see easily that the complexity of a parameter modulo the 2×2 permanental ideal of a $2 \times n$ generic matrix is exactly $n - 1$.

Furthermore, when $m, n \geq 3$, again according to [LS], for $\sum_{ij} c_{ij} X_{ij}$ to be a parameter modulo the permanental ideal P, it is necessary that for each i, some c_{ij} is nonzero, similarly that for each j, some c_{ij} is nonzero, and lastly that for all $i < i' \leq m$, $j < j' \leq n$, at least one of $c_{ij}, c_{ij'}, c_{i'j}, c_{i'j'}$ is nonzero.

We say that a 0-1 matrix is *balanced* if every 2×2 submatrix (not necessarily contiguous) contains at least one unit element, and we let $f(m, n)$ be the minimal number of ones in a balanced $m \times n$ matrix. Note that

$$f(m, n) = mn - Z(m, n, 2, 2) + 1$$

where $Z(m, n, r, s)$ is the Zarankiewicz number.

It turns out that $f(m, n)$ equals the smallest possible complexity of a single parameter, as we prove below. It is clear from above that in the case $2 = m \leq n$, both of these numbers are $n - 1$, and similarly when

$2 = n \leq m$, both of these numbers are $m - 1$. In the sequel, we will assume (without loss of generality) that $3 \leq m, n$.

We first need a lemma:

Lemma 1. *Let $m, n \geq 3$, and let A be a balanced $m \times n$ matrix with f_A ones. Then there exists a balanced $m \times n$ matrix B with exactly f_A ones such that every row and every column of B contains at least one nonzero entry.*

Proof. Suppose that one of the rows or columns of A is zero. Without loss of generality we may assume that the first row of A is zero. As A is balanced, each of the rows $2, 3, \ldots, m$ must have at least $n - 1$ ones. Thus after possibly permuting the rows and columns of A, the first three rows are of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ * & 1 & \cdots & 1 & ** \end{bmatrix}.$$

Here, $*, **$ are either 1 or 0, but they are not both 0 since A is balanced. Let B be obtained from A by switching the $(1,1)$ and $(2,2)$ entries. Then B is still balanced with f_A ones, and every row, every column of B has at least one 1. Q.E.D.

Now we can show the connection between the complexity of parameters and $f(m, n)$:

Proposition 1. *Whenever $2 \leq m, n$, $f(m, n)$ equals the minimal possible complexity of a parameter modulo P.*

Proof. By the earlier discussion, we may take $3 \leq m \leq n$ without loss of generality. For each parameter $\sum_{ij} c_{ij} X_{ij}$, we form the balanced matrix A whose (i, j) entry equals 0 if $c_{ij} = 0$ and equals 1 otherwise. Notice that the balanced matrix A constructed in this way has the additional property that every row and every column of A contains a nonzero entry. Conversely, given a balanced matrix A, we first use Lemma 1 to convert it (non-uniquely) into a balanced matrix B such that every row and every column of B contains a nonzero entry, we then construct a parameter $\sum c_{ij} X_{ij}$ modulo the 2×2 permanental ideal by setting c_{ij} to be the ij entry of B. This element is indeed a parameter by [LS]. Q.E.D.

Thus finding $f(m, n)$, the minimal number of ones in A, is the same as finding the sparsest possible parameter for the polynomial ring modulo the permanental ideal. Hence all the values and bounds on $Z(m, n, 2, 2)$ listed at the beginning of the paper apply also for $mn + 1$ minus the smallest possible complexity of a parameter. Clearly, no parameter is sparse.

§2. Chow forms

There is another way to analyze the complexity of ideals, due to Eisenbud and Sturmfels [ES]:

Theorem 1 ([ES, Theorem 2.7]). *The complexity of an ideal I equals the least number of variables appearing in any initial monomial of the Chow form of the ring modulo I (under any monomial ordering).*

Some helpful references for Chow forms are [Sh], [ES], [GS].

Of course, calculating the complexity of an ideal is much more than calculating the complexity of a single parameter. However, the complexity of the ideal divided by the number of parameters gives an upper bound on the complexity of a parameter, and so by the previous section this is a step towards computing $Z(m, n, 2, 2)$. Thus, the problem is first to calculate the Chow form of the ring modulo P, and secondly, to find a monomial ordering on the variables under which the initial monomial of the Chow form involves the fewest number of variables.

