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On the Navier-Stokes equations with initial data nondecaying at space infinity

Yasunori Maekawa and Yutaka Terasawa

Abstract.

We will consider the Cauchy problem for the incompressible homogeneous Navier-Stokes equations in the d-dimensional Eucledian space with initial data in uniformly local L^p (L^p_{uloc}) spaces where p is larger than or equal to d. For the construction of the local mild solution of this, $L^p_{uloc} - L^q_{uloc}$ estimates for some convolution operators are important. So we explain these estimates here.

§1. Introduction

In this paper, we consider the Cauchy problem for the incompressible homogeneous Navier-Stokes equations with viscosity 1 in $\mathbb{R}^d$ where $d \geq 2$. The system of the equations is of the form

$$\text{(NS)} \qquad \left\{ \begin{array}{rcll} u_t - \Delta u + (u, \nabla)u + \nabla p & = & 0, & t > 0, \quad x \in \mathbb{R}^d, \\ \nabla \cdot u & = & 0, & x \in \mathbb{R}^d, \\ u(0, x) & = & u_0(x), & x \in \mathbb{R}^d, \end{array} \right.$$

where $u = (u^1, u^2, \cdots, u^d)$ is the unknown velocity vector field, p is the unknown pressure scalar field and $u_0 = (u_0^1, u_0^2, \cdots, u_0^d)$ is the given initial velocity satisfying $\nabla \cdot u_0 = 0$.

Our main purpose here is to solve (NS) for initial data which may not decay at space infinity but not necessarily be locally bounded. There are many works which construct mild solutions of the Navier-Stokes equations on various function spaces (e.g. [15], [2], [7], [14], [10], [11], [17], [5], [4], [6], [8], [16], [23], [3]). E. B. Fabes, B. F. Jones and N. M. Rivière [7], T. Kato [14], Y. Giga and T. Miyakawa [10] constructed mild solutions of (NS) with initial data in L^p space where p is larger than the space

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dimension d. Moreover in [14] and [10], the case $p = d$ is discussed. However, all functions in L^p spaces decay at space infinity when p is finite. When one considers nondecaying flows at space infinity as we would like to do, the function space for initial data should be a space of functions which do not decay at space infinity. The L^∞ space, considered by J. R. Cannon and G. H. Knightly [2], M. Cannone [4], Y. Giga, K. Inui and S. Matsui [8] is of course such a kind of function spaces, and both the Besov spaces with negative regularity considered by M. Cannone and Y. Meyer [5], M. Cannone [4], H. Kozono and M. Yamazaki [17], etc. and the function space considered by H. Koch and D. Tataru [16] are such kinds of function spaces too. However there is no work constructing mild solutions with initial data in uniformly local type spaces, which naturally contain functions which may not decay at space infinity. (For the definition of uniformly local Triebel-Lizorkin spaces and Besov Spaces, see, e.g., [27]). In this paper, we shall construct the mild solutions of (NS) with initial data in uniformly local L^p spaces where p is grater than or equal to the space dimension d. The method is quite similar to that of E. B. Fabes, B. F. Jones and N. M. Rivière [7], T. Kato [14], Y. Giga and T. Miyakawa [10] except that we use the convolution type estimate we newly obtain instead of Young's inequality for convolutions. Uniformly local L^p spaces consist of functions which are locally in L^p and its L^p norm in any Eucledian ball with radius 1 are uniformly bounded. When p is finite, they obviously contain functions which do not decay at space infinity but are not necessarily locally bounded.

Uniformly local L^p spaces were used by J. Ginibre and G. Velo [12] for complex Ginzburg-Landau equations and used by P. G. Lemarié-Rieusset [18], [19] and Y. Taniuchi [25] for equations of the fluid mechanics. In his work [18] and [19], Lemarié-Rieusset constructed in the three dimensional Euclidean space a suitable weak solution which is *local in time* with arbitrary initial data in uniformly local L^2 space. Furthermore he constructed a suitable weak solution which is *global in time* with arbitrary initial data in the closure of compactly supported smooth functions in uniformly local L^2 space. Y. Taniuchi [25] obtained uniformly local L^p estimates of vorticity equations. However he only considers L^p_{uloc}-L^p_{uloc} type estimates of convolution type operators, while we also treat $L^p_{uloc} - L^q_{uloc}$ type estimates of convolution type operators in which the indices p and q may be different. Let us be more precise. We consider the equations (NS) with initial data in $L^p_{uloc,\rho}$ space for any positive number ρ and any $p \in [d, \infty]$. When p is finite, the space $L^p_{uloc,\rho}(\mathbb{R}^d)$ is defined as follows.

$$(1)\quad L^p_{uloc,\rho}(\mathbb{R}^d) := \{f \in L^1_{loc}(\mathbb{R}^d);\ ||f||_{L^p_{uloc,\rho}} := \sup_{x\in\mathbb{R}^d}\Big(\int_{|x-y|<\rho}|f(y)|^p dy\Big)^{\frac{1}{p}} < \infty\}.$$

For simplicity of notations we set $L^\infty_{uloc,\rho}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$. When p is finite, the space $L^p_{uloc,\rho}$ naturally contains both the space L^p and the space L^∞. The space $L^p_{uloc,\rho}$ also contains all L^p-periodic functions, i.e., periodic functions which are locally p-th integrable in $\mathbb{R}^d$. We include the parameter ρ here, since the existence time estimate of the mild solutions can be different if ρ is different. Moreover varying ρ, we can reproduce T. Kato's global existence result for small initial data; more precisely, one can construct a unique mild solution globally in time if L^d norm of the initial data are sufficiently small.

