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Sobolev's imbedding theorem in the limiting case with Lorentz space and BMO

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Abstract.

We shall prove a Gagliardo-Nirenberg type interpolation inequality with Lorentz space and BMO of functions of bounded mean oscillation in the critical case. Moreover, we obtain a Trudinger type inequality and a Brezis-Gallouet-Wainger type inequality as an application of the Gagliardo-Nirenberg type inequality.

§1. Introduction

We consider a Gagliardo-Nirenberg type inequality in $\mathbb{R}^n$. It is well known that Sobolev space $H^{n/p,p}(\mathbb{R}^n)$, $1 < p < \infty$, is continuously embedded into $L^q(\mathbb{R}^n)$ for all q with $p \leqq q < \infty$. However, we cannot take $q = \infty$ in such an embedding. T.Ogawa [15] and T.Ogawa-T.Ozawa [16] treated Hilbert space $H^{n/2,2}(\mathbb{R}^n)$ and then T.Ozawa [19] gave the following general embedding theorem in Sobolev space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives which states that

$$\|\Phi_p(\alpha|u|^{p'})\|_{L^1(\mathbb{R}^n)} \leqq C\|u\|^p_{L^p(\mathbb{R}^n)} \tag{1.1}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$, where

$$\Phi_p(\xi) := \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geqq p-1\}.$$

The advantage of (1.1) gives the scale invariant form. In order to prove the above Trudinger type inequality, T.Ozawa [19] showed the following Gagliardo-Nirenberg type interpolation inequality which is equivalent to

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(1.1). For $1 < p < \infty$, there is a constant C depending only on n and p such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq C q^{1/p'} \|u\|_{L^p(\mathbb{R}^n)}^{p/q} \|(-\Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{1.2}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all q with $p \leqq q < \infty$. However, the usage of the Gagliardo-Nirenberg inequality to derive the embedding into Orlitz space is originally due to T.Ogawa [15]. Our goal is obtaining a Gagliardo-Nirenberg type inequality which is located in the extreme case of (1.2) in some sense. That is, we shall prove (1.2) of which the critical Sobolev norm is replaced by the BMO norm. Furthermore, we use Lorentz space instead of Lebesgue space as the functional space. Here, we note that Lorentz space includes Lebesgue space. As corollaries of the Gagliardo-Nirenberg type inequality with BMO, we have a corresponding Trudinger type inequality. The method of the proof is also based on T.Ogawa [15]. The Trudinger type inequality with BMO is regarded as the whole space version of the well known inequality due to F.John-L.Nirenberg [11] which is treated in the case of an arbitrary domain with the finite measure. Finally, as another corollary of the Gagliardo-Nirenberg type inequality with BMO, we obtain a logarithmic type Sobolev inequality including BMO, that is, the Brezis-Gallouet-Wainger type inequality proved by H.Brezis-T.Gallouet [6] for an arbitrary domain in $\mathbb{R}^2$ originally and by H.Brezis-S.Wainger [7] for the whole space $\mathbb{R}^n$. Moreover, we refer to H.Engler [8] concerning another proof of the Brezis-Gallouet-Wainger inequality. With respect to the derivation of the Brezis-Gallouet-Wainger type inequality with BMO, we consider the heat equation with the fractional derivative and the initial value belonging to the Schwartz class. the L^p-L^q estimate of the Gauss kernel corresponding to this heat equation is the key of the proof.

§2. Preliminaries

In this section, in order to state the main theorems, let us recall the definition of the rearrangement of a measurable function to define BMO and Lorentz space. Concerning the rearrangement and its fundamental properties, we refer to C.Bennett-R.Sharpley [4]. For a measurable function u on $\mathbb{R}^n$, $a_u : [0,\infty) \to [0,\infty]$ denotes the distribution function of u, i.e.,

$$a_u(\lambda) := |\{x \in \mathbb{R}^n \,;\, |u(x)| > \lambda\}| \quad \text{for } \lambda \geqq 0.$$

Then $u^* : [0,\infty) \to [0,\infty]$ is defined as follows.

$$u^*(t) := \inf\{\lambda > 0 \,;\, a_u(\lambda) \leqq t\} \quad \text{for } t \geqq 0.$$

We call u^* the rearrangement of u. Moreover, u^{**} denotes the average function of u^* , i.e.,

$$u^{**}(t) := \frac{1}{t}\int_0^t u^*(\tau)d\tau \quad \text{for } t > 0.$$

In what follows, we assume that $u^*(t) < \infty$ for all $t > 0$. Then u^* is right-continuous and nonincreasing on $(0, \infty)$. Hence, u^{**} is continuous and noincreasing on $(0, \infty)$ with

$$u^*(t) \leqq u^{**}(t) \quad \text{for } t > 0.$$

Furthermore, the rearrangement preserves the L^p-norm, i.e.,

$$\|u\|_{L^p}^p = \int_0^\infty u^*(t)^p dt \tag{2.1}$$

holds for $1 \leqq p < \infty$. Moreover,

$$\|u\|_{L^\infty} = \sup_{t>0} u^*(t) = \lim_{t\downarrow 0} f^*(t) = \sup_{t>0} u^{**}(t) = \lim_{t\downarrow 0} u^{**}(t).$$