In general, the computation of Chow forms is difficult, in technical terms even NP-hard (see [ES] for discussion). Even in the case of determinantal ideals, which tend to be much better behaved than permanental ideals, the Chow forms are difficult to compute. Glassbrenner and Smith [GS] analyzed the complexity of determinantal ideals by using the theorem of Eisenbud and Sturmfels quoted above. For the ideal of 2×2 minors of a generic $m \times n$ matrix, Glassbrenner and Smith [GS] determined that the parameter complexity is exactly mn. As the number of parameters in a parameter system for this ideal is $m+n-1$, this implies that we can choose the first parameter with at most $\frac{mn}{m+n-1}$ non-zero coefficients c_{ij}. However, as the determinantal ideal is prime, we may choose the first parameter to be any one of the variables, and hence the smallest possible complexity of a parameter is exactly 1. In contrast, the results on the Zarankiewicz problem quoted earlier show that for the 2×2 permanental ideal the complexity of a parameter is much larger.

Even though the complexity of the permanental Chow form and the complexity of permanental parameters are much larger than the corresponding complexities for determinants, here is at least one algebro-geometric problem that turns out to be easier for permanents than for determinants: namely, the computation of Chow forms.

The Chow form of an ideal is the product of the Chow forms of its minimal primes. In addition, the Chow form of the ideal I plus an ideal generated by variables is simply the Chow form of I. Putting these facts together, we get:

Proposition 2 (see [Sh]). *The Chow form of the permanental ideal P is the product of the Chow forms of all the ideals $J_{ii'jj'}$, with $i < i' \leq m, j < j' \leq n$, where $J_{ii'jj'} = (X_{ij}X_{i'j'} + X_{i'j}X_{ij'})$ (generated by a 2×2 permanent of X).*

The calculation of the Chow form of a principal ideal is straightforward (see for example [Sh]). In particular, to define the Chow form of $X_{ij}X_{i'j'} + X_{i'j}X_{ij'}$, we first introduce 12 new variables C_{lkp}, l varying from 1 to the dimension of the polynomial ring in the four given variables modulo the quadric (which is 3), and kp varying over the subscripts of the variables X above. Let $M_{ii'jj'}$ be the 3×4 generic matrix with indeterminates C_{lkp} each of whose rows contains the variables with the same first subscript and whose columns have the matching rest of the subscripts. Explicitly,

$$M_{ii'jj'} = \begin{bmatrix} C_{1ij} & C_{1i'j} & C_{1ij'} & C_{1i'j'} \\ C_{2ij} & C_{2i'j} & C_{2ij'} & C_{2i'j'} \\ C_{3ij} & C_{3i'j} & C_{3ij'} & C_{3i'j'} \end{bmatrix}.$$

The Chow form of $X_{ij}X_{i'j'} + X_{i'j}X_{ij'}$ is given by

$$R_{ii'jj'} = \Delta_{ij}\Delta_{i'j'} + \Delta_{i'j}\Delta_{ij'},$$

where Δ_{kp} is the determinant of the submatrix of $M_{ii'jj'}$ after removing the column corresponding to kp. We thus have:

Theorem 2. *The Chow form of P is $\prod_{i,i',j,j'} R_{ii'jj'}$.*

One can verify that each $R_{ii'jj'}$ is a linear combination of 66 distinct monomials of degree 6. Hence the Chow form is the product of $\binom{m}{2}\binom{n}{2}$ factors, each of which is a linear combination of 66 monomials of degree 6. Thus while the Chow form is relatively easy to get at, its expansion is far from computationally trivial.

By the Eisenbud-Sturmfels result (Theorem 1), we now have:

Theorem 3. *The parameter complexity of the ideal P equals the minimal number of distinct variables C_{lkp} appearing in any monomial in the expansion of the Chow form of P.*

This new formulation raises more questions than answers:

Question 1. *What monomials appear in the expansion of the Chow form of P? Is there a combinatorial representation of these monomials?*

Question 2. *What is the smallest possible number of distinct variables C_{lkp} such that a monomial appearing in the Chow form of P is a*

power product of exactly these variables? Also, what is the smallest possible number of distinct variables C_{1kp} such that a monomial appearing in the Chow form of P is a power product of these variables and variables $C_{2k'p'}, C_{3k'p'}$?