To solve (NS) we convert the equations to the integral equation of the form

$$(2)\qquad u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)ds.$$

Here $e^{t\Delta}$ is the heat semigroup, $\mathbf{P}$ is the Helmholtz projection, and $f\otimes g := (f_i g_j)_{1\le i,j\le d}$ is a tensor product of $f = (f_1, f_2, \cdots, f_d)$ and $g = (g_1, g_2, \cdots, g_d)$. The solution of the integral equation (2) is called the mild solution of (NS) with initial data u_0. The precise meaning and well-definedness of each term follows from the $L^p_{uloc} - L^q_{uloc}$ type estimates of convolution type operators which we shall obtain. Now we would like to state our main results.

Theorem 1.1. *(Existence and uniqueness)*
(i) *Let $p \in (d,\infty]$ and $\rho > 0$. Then, for all $u_0 \in (L^p_{uloc,\rho}(\mathbb{R}^d))^d$ so that $\nabla\cdot u_0 = 0$, there exist a positive T and a unique mild solution $u \in L^\infty((0,T);(L^p_{uloc,\rho})^d)\cap C((0,T);(L^p_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0,T)\times\mathbb{R}^d$. The existence time T can be taken as it satisfies*

$$T^{\frac{1}{2}+\frac{d}{2p}}\rho^{-\frac{2d}{p}} + T^{\frac{1}{2}-\frac{d}{2p}} \ge \frac{\gamma}{||u_0||_{L^p_{uloc,\rho}}},$$

where γ is a positive constant depending only on d and p.

(ii) *Let $\rho > 0$. For all $u_0 \in (\overline{\cup_{p>d}L^p_{uloc,\rho}(\mathbb{R}^d)}^{||\cdot||_{L^d_{uloc,\rho}}})^d$ so that $\nabla\cdot u_0 = 0$, there exist a positive T and a mild solution $u \in L^\infty((0,T);(L^d_{uloc,\rho})^d)\cap C((0,T);(L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0,T)\times\mathbb{R}^d$. This solution may be chosen so that for all $T' \in (0,T)$ we have*

$$\sup_{0<t<T'} t^{\frac{1}{2}}||u(t,\cdot)||_{L^\infty} < \infty, \textit{ and } \lim_{t\to 0} t^{\frac{1}{2}}||u(t,\cdot)||_{L^\infty} = 0.$$

With this extra condition on the L^∞ norm, such a solution is unique. The existence time T can be taken as it satisfies

$$T \geq \min\{\rho^2,\ \alpha\},$$

where α is a positive number satisfying

$$\sup_{0<t<\alpha} t^{\frac{1}{4}}||e^{t\Delta}u_0||_{L^{2d}_{uloc,\rho}} \leq \gamma$$

where γ is a positive constant depending only on d.

(iii) *Let $\rho > 0$. There exists $\epsilon > 0$ such that for all $u_0 \in (L^d_{uloc,\rho})^d$ with $||u_0||_{L^d_{uloc,\rho}} \leq \epsilon$, there exist a positive T and a unique mild solution $u \in L^\infty((0,T);(L^d_{uloc,\rho})^d) \cap C((0,T);(L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0,T)\times\mathbb{R}^d$ so that $u(0,\cdot) = u_0$. The existence time T can be taken as it satisfies*

$$T^{\frac{1}{4}}\rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{||u_0||_{L^d_{uloc,\rho}}},$$

where γ is a positive constant depending only on d.

H. Koch and D. Tataru [16] showed that for any given $T > 0$, one can construct a local mild solution of (NS) which exists at least until time T if the bmo^{-1} norm of the initial data is sufficiently small. Especially, they constructed local mild solutions for any initial data in vmo^{-1}. They also showed that one can construct global mild solutions for small initial data in BMO^{-1}. The definitions of bmo^{-1}, vmo^{-1} and BMO^{-1} are the following.

For $f \in \mathcal{S}'(\mathbb{R}^n)$ (i.e., f is a tempered distribution), we set

$$||f||_{BMO_T^{-1}} := \sup_{x\in\mathbb{R}^n, 0<R^2<T} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \int_0^{R^2} |e^{t\Delta}f(y)|^2 dtdy \right)^{\frac{1}{2}},$$

where $B(x,R)$ is a ball centered at x with radius R and $|B(x,R)|$ is the Lebesgue measure of the ball $B(x,R)$. Then

$$\begin{aligned}
BMO^{-1} &:= \left\{f \in \mathcal{S}'(\mathbb{R}^n); ||f||_{BMO^{-1}} := ||f||_{BMO_\infty^{-1}} < \infty\right\},\\
bmo^{-1} &:= \left\{f \in \mathcal{S}'(\mathbb{R}^n); ||f||_{bmo^{-1}} := ||f||_{BMO_1^{-1}} < \infty\right\},\\
vmo^{-1} &:= \left\{f \in bmo^{-1}; \lim_{T\to 0} ||f||_{BMO_T^{-1}} = 0\right\}.
\end{aligned}$$

The inclusion relations of $L^p_{uloc,\rho}$, bmo^{-1}, vmo^{-1} and BMO^{-1} are as follows.

$$\begin{aligned} L^p_{uloc,\rho} &\subset vmo^{-1} \text{ if } p > d, \\ L^p_{uloc,\rho} &\subset bmo^{-1} \text{ if } p \geq d, \\ L^d &\subset BMO^{-1}. \end{aligned}$$

These relations can be proved using the result of M. E. Taylor [26], but we will reproduce them in a different way in Section 2. Related to mild solutions constructed by H. Koch and D. Tataru [16], H. Miura [21] showed some uniqueness theorems of mild solutions of (NS).