Concerning the relations between u^* and u^{**}, the following inequality is well known. Let $1 < p < \infty$. Then there hold

$$\int_0^\infty u^{**}(\tau)^p d\tau \leqq p'^{\,p} \int_0^\infty u^*(\tau)^p d\tau. \tag{2.2}$$

(2.2) is a variant of the Hardy inequality (see G.Hardy-J.Littlewood-G.Pólya [10]). Here, we note the well known inequality as follows. Let $1 \leqq p \leqq \infty$. Then there hold

$$\begin{aligned} \|u\|_{L^p_w} : &= \sup_{t>0} t^{1/p} u^*(t) \\ &\leqq \sup_{t>0} t^{1/p} u^{**}(t) \\ &\leqq \|u\|_{L^p}. \end{aligned}$$

Here, let us define Lorentz space with the rearrangement which appear in the main theorems. For $1 \leqq p,\ q < \infty$, Lorentz space is defined as follows.

$$L(p,q) := \left\{ f \in L^1_{loc}(\mathbb{R}^n); \|u\|_{L(p,q)} := \left(\int_0^\infty \left(t^{1/p} u^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

For $1 \leqq p \leqq \infty,\ q = \infty$, we define $L(p,q)$ as follows.

$$L(p,q) := \left\{ u \in L^1_{loc}(\mathbb{R}^n)\ ;\ \|u\|_{L(p,q)} := \sup_{0<t<\infty} t^{1/p} u^{**}(t) < \infty \right\}.$$

By considering (2.1) and (2.2), we can easily checked that $L(p,p) = L^p$ and $L(p,\infty) = L^p_w$ for $1 < p \leqq \infty$. Moreover, we see that the inclusion relation holds with respect to the second index q in $L(p,q)$. That is, the continuous imbedding $L(p,q_1) \hookrightarrow L(p,q_2)$ holds for $1 \leqq p < \infty$ and $1 \leqq q_1 \leqq q_2 \leqq \infty$. Furthermore,

$$\|u\|_{L(p,q_2)} \leqq \left(\frac{q_1}{p}\right)^{1/q_1 - 1/q_2} \|u\|_{L(p,q_1)} \tag{2.3}$$

holds for all $u \in L(p,q_1)$. In the end of this section, we define the function space BMO. We introduce the sharp function of a locally integrable function relative to an arbitrary domain Ω which is defined by

$$u^{\#}_{\Omega}(x) := \begin{cases} \sup\limits_{\substack{Q\subset\Omega \\ x\in Q}} \dfrac{1}{|Q|}\displaystyle\int_Q |u(y) - u_Q|dy, & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

$$u_Q := \frac{1}{|Q|}\int_Q u(y)dy,$$

where Q is a open cube having its sides parallel to the coordinate axes. Then we call a locally integrable function u belongs to BMO if $u^{\#}_{\mathbb{R}^n}$ is in $L^\infty(\mathbb{R}^n)$. We denote the BMO norm by $\|u\|_{BMO} := \|u^{\#}_{\mathbb{R}^n}\|_{L^\infty(\mathbb{R}^n)}$.

§3. Main theorems

We state main theorems in this section. Firstly, the Gagliardo-Nirenberg type inequalities with BMO are obtained as follows.

Theorem 3.1. *Let* $1 < p_1 < \infty$.
(i) *There exists a constant* C *depending only on* n *and* p_1 *such that*

$$\|u\|_{L(q_1,q_2)} \leqq C q_1^{1+1/q_2} \|u\|^{p_1/q_1}_{L(p_1,p_2)} \|u\|^{1-p_1/q_1}_{BMO}$$

holds for all $u \in L(p_1,p_2) \cap BMO$, *where* p_2, q_1 *and* q_2 *are satisfying* $p_1 \leqq q_1 < \infty$ *and* $1 \leqq p_2 \leqq q_2 \leqq \infty$.
(ii) *There exists a constant* C *depending only on* n *and* p_1 *such that*

$$\|u\|_{L(q_1,q_2)} \leqq C\frac{q_1^{2+1/q_2}}{q_1 - p_1} \|u\|^{p_1/q_1}_{L(p_1,p_2)} \|u\|^{1-p_1/q_1}_{BMO}$$

holds for all $u \in L(p_1,p_2) \cap BMO$, *where* p_2, q_1 *and* q_2 *are satisfying* $p_1 < q_1 < \infty$ *and* $1 \leqq q_2 < p_2 \leqq \infty$.
(iii) *There exists a constant* C *depending only on* n *such that*

$$\|u\|_{L(q_1,q_2)} \leqq C\, q_1^{1+1/q_2} \|u\|^{1/q_1}_{L^1} \|u\|^{1-1/q_1}_{BMO}$$

holds for all $u \in L^1 \cap BMO$, where q_1 and q_2 are satisfying $1 \leqq q_1 < \infty$ and $1 \leqq q_2 \leqq \infty$.