By Theorem 3) the answer to the first part of Question 2 is exactly the complexity of the ideal P. Furthermore, this number divided by 3 is an upper bound on the complexity of one parameter modulo P, and hence also an upper bound on $f(m,n)$.

A further question is then:

Question 3. *Does this upper bound on $mn+1-f(m,n)$ give a new lower bound on the Zarankiewicz number $Z(m,n,2,2)$?*

§3. Zarankiewicz problem in three dimensions and hypergraphs

It turns out that the monomials appearing in the Chow form of P are related to a certain *three-dimensional* Zarankiewicz problem. We now discuss this connection.

We will call a $3 \times m \times n$ 0-1 matrix *balanced* if (a) it contains no zero submatrix of size $2 \times 2 \times 1$, $2 \times 1 \times 2$, or $1 \times 2 \times 2$, and (b) none of the mn columns $\{A_{1,i,j}, A_{2,i,j}, A_{3,i,j}\}$ consists entirely of zeros. We define $g(m,n)$ to be the minimum number of ones in any balanced matrix of size $3 \times m \times n$.

Just as the (2-dimensional) Zarankiewicz problem can be phrased in the language of graph theory, so the above condition (a) can be expressed in terms of hypergraphs. Here we are looking for the minimum number of edges in the complete tripartite 3-graph $K_{3,m,n}$ with the property that the complement does not contain the tripartite 3-graph $K_{2,2,1}$. Condition (b) seems perhaps a little less natural. It is interesting to note, however, that condition (b) is similar to the extra condition that arose in our combinatorial interpretation of the (2-dimensional) Zarankiewicz problem: namely, that each column of the matrix should contain at least one 1. Lemma 1 showed that this extra condition was, in fact, redundant. However, this does not appear to follow easily in the 3-dimensional case.

For any monomial γ in the variables C_{lkp} ($l = 1 \ldots 3$, $k = 1 \ldots m$, $p = 1 \ldots n$), we can form a $3 \times m \times n$ 0-1 matrix, where a 1 in position (l,k,p) indicates that γ is divisible by C_{lkp}. We then have the following:

Theorem 4. *Any monomial that appears in the expansion of the Chow form of P determines a balanced $3 \times m \times n$ submatrix. Hence, the parameter complexity of the ideal P is greater than or equal to $g(m,n)$.*

Manifestly, a $3 \times m \times n$ matrix is balanced if and only if every $3 \times 2 \times 2$ submatrix of it is balanced. Thus, to prove the theorem, it is enough to prove the following lemma.

Lemma 2. *Any monomial appearing in the expansion of* $R_{ii'jj'} = \Delta_{ij}\Delta_{i'j'} + \Delta_{i'j}\Delta_{ij'}$ *determines a* $3 \times 2 \times 2$ *balanced submatrix of the matrix* (C_{lkp}).

Proof. For condition (a), there are three things to check. First consider the case of a $1 \times 2 \times 2$ submatrix. Without loss of generality, such a matrix corresponds to the four monomials $C_{1ij}, C_{1i'j}, C_{1ij'}, C_{1i'j'}$. Clearly, each of the four determinants Δ above will involve one of these monomials. Now consider a $2 \times 1 \times 2$ submatrix. Without loss of generality, such a matrix corresponds to the four monomials $C_{1ij}, C_{2ij}, C_{1ij'}$, $C_{2ij'}$. Any term in $\Delta_{i'j'}$ or in $\Delta_{i'j}$ contains one of these four monomials. Thus any term of $R_{ii'jj'}$ will contain one of these four, as well. The case of $2 \times 2 \times 1$ submatrices is similar.

To verify condition (b), we consider the monomials $C_{1ij}, C_{2ij}, C_{3ij}$. Clearly, any term in $\Delta_{i'j}$, $\Delta_{ij'}$, or $\Delta_{i'j'}$ contains one of these three monomials. Thus any term of $R_{ii'jj'}$ will also contain at least one of the three monomials above. Hence, condition (b) is satisfied. Q.E.D.