Although our function spaces $L^p_{uloc,\rho}$ when $d \leq p \leq +\infty$ are contained in bmo^{-1}, our results are not included in H. Koch and D. Tataru [16], since they impose the condition such as

$$\sup_{0<t<T} t^{\frac{1}{2}} ||u(t,\cdot)||_{L^\infty} < \infty$$

for solutions. We shall show that the uniqueness of the solution holds without this condition, if the initial data belongs to $L^p_{uloc}(\mathbb{R}^n)$ for some $p \in (n, \infty]$. Another advantage is that the definition of $L^p_{uloc,\rho}$ is very simple and it obviously contains some functions which may have singularities and may not decay at space infinity. Moreover, the convergence of mild solutions to initial data when time goes to zero are relatively simple in our case. For describing the convergence of mild solutions to initial data, we define the subspace $\mathcal{L}^p_{uloc,\rho}$ as the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^d)$ in the space $L^p_{uloc,\rho}$, i.e.,

$$\mathcal{L}^p_{uloc,\rho} := \overline{BUC(\mathbb{R}^d)}^{||\cdot||_{L^p_{uloc,\rho}}}. \tag{3}$$

Remark that the subspace $\mathcal{L}^\infty_{uloc,\rho}(\mathbb{R}^d)$ is the space $BUC(\mathbb{R}^d)$. The space $\mathcal{L}^p_{uloc,\rho}$ is useful since we can show that the solutions converge to the initial data in $L^p_{uloc,\rho}$ norm if the initial data belong to $\mathcal{L}^p_{uloc,\rho}$. In fact, we have the following theorem whose proof we do not state here.

Theorem 1.2. *(Convergence to initial data)*
(i) Let $p \in (d, \infty]$ and $\rho > 0$. Let $u \in L^\infty((0,T); (L^p_{uloc,\rho})^d)$ be a unique mild solution with initial data $u_0 \in L^p_{uloc,\rho}$. Then, for any compact set $K \subset \mathbb{R}^d$, we have

$$\lim_{t\to 0}||u(t)-u_0||_{L^p(K)}=0 \tag{4}$$

holds. Moreover,

$$\lim_{t\to 0}||u(t)-u_0||_{L^p_{uloc,\rho}}=0 \tag{5}$$

holds if and only if $u_0\in\mathcal{L}^p_{uloc,\rho}$.

(ii) Let $\rho>0$. *Let* u *be a unique mild solution in* $L^\infty((0,T);(L^d_{uloc,\rho})^d)$ *which satisfies* $t^{\frac12}u(t)\in L^\infty((0,T);(L^\infty)^d)$, $\lim_{t\to 0}t^{\frac12}||u(t)||_{L^\infty}=0$. *Then,*

$$\lim_{t\to 0}||u(t)-u_0||_{L^d(K)}=0 \tag{6}$$

holds. Moreover, under the above condition,

$$\lim_{t\to 0}||u(t)-u_0||_{L^d_{uloc,\rho}}=0 \tag{7}$$

holds if and only if $u_0\in\mathcal{L}^d_{uloc,\rho}$.

One of the keys to our results is $L^p_{uloc,\rho}$-$L^q_{uloc,\rho}$ estimates we newly obtain, which are the followings.

Let $1\le q\le p\le\infty$. Then for any $f\in L^p_{uloc,\rho}$, we have

$$||e^{t\Delta}f||_{L^p_{uloc,\rho}}\le\Big(\frac{C_1}{\rho^{d(\frac1q-\frac1p)}}+\frac{C_2}{t^{\frac d2(\frac1q-\frac1p)}}\Big)||f||_{L^q_{uloc,\rho}}, \tag{8}$$

$$||\nabla e^{t\Delta}f||_{L^p_{uloc,\rho}}\le\Big(\frac{C_3}{t^{\frac12}\rho^{d(\frac1q-\frac1p)}}+\frac{C_4}{t^{\frac d2(\frac1q-\frac1p)+\frac12}}\Big)||f||_{L^q_{uloc,\rho}}. \tag{9}$$

For $F\in(L^p_{uloc,\rho})^{d\times d}$,

$$||e^{t\Delta}\mathbf{P}\nabla\cdot F||_{L^p_{uloc,\rho}}\le\Big(\frac{C_5}{t^{\frac12}\rho^{d(\frac1q-\frac1p)}}+\frac{C_6}{t^{\frac d2(\frac1q-\frac1p)+\frac12}}\Big)||F||_{L^q_{uloc,\rho}}, \tag{10}$$

holds. Here $e^{t\Delta}$ is the heat semigroup, $\mathbf{P}$ is the Helmholtz projection, and C_1, C_3 , C_5 are positive constants depending only on d, and C_2, C_4, C_6 are positive constants depending only on d, p and q.

Let us state the outline of the proof. By rescaling, we may assume that $\rho = 1$. To obtain this estimate, we decompose $\mathbb{R}^d$ into countable cubes whose centers are lattice points, i.e.,

$$\mathbb{R}^d = \cup_{k \in \mathbb{Z}^d} S(k, \frac{1}{2}),$$

where $S(x, \theta) := \{y \ ; \ \max_{1 \le i \le d} |y_i - x_i| \le \theta\}$. We can decompose any measurable function f in $\mathbb{R}^d$ into

$$f(x) = \sum_{k \in \mathbb{Z}^d} \chi_{S(k, \frac{1}{2})}(x) f(x), \ a.e. \ x \in \mathbb{R}^d,$$

where $\chi_A(x)$ is the characteristic function of a subset A in $\mathbb{R}^d$. We decompose both a convolution kernel and a convoluted function in this way. Using Young's inequality for convolutions and the relation supp $f_1 * f_2 \subset$ suppf_1 + suppf_2 when $f_1 * f_2$ is well-defined, we can obtain the desired estimates. In the proof, we also use the technique of estimating certain sums by certain integrals. After our work was completed, we were informed of the work by J. M. Arriera, A. Rodriguez-Bernal, J. W. Cholewa and T. Dlotko [1]. They treated linear parabolic equations in uniformly local L^p spaces. In particular, they also established $L^p_{uloc} - L^q_{uloc}$ estimates for the heat semigroup and their proof is similar to ours. However, we shall obtain $L^p_{uloc} - L^q_{uloc}$ estimates for more general convolution kernels including $e^{t\Delta}$, $\nabla e^{t\Delta}$ and $e^{t\Delta}\mathbf{P}\nabla\cdot$. For details, see section 3.2.

This paper is organized as follows. In section 2, we will state some properties of $L^p_{uloc,\rho}$ spaces and $\mathcal{L}^p_{uloc,\rho}$ spaces. In section 3, we will prove our key $L^p_{uloc,\rho}$- $L^q_{uloc,\rho}$ estimates of convolution operators with integrable functions satisfying some conditions. Using these estimates, we can construct the local mild solutions which are smooth in time and space and can show Theorem 1.2. For this part, see [20].

§2. $L^p_{uloc,\rho}$ space and $\mathcal{L}^p_{uloc,\rho}$ space

In this section, we state several properties of the function spaces $L^p_{uloc,\rho}$ and $\mathcal{L}^p_{uloc,\rho}$. When $\rho = 1$, we write L^p_{uloc}, $\mathcal{L}^p_{uloc}$ instead of $L^p_{uloc,1}$, $\mathcal{L}^p_{uloc,1}$, respectively. We will often use the estimates for the operator $e^{t\Delta}$ in section 3.2 (Remark that the estimates obtained in section 3.2 are independent of the results in this section). The inclusion relations for $L^p_{uloc,\rho}$ are as follows.

Proposition 2.1.
(i) *For any $\rho_1, \rho_2 > 0$, we have*

$$L^p_{uloc,\rho_1} = L^p_{uloc,\rho_2}$$

with equivalent norms.
(ii) *For $1 \leq p_1 \leq p_2 \leq \infty$ and $\rho > 0$, we have*

$$L^{p_2}_{uloc,\rho} \subset L^{p_1}_{uloc,\rho}.$$

(iii) *For any $\rho > 0$ and $1 \leq p \leq \infty$, we have*

$$L^p \subset L^p_{uloc,\rho},$$

$$L^\infty \subset L^p_{uloc,\rho}.$$

(iv) *Let $\rho > 0$ and $p > d$. Then, we have*

$$L^p_{uloc,\rho} \subset vmo^{-1} \subset bmo^{-1}.$$

(v) *Let $\rho > 0$. Then, we have*

$$L^d_{uloc,\rho} \subset bmo^{-1}.$$

Proof. The proofs of (i), (ii) and (iii) are easy, so we omit them. We use $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for $e^{t\Delta}$ in section 3.2 to prove the assertions (iv) and (v). By Hölder's inequality,

$$\begin{aligned}\int_{|x-x_0|\leq\sqrt{s}} |e^{s\Delta}f|^2 dx &\leq \Big(\int_{|x-x_0|\leq\sqrt{s}} dx\Big)^{1-\frac{2}{p}} ||e^{s\Delta}f||^2_{L^p_{uloc,\sqrt{s}}} \\ &\leq Cs^{\frac{d}{2}(1-\frac{2}{p})}||f||^2_{L^p_{uloc,\sqrt{s}}},\end{aligned}$$

where C is a positive constant depending only on d and p. So we have

$$\begin{aligned}||f||^2_{BMO_R^{-1}} &\leq \sup_{0<t<R}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{\frac{d}{2}}}\int_0^t\int_{|x-x_0|\leq\sqrt{s}}|e^{s\Delta}f|^2dxds \\ &\leq C\sup_{0<t<R}\sup_{x_0\in\mathbb{R}^d}\frac{1}{t^{\frac{d}{2}}}\int_0^t s^{\frac{d}{2}(1-\frac{2}{p})}||f||^2_{L^p_{uloc,\sqrt{s}}}ds \\ &\leq C\sup_{0<t<R} t^{1-\frac{d}{p}}||f||^2_{L^p_{uloc,R}} \leq CR^{1-\frac{d}{p}}||f||^2_{L^p_{uloc,R}}.\end{aligned}$$

This implies that $L^p_{uloc,\rho} \subset bmo^{-1}$ if $p \geq d$ and $L^p_{uloc,\rho} \subset vmo^{-1}$ if $p > d$. So (iv) and (v) hold.

Remark 2.1. *The space L^∞ is not dense in L^p_{uloc}, hence not in $L^p_{uloc,\rho}$. More strongly, we have that the subset $\cup_{q>p} L^q_{uloc}$ is not dense in L^p_{uloc}. For example, let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a function such that* supp $\phi \subset B(0,1)$, $\int |\phi|^p = 1$ *and* $\phi \geq 0$ *where $B(0,1)$ is a unit ball whose center is the origin. Choose any countable points $\{x_n\}_{n\geq 1}$ such that $B(x_n,1) \cap B(x_m,1) = \emptyset$. Set $\phi_n(x) = n^{\frac{d}{p}} \phi(n(x-x_n))$. Then* supp $\phi_n \subset B(x_n, \frac{1}{n})$. *So if we set*

$$\tilde{\phi}(x) = \begin{cases} \phi_n(x), & \text{for } x \in B(x_n,1) \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$||\tilde{\phi}||_{L^p_{uloc}} = \sup_n ||\phi_n||_{L^p(B(x_n,1))} = 1.$$