Given the inequality of Theorem 4, one might wonder if the complexity of ideal P is actually *equal* to $g(m,n)$. Indeed, the following converse of Lemma 2 is true. Given any $3 \times 2 \times 2$ balanced matrix M, there is a monomial occurring in the expansion of $\Delta_{ij}\Delta_{i'j'} + \Delta_{i'j}\Delta_{ij'}$ all of whose variables correspond to 1's in M. (One can check this, for example, by a tedious examination of cases.) Suppose that A is a balanced $3 \times m \times n$ matrix. It would follow from the converse of Lemma 2 that one of the $66^{\binom{m}{2}\binom{n}{2}}$ terms in the expansion of the Chow form of P consists entirely of variables corresponding to 1's in A. Thus if no monomial in this expansion cancels out entirely, then the parameter complexity of P is exactly $g(m,n)$. Unfortunately, it is not clear to us whether or not such cancelation can occur.

In any case, each of the three $1 \times m \times n$ submatrices of the $3 \times m \times n$ balanced matrix is a balanced $m \times n$ matrix. Thus $g(m,n) \geq 3f(m,n)$. This finally expands on the last question:

Question 4. *Is* $mn + 1 - \frac{1}{3}g(m,n)$ *a new lower bound on the Zarankiewicz number* $Z(m,n,2,2)$?

Acknowledgments. We thank Doron Zeilberger for discussions regarding this material, and MSRI for hospitality and support during the Fall of 1998 when this joint work started. The third author also thanks the National Science Foundation for partial support.

References

[BR] H. J. Bruck and R. H. Ryser, The nonexistence of certain finite projective planes, *Canad. J. Math.*, **1** (1949), 88–93.

[B] P. Bürgisser, The computational complexity to evaluate immanants and representations of general linear groups, preprint, 1998.

[Č] K. Čulík, Teilweise Lösung eines verallgemeinerten Problems von K. Zarankiewicz, *Ann. Soc. Polon. Math.*, **3** (1956), 165–168.

[ES] D. Eisenbud and B. Sturmfels, Finding sparse systems of parameters, *J. of Pure and Appl. Algebra*, **94** (1994), 143–157.

[F] Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Comb. Theory*, **75** (1996), 141–144.

[GS] D. Glassbrenner and K. E. Smith, Sparse systems of parameters for determinantal varieties, *Adv. in Appl. Math.*, **19** (1997), 529–558.

[G] R. K. Guy, A problem of Zarankiewicz, in Theory of Graphs, Proceedings of the Colloquium held at Tihany, Hungary, September 1966, edited by P. Erdös and G. Katona. Academic Press, New York, 1968, 119–150.

[HMR] S. Hartmann, J. Mycielski and C. Ryll-Nardzewski, Systèmes spéciaux de points á coordonées entières, *Colloq. Math.*, **3** (1954), 84–85.

[HC] C. Hyltén-Cavallius, On a combinatorial problem, *Colloq. Math.*, **6** (1958), 59–65.

[I] R. W. Irving, A bipartite Ramsey problem and the Zarankiewicz numbers, *Glasgow Math. J.*, **19** (1978), 13–26.

[KST] T. Kövári, V. T. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.*, **3** (1954), 50–57.

[LS] R. C. Laubenbacher and I. Swanson, Permanental ideals, 1998, to appear in *J. Symbolic Comput.*.

[M] M. Mörs, A new result on the problem of K. Zarankiewicz, *J. Comb. Theory*, **31** (1981), 126–130.

[Re] I. Reiman, Über ein Problem von K. Zarankiewicz, *Acta Math. Acad. Sci. Hungar.*, **9** (1958), 269–279.

[Ro] S. Roman, A problem of Zarankiewicz, *J. Comb. Theory*, **18** (1975), 187–198.

[Sh] I. R. Shafarevich, Basic algebraic geometry. Translated from the Russian by K. A. Hirsch. Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974. Springer Study Edition. Springer-Verlag, Berlin-New York, 1977.

[V] L. G. Valiant, The complexity of computing the permanent, *Theoret. Comp. Sci.*, **8** (1979), 189–201.

[Za] K. Zarankiewicz, Problem P 101, *Colloq. Math.*, **2** (1951), 301.

[Zn] Š. Znám, Two improvements of a result concerning a problem of K. Zarankiewicz, *Colloq. Math.*, **13** (1965), 255–258.

Marko Petkovšek
Department of Mathematics
University of Ljubljana
Jadranska 19
1000 Ljubljana, Slovenia
E-mail address: marko.petkovsek@fmf.uni-lj.si

James Pommersheim
Department of Mathematics
Pomona College
Claremont, CA 91711
U. S. A.
E-mail address: jpommersheim@pomona.edu

Irena Swanson
Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-8001
U. S. A.
E-mail address: iswanson@nmsu.edu