Let $q > p$. For any $g \in L^q_{uloc}(\mathbb{R}^d)$, we have $||\tilde{\phi} - g||_{L^p_{uloc}} \geq 1$. Indeed,

$$\begin{aligned} ||\tilde{\phi} - g||_{L^p_{uloc}} &\geq ||\phi_n - g||_{L^p(B(x_n,1))} \\ &\geq ||\phi_n - g||_{L^p(B(x_n,\frac{1}{n}))} \\ &\geq \big| ||\phi_n||_{L^p(B(x_n,\frac{1}{n}))} - ||g||_{L^p(B(x_n,\frac{1}{n}))} \big| \\ &\geq 1 - |B(x_n,\tfrac{1}{n})|^{\frac{1}{p}(1-\frac{p}{q})} ||g||_{L^q_{uloc}} \\ &\geq 1 - \frac{C}{n^{d(\frac{1}{p}-\frac{1}{q})}} ||g||_{L^q_{uloc}} \\ &\to 1 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that $\tilde{\phi}$ does not belong to $\overline{\cup_{q>p} L^q_{uloc}}^{||\cdot||_{L^p_{uloc}}}$.

We have the following characterizations of $\mathcal{L}^p_{uloc,\rho}$.

Proposition 2.2.
For any $\rho > 0$, the following three statements are equivalent.
(i) $f \in \mathcal{L}^p_{uloc,\rho}$.
(ii) $\lim_{|y|\to 0} ||f(\cdot + y) - f(\cdot)||_{L^p_{uloc,\rho}} = 0$.
(iii) $\lim_{t\to 0+} ||e^{t\Delta} f - f||_{L^p_{uloc,\rho}} = 0$.

Proof. Without loss of generality, we may assume that $\rho = 1$. For $p = \infty$, the above assertion is well-known so we omit it.

Let $p \in [1, \infty)$. From the estimates (25) and (26) in section 3.2, $e^{t\Delta} f$ is well-defined and belongs to BUC when $t > 0$ for any $f \in L^p_{uloc}$. This implies $(iii) \Rightarrow (i)$.

If $f \in \mathcal{L}^p_{uloc}$, there exists a sequence $\{f_n\}_{n \geq 1} \subset BUC$ such that $f_n \to f$ in L^p_{uloc}. So we have

$$\begin{aligned} ||f(\cdot + y) - f(\cdot)||_{L^p_{uloc}} &\leq ||f(\cdot + y) - f_n(\cdot + y)||_{L^p_{uloc}} \\ &\quad + ||f_n(\cdot + y) - f_n(\cdot)||_{L^p_{uloc}} \\ &\quad + ||f_n(\cdot) - f(\cdot)||_{L^p_{uloc}} \\ &\leq 2||f(\cdot) - f_n(\cdot)||_{L^p_{uloc}} \\ &\quad + |B|^{\frac{1}{p}} ||f_n(\cdot + y) - f_n(\cdot)||_{L^\infty}, \end{aligned}$$

where $|B|$ is a Lebesgue measure of the unit ball in $\mathbb{R}^d$. The above inequality shows that $(i) \Rightarrow (ii)$ holds. Let us show $(ii) \Rightarrow (iii)$. Since

$$e^{t\Delta} f - f = \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4}} \big(f(x - \sqrt{t}z) - f(x)\big) dz,$$

we have

$$||e^{t\Delta} f - f||_{L^p_{uloc}} \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}} ||f(\cdot - \sqrt{t}z) - f(\cdot)||_{L^p_{uloc}} dz.$$

So (iii) follows from (ii) and Lebesgue's convergence theorem.

Remark 2.2. *This characterization was also obtained in* [1].

Remark 2.3. *The characterization* (ii) *in the above proposition is convenient to check whether a function f belongs to $\mathcal{L}^p_{uloc,\rho}$ or not. The equivalence between* (i) *and* (iii) *is useful when we consider the convergence of a mild solution to the initial data.*

§3. Mild solutions in $L^p_{uloc,\rho}$ and $\mathcal{L}^p_{uloc,\rho}$

We will state main results of this paper in this section. In section 3.1, we will define mild solutions, i.e., solutions of the integral equations for the Navier-Stokes equations.

3.1. Definition of mild solutions

First, we define the convolution operator $e^{t\Delta}\mathbf{P}\nabla\cdot$ which appears in the nonlinear term of the integral equations of the Navier-Stokes equations. Notice that we regard $e^{t\Delta}\mathbf{P}\nabla\cdot$ as one operator. So we do not treat the operators $e^{t\Delta}\nabla\cdot$ and $\mathbf{P}$ separately.

Definition 3.1. *We define the operator $e^{t\Delta}\mathbf{P}\nabla\cdot$ (where $\mathbf{P}$ is the Helmholtz projection) as follows.*

For any $F \in (L^p_{uloc}(\mathbb{R}^d))^{d\times d}$, $1 \le p \le \infty$, we define

$$(e^{t\Delta}\mathbf{P}\nabla\cdot)F := \sum_{i=1}^{d}(e^{t\Delta}\mathbf{P}\frac{\partial}{\partial x_i})F\mathbf{e_i}, \tag{11}$$

where $\{\mathbf{e_i}\}_{i=1}^d$ is the standard bases in $\mathbb{R}^d$ and the matrix-valued convolution kernel of the operator $(e^{t\Delta}\mathbf{P}\frac{\partial}{\partial x_i})$ is defined as

$$\Big(\frac{1}{t^{\frac{d}{2}}}\partial_i\big(K_{j,k}(\frac{x}{\sqrt{t}})\big) + \partial_i G_t(x)\delta_{j,k}\Big)_{1\le j,k\le d},$$

where $G_t(x)$ is the Gauss kernel

$$G_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}}\exp(-\frac{|x|^2}{4t})$$

and

$$K_{j,k} := -\mathcal{F}^{-1}(\frac{\xi_j\xi_k}{|\xi|^2}\exp(-|\xi|^2)).$$

Here $\mathcal{F}^{-1}$ is the inverse Fourier transform

$$\mathcal{F}^{-1}g(x) := (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} g(\xi)e^{ix\cdot\xi}d\xi.$$

The following lemma is the pointwise estimate of the function $K_{j,k}(x)$. This pointwise estimate was originally obtained by C. W. Oseen [22] (see also P. G. Lemarié-Rieusset [19], and Y. Shibata and S. Shimizu [24]). But we shall give the proof of this estimate for the sake of completeness.

Lemma 3.1. *Let $K_{j,k}(x)$ be the function defined as above. Then, for any non-negative multi-index $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that*

$$|\partial_x^\alpha K_{j,k}(x)| \le \frac{C_\alpha}{(1+|x|)^{d+|\alpha|}}, \tag{12}$$

where C_α depends only on d and α.

Next we define the mild solutions of (NS) as the solutions of the integral equations associated with (NS).

Definition 3.2. *Let $p \geq d$. The function $u \in L^\infty((0,T);(L^p_{uloc,\rho})^d)$ is called a mild solution of the Navier-Stokes equations on $(0,T)\times\mathbb{R}^d$ if there exists $u_0 \in L^p_{uloc,\rho}$ with $\operatorname{div} u_0 = 0$ such that*

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)ds \tag{13}$$

holds.

Remark 3.1. *By the $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates in the following section, the term $e^{t\Delta}u_0$ and $e^{(t-s)\Delta}\mathbf{P}\nabla\cdot(u\otimes u)$ can be defined pointwise if $u_0 \in (L^p_{uloc,\rho})^d$ and $u \in (L^p_{uloc,\rho})^d$ where $p \geq 1$.*

3.2. $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates

In this section, we shall prove the key estimates for the convolution operators, $e^{t\Delta}$, $\nabla e^{t\Delta}$, and $e^{t\Delta}\mathbf{P}\nabla\cdot$. Using these, we can construct the local mild solutions of the Navier-Stokes equations in a similar method to T.Kato [14] and can get Theorem 1.1. (See also P. G. Lemarié-Rieusset [19].) We shall omit the details. Here we shall give the proof of the estimates for a certain class of convolution operators, which includes the three operators mentioned above. We say that a function is radial decreasing if it is radial symmetric and nonincreasing.

Theorem 3.1.
Let $1 \leq q \leq p \leq \infty$. Let $F(x)$, $H(x)$ be two real-valued functions in $\mathbb{R}^d$ and let $|F(x)| \leq H(x)$ hold. Furthermore, assume that H is a bounded, integrable and radial decreasing function in $\mathbb{R}^d$.

We set $F_{t,m}(x) = t^{-\frac{d}{2}-m}F(x/t^{\frac{1}{2}})$ for $t>0$, $m \geq 0$.

Then, for any function $g \in L^q_{uloc,\rho}(\mathbb{R}^d)$, we can define pointwise

$$F_{t,m} * g(x) = \int_{\mathbb{R}^d} F_{t,m}(x-y)g(y)dy.$$

Furthermore we have the estimate

$$(14)\qquad ||F_{t,m} * g||_{L^p_{uloc,\rho}} \le \Big(\frac{C_1||H||_{L^1}}{t^m \rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2||H||_{L^r}}{t^{m+\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)||g||_{L^q_{uloc,\rho}},$$

where r is the number satisfying $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$, and C_1, C_2 are positive constants depending only on d.

Proof. It suffices to prove that

$$(15)\qquad ||H_{t,0} * g||_{L^p_{uloc,\rho}} \le \Big(\frac{C_1||H||_{L^1}}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2||H||_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)||g||_{L^q_{uloc,\rho}},$$

Moreover, by rescaling we may assume that $\rho = 1$. Indeed, if we set $f_\rho(x) = \rho f(\rho x)$, then we easily see the relations

$$\begin{aligned}
(16)\qquad ||f_\rho||_{L^p} &= \rho^{1-\frac{d}{p}}||f||_{L^p},\\
(17)\qquad ||f_\rho||_{L^p_{uloc}} &= \rho^{1-\frac{d}{p}}||f||_{L^p_{uloc,\rho}},\\
(18)\qquad ||f * g||_{L^p_{uloc,\rho}} &= \rho^{\frac{d}{p}-1}||(f * g)_\rho||_{L^p_{uloc}} = \rho^{\frac{d}{p}+d-2}||f_\rho * g_\rho||_{L^p_{uloc}}.
\end{aligned}$$

So if we have the estimate (15) for $\rho = 1$, then

$$\begin{aligned}
||H_{t,0} * g||_{L^p_{uloc,\rho}} &= \rho^{\frac{d}{p}+d-2}||(H_{t,0})_\rho * g_\rho||_{L^p_{uloc}}\\
&\le \rho^{\frac{d}{p}+d-2}\Big(C_1||H_\rho||_{L^1} + \frac{C_2||H_\rho||_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)||g_\rho||_{L^q_{uloc}}\\
&= \rho^{\frac{d}{p}+d-2}\Big(C_1\rho^{1-d}||H||_{L^1}\\
&\quad +\rho^{1-\frac{d}{r}}\frac{C_2||H||_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)\rho^{1-\frac{d}{q}}||g||_{L^q_{uloc,\rho}}\\
&= \Big(\frac{C_1||H||_{L^1}}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2||H||_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)||g||_{L^q_{uloc,\rho}}.
\end{aligned}$$

First we decompose $\mathbb{R}^d$ into countable cubes whose centers are lattice points, sides are of length one and parallel to the axes, i.e.,

$$\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} S(k, \frac{1}{2}),$$

where $S(x,\theta) := \{y\ ;\ \max_{1\le i\le d}|y_i - x_i| \le \theta\}$. Remark that $|S(k,\frac{1}{2}) \cap S(l,\frac{1}{2})| = 0$ if $k \ne l$, where $|A|$ is a Lebesgue measure of any Euclidean subset A.

One can easily check that there exist positive constants C and C' depending only on d such that

(19)
$$C \sup_{x\in\mathbb{R}^d} ||f||_{L^p(B(x,1))} \le \sup_{k\in\mathbb{Z}^d} ||f||_{L^p(S(k,\frac{1}{2}))} \le C' \sup_{x\in\mathbb{R}^d} ||f||_{L^p(B(x,1))},$$

where $B(x,\theta) := \{y \ ; \ |y-x| < \theta\}$. We now decompose the function $H_{t,0}(x)$ as

$$H_{t,0}(x) = \sum_{k\in\mathbb{Z}^d} \chi_{S(k,\frac{1}{2})}(x) H_{t,0}(x), \ \textit{a.e. } x,$$

and

$$g(x) = \sum_{k\in\mathbb{Z}^d} \chi_{S(k,\frac{1}{2})}(x) g(x), \ \textit{a.e. } x.$$

Let the number r satisfy $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$.

Note that supp $f * g \subset$ supp f+supp g when $f * g$ is well-defined. For any $k \in \mathbb{Z}^d$, we have

$$\begin{aligned} & ||H_{t,0} * g||_{L^p(S(k,\frac{1}{2}))} \\ = \ & ||\Big(\sum_{k'\in\mathbb{Z}^d} \chi_{S(k',\frac{1}{2})} H_{t,0}\Big) * \Big(\sum_{k''\in\mathbb{Z}^d} \chi_{S(k'',\frac{1}{2})} g\Big)||_{L^p(S(k,\frac{1}{2}))} \\ = \ & || \sum_{k',k''\in\mathbb{Z}^d} \big(\chi_{S(k',\frac{1}{2})} H_{t,0}\big) * \big(\chi_{S(k'',\frac{1}{2})} g\big)||_{L^p(S(k,\frac{1}{2}))}. \end{aligned} \tag{20}$$

Since $\big(\chi_{S(k',\frac{1}{2})} H_{t,0}\big) * \big(\chi_{S(k'',\frac{1}{2})} g\big)$ is well-defined and

$$\mathrm{supp}(\chi_{S(k',\frac{1}{2})} H_{t,0}\big) * \big(\chi_{S(k'',\frac{1}{2})} g\big) \subset S(k'+k'',1)$$

holds, we have, by Young's inequality,

$$\begin{aligned} (\textit{R.H.S.}) \text{ of } (20) \ & \le \sum_{\substack{k',k''\in\mathbb{Z}^d \\ \max|k'_i+k''_i-k_i|\le 1}} ||\chi_{S(k',\frac{1}{2})} H_{t,0}||_{L^r} ||\chi_{S(k'',\frac{1}{2})} g||_{L^q} \\ & \le \sum_{k'\in\mathbb{Z}^d} 3^d ||\chi_{S(k',\frac{1}{2})} H_{t,0}||_{L^r} \sup_{k''\in\mathbb{Z}^d} ||g||_{L^q(S(k'',\frac{1}{2}))}. \end{aligned} \tag{21}$$

So we have to estimate the quantity $\sum_{k'\in\mathbb{Z}^d} ||\chi_{S(k',\frac{1}{2})} H_{t,0}||_{L^r}$.

$$\begin{aligned} \sum_{k\in\mathbb{Z}^d} ||\chi_{S(k,\frac{1}{2})} \frac{1}{t^{\frac{d}{2}}} H(\frac{x}{\sqrt{t}})||_{L^r} & = \sum_{k\in\mathbb{Z}^d} \Big(\int_{S(k,\frac{1}{2})} (\frac{1}{t^{\frac{d}{2}}} H(\frac{x}{\sqrt{t}})^r dx\Big)^{\frac{1}{r}} \\ & = \sum_{k\in\mathbb{Z}^d} \Big(\int_{S(\frac{k}{\sqrt{t}},\frac{1}{2\sqrt{t}})} (\frac{1}{t^{\frac{d}{2}}} H(y))^r t^{\frac{d}{2}} dy\Big)^{\frac{1}{r}}. \end{aligned} \tag{22}$$

We set $y_{k,t} \in \mathbb{R}^d$ as the closest point to the origin in $S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})$. In the right-hand side of (22), since $H(y)$ is radial decreasing, if k satisfies $\max_{1\le i\le d}|k_i| \ge 2$, we can estimate as

$$(\int_{S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})} H(y)^r dy)^{\frac{1}{r}} \le (\frac{1}{\sqrt{t}})^{\frac{d}{r}} H(y_{k,t}).$$

The sum of the other terms (that is, k satisfies $\max_{1\le i\le d}|k_i| \le 1$) can be bounded from above $3^d \|\frac{1}{t^{\frac{d}{2}}} H(\frac{x}{\sqrt{t}})\|_{L^r}$. So the right-hand side of (22) can be estimated as

$$\begin{aligned}
(R.H.S.) \text{ of } (22) &\le \frac{1}{t^{\frac{d}{2}-\frac{d}{2r}}} \sum_{\substack{k\in\mathbb{Z}^d \\ \max|k_i|\ge 2}} (\frac{1}{\sqrt{t}})^{\frac{d}{r}} H(y_{k,t}) + 3^d \|\frac{1}{t^{\frac{d}{2}}} H(\frac{\cdot}{\sqrt{t}})\|_{L^r} \\
&\le \sum_{\substack{k\in\mathbb{Z}^d \\ \max|k_i|\ge 2}} (\frac{1}{\sqrt{t}})^d H(y_{k,t}) + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}} \|H\|_{L^r}. \qquad (23)
\end{aligned}$$

We now consider the estimate of the term

$$\sum_{\substack{k\in\mathbb{Z}^d \\ \max|k_i|\ge 2}} (\frac{1}{\sqrt{t}})^d H(y_{k,t}).$$

We draw a line from the origin to the point $y_{k,t}$. This line intersects several cubes of the form $\big(S(\frac{l}{\sqrt{t}}, \frac{1}{2\sqrt{t}})\big)^i$ (inside of the cube) where $l \in \mathbb{Z}^d$. We order these cubes as follows. The first cube is the cube $S(0, \frac{1}{2\sqrt{t}})$. The second cube is the cube which the line meets when it goes out of the first cube, and so on. We correspond the cube $S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})$ to the second last cube in this order. We denote this correspondence I_t, i.e., $I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))$ is the second last cube in this order.

Then all points in the cube $I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))$ are closer to the origin than the point $y_{k,t}$. Hence

$$(\frac{1}{\sqrt{t}})^d H(y_{k,t}) \le \int_{I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))} H(y)dy,$$

since H is radial decreasing. It is easy to see that

$$\#\{k' \in \mathbb{Z}^d \ ; \ I_t(S(\frac{k'}{\sqrt{t}}, \frac{1}{2\sqrt{t}})) = S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))\} \le 5^d$$

for any $k \in \mathbb{Z}^d$ with $\max_{1\le i\le d}|k_i| \ge 2$. So

$$(R.H.S.)\text{ of }(23) \le \sum_{\substack{k\in\mathbb{Z}^d\\ \max|k_i|\ge 2}} \int_{I_t(S(\frac{k}{\sqrt{t}},\frac{1}{2\sqrt{t}}))} H(y)dy + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}}||H||_{L^r}$$

$$\le 5^d \int_{\mathbb{R}^d} H(y)dy + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}}||H||_{L^r}$$

$$\le 5^d||H||_{L^1} + \frac{3^d}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}||H||_{L^r}. \tag{24}$$

This completes the proof.

Proposition 3.1. *For $F_1(x) := \exp(-|x|^2/4)$, $F_2(x) := |x|\exp(-|x|^2/4)$, and $F_3(x) := \frac{1}{(1+|x|)^{d+1}}$, there exist positive, bounded, integrable and radial decreasing functions H_1, H_2, H_3 satisfying $F_i \le H_i$ $(i = 1, 2, 3)$.*

Proof. Clearly, we can take $H_1 = F_1$ and $H_3 = F_3$. For F_2, if we compute the derivative of $|x|\exp(-|x|^2/4)$, we can see that it suffices to set $H_2(x) = \sqrt{\frac{2}{e}}$ if $|x| < \sqrt{2}$, $H_2(x) = |x|\exp(-|x|^2/4)$ if $|x| \ge \sqrt{2}$.

From Theorem 3.1 and the proposition above, we can establish $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for the linear term and the nonlinear term in the integral equations (13).

Corollary 3.1.
Let $1 \le q \le p \le \infty$. Then for any $f \in L^p_{uloc,\rho}$, we have

$$||e^{t\Delta}f||_{L^p_{uloc,\rho}} \le \left(\frac{C_1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\right)||f||_{L^q_{uloc,\rho}}, \tag{25}$$

$$||\nabla e^{t\Delta}f||_{L^p_{uloc,\rho}} \le \left(\frac{C_3}{t^{\frac{1}{2}}\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_4}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\right)||f||_{L^q_{uloc,\rho}}. \tag{26}$$

For $F \in (L^p_{uloc,\rho})^{d\times d}$,

$$||e^{t\Delta}\mathbf{P}\nabla\cdot F||_{L^p_{uloc,\rho}} \le \left(\frac{C_5}{t^{\frac{1}{2}}\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_6}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\right)||F||_{L^q_{uloc,\rho}}, \tag{27}$$

holds. Here, C_1, C_3, C_5 are positive constants depending only on d, and C_2, C_4, C_6 are positive constants depending only on d, p and q.

Proof. Combining Theorem 3.1, Proposition 3.1, and Lemma 3.1, we can easily get the estimates above. We omit the details.

Remark 3.2. *From Lemma 3.1 and pointwise estimates for derivatives of the Gauss kernel, we also obtain $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for derivatives of the convolution operators, that is, it follows that*

$$||\partial_t^\alpha \partial_x^\beta e^{t\Delta} f||_{L^p_{uloc,\rho}} \leq \frac{C}{t^{|\alpha|+\frac{|\beta|}{2}}}\left(\frac{1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{1}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\right)||f||_{L^q_{uloc,\rho}}, \tag{28}$$

$$||\partial_t^\alpha \partial_x^\beta e^{t\Delta} \mathbf{P}\nabla \cdot F||_{L^p_{uloc,\rho}} \leq \frac{C'}{t^{|\alpha|+\frac{|\beta|}{2}+\frac{1}{2}}}\left(\frac{1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{1}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\right)||F||_{L^q_{uloc,\rho}}, \tag{29}$$

where C, C' are positive constants depending only on d, p, q, α and β. Here $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ is any multi-index.

Remark 3.3. *The $L^p_{uloc} - L^q_{uloc}$ estimates for the heat semigroup (28) are obtained in* [1]. *When $p = q = \infty$, the estimate (29) are obtained in* [8].

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Yutaka Terasawa
e-mail: yutaka@math.sci.hokudai.ac.jp
Department of Mathematics
Hokkaido University
Sapporo, 060-0810
Japan
Yasunori Maekawa
e-mail: yasunori@math.sci.hokudai.ac.jp
Department of Mathematics
Hokkaido University
Sapporo, 060-0810
Japan