By putting $q_1 = q_2$ in Theorem 3.1, we have the following Corollary 3.2.

Corollary 3.2. (i) *For every $1 \leqq p_1 < \infty$, there exists a constant C depending only on n and p_1 such that*

$$\|u\|_{L^q} \leqq C\, q\, \|u\|_{L(p_1,p_2)}^{p_1/q} \|u\|_{BMO}^{1-p_1/q} \tag{3.1}$$

holds for all $u \in L(p_1,p_2) \cap BMO$, where p_2 and q are satisfying $1 \leqq p_2 \leqq p_1 \leqq q < \infty$.
(ii) *For every $1 < p_1 < \infty$, there exists a constant C depending only on n and p_1 such that*

$$\|u\|_{L^q} \leqq C \frac{q^2}{q - p_1} \|u\|_{L(p_1,\infty)}^{p_1/q} \|u\|_{BMO}^{1-p_1/q} \tag{3.2}$$

holds for all $u \in L(p_1,\infty) \cap BMO$ and for all $p_1 < q < \infty$.

Remark 3.3. *The above theorems also may be obtained from the interpolation theory if we don't care the sharp constant. However, what we emphasize most is that the orders with respect to q_1 in Theorem 3.1 or q in Corollary 3.2, that is, q_1^{1+1/q_2} as $q_1 \to \infty$ or q^1 as $q \to \infty$ are optimal respectively. The optimality is easily shown by considering the logarithmic function restricted in a ball centered at the origin.*

Moreover, from Corollary 3.2, we obtain Trudinger type inequalities equivalent to (3.1) and (3.2) as follows:

Corollary 3.4. (i) *For every $1 \leqq p_1 < \infty$, there exists a constant C depending only on n and p_1 such that the following holds. For arbitrary $0 < \alpha < C$, there exists a constant $\tilde{C}$ depending only on n, p_1 and α such that*

$$\int_{\mathbb{R}^n} \Phi_{p_1}\left(\alpha \frac{|u(x)|}{\|u\|_{BMO}}\right) dx \leqq \tilde{C} \left(\frac{\|u\|_{L(p_1,p_2)}}{\|u\|_{BMO}}\right)^{p_1}$$

holds for all $u \in L(p_1,p_2) \cap BMO \setminus \{0\}$ and for all $1 \leqq p_2 \leqq p_1$, where Φ_{p_1} is defined by

$$\Phi_{p_1}(\xi) := \sum_{\substack{j \geqq p_1 \\ j \in \mathbb{N}}} \frac{\xi^j}{j!} \qquad \textit{for } \xi \in \mathbb{R}.$$

(ii) *For every* $1 < p_1 < \infty$, *there exists a constant* C *depending only on* n *and* p_1 *such that the following holds. For arbitrary* $0 < \alpha < C$, *there exists a constant* $\tilde{C}$ *depending only on* n, p_1 *and* α *such that*

$$\int_{\mathbb{R}^n} \tilde{\Phi}_{p_1}\left(\alpha \frac{|u(x)|}{\|u\|_{BMO}}\right) dx \leqq \tilde{C} \left(\frac{\|u\|_{L(p_1,\infty)}}{\|u\|_{BMO}}\right)^{p_1}$$

holds for all $u \in L(p_1,\infty) \cap BMO \setminus \{0\}$, *where* $\tilde{\Phi}_{p_1}$ *is defined by*

$$\tilde{\Phi}_{p_1}(\xi) := \sum_{\substack{j>p_1 \\ j\in\mathbb{N}}} \frac{\xi^j}{j!} \qquad \text{for } \xi \in \mathbb{R}.$$

Finally, we shall state the application to the Brezis-Gallouet-Wainger type inequality with BMO. In fact, from Corollary 3.2, we can obtain the inequality as follows:

Theorem 3.5. (i) *For every* $1 \leqq p_1 < \infty, 1 \leqq q \leqq \infty$ *and* $n/q < m < \infty$, *there exists a constant* C *depending only on* n, p_1, q *and* m *such that*

$$\|u\|_{L^\infty} \leqq C\Big[1 + \big(\|u\|_{L(p_1,p_2)} + \|u\|_{BMO}\big)\log(e + \|(-\Delta)^{m/2}u\|_{L^q})\Big]$$

holds for all $u \in L(p_1,p_2) \cap BMO$ *with* $(-\Delta)^{m/2}u \in L^q$, *where* p_2 *is satisfying* $1 \leqq p_2 \leqq p_1$.
(ii) *For every* $1 < p_1 < \infty, 1 \leqq q \leqq \infty$ *and* $n/q < m < \infty$, *there exists a constant* C *depending only on* n, p_1, q *and* m *such that*

$$\|u\|_{L^\infty} \leqq C\Big[1 + \big(\|u\|_{L(p_1,\infty)} + \|u\|_{BMO}\big)\log(e + \|(-\Delta)^{m/2}u\|_{L^q})\Big]$$

holds for all $u \in L(p_1,\infty) \cap BMO$ *with* $(-\Delta)^{m/2}u \in L^q$.

§4. Outline of proof

Firstly, we state the outline of the proof of the Gagliardo-Nirenberg type inequality with BMO, i.e., Theorem 3.1. We define a new function space as follows.

$$W := \{u \in L^1_{loc}(\mathbb{R}^n)\ ;\ \|u\|_W := \sup_{t>0}(u^{**}(t) - u^*(t)) < \infty\}.$$

Then in order to prove Theorem 3.1, it is enough to show the following propositions, i.e.,

Proposition 4.1. *Let $1 \leqq p < \infty$. Then there exists a constant C depending only p such that*

$$\|u\|_{L(q,1)} \leqq C\,\frac{q^3}{q-p}\|u\|_{L(p,\infty)}^{p/q}\|u\|_W^{1-p/q}$$

holds for all $u \in L(p,\infty) \cap W$ and for all $p < q < \infty$.

Proposition 4.2. *There exists a constant C depending only on n such that*

$$\|u\|_W \leqq C\|u\|_{BMO}$$

holds for all $u \in \{\tilde{u} \in BMO$: There exists $1 < p < \infty$ such that $\tilde{u} \in L(p,\infty)$ or $\tilde{u} \in L^1\}$.

In fact, from (2.3) and Proposition 4.1, we have the following Corollary 4.3 immediately.

Corollary 4.3. *Let $1 \leqq p_1 < \infty$.*
(i) *There exists a constant C depending only on p_1 such that*

$$\|u\|_{L(q_1,q_2)} \leqq C q_1^{1+1/q_2}\|u\|_{L(p_1,p_2)}^{p_1/q_1}\|u\|_W^{1-p_1/q_1}$$

holds for all $u \in L(p_1,p_2) \cap W$, where p_2, q_1 and q_2 are satisfying $p_1 \leqq q_1 < \infty$ and $1 \leqq p_2 \leqq q_2 \leqq \infty$.
(ii) *There exists a constant C depending only on p_1 such that*

$$\|u\|_{L(q_1,q_2)} \leqq C\frac{q_1^{2+1/q_2}}{q_1-p_1}\|u\|_{L(p_1,p_2)}^{p_1/q_1}\|u\|_W^{1-p_1/q_1}$$

holds for all $u \in L(p_1,p_2) \cap W$, where p_2, q_1 and q_2 are satisfying $p_1 < q_1 < \infty$ and $1 \leqq q_2 < p_2 \leqq \infty$.

It is clear that we obtain Theorem 3.1 from Corollary 4.3 and Proposition 4.2. Concerning the proof of Proposition 4.1, we divide the Lorentz norm into two parts with one parameter and estimate each term by the definition of Lorentz space. Finally, by optimizing the estimated value with respect to the parameter, we have the desired interpolation inequality. Furthermore, Proposition 4.2 is obtained as a corollary of the following theorem proved by C.Bennett-R.Sharpley [4].

Theorem 4.4. *There exists C depending only on n such that*

$$u^{**}(t) - u^*(t) \leqq C\left(u_Q^{\#}\right)^*(t)$$

holds for all cubes Q in $\mathbb{R}^n$, $u \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} u \subset \overline{Q}$ *and $0 < t < |Q|/6$.*

Next, concerning the proof of the Brezis-Gallouet-Wainger type inequality with BMO, i.e., Theorem 3.5, we consider the heat equation with the fractional derivative, i.e.,

$$\frac{\partial u}{\partial t} = -(-\Delta)^{m/2}u \quad \text{in } \mathbb{R}^n \times (0,\infty),$$
$$u(\cdot,0) = \Phi \quad \text{in } \mathbb{R}^n,$$

where Φ is a function which belongs to the Schwartz class and $0 < m < \infty$. Then the solution to this heat equation is represented by

$$u(\cdot,t) = G_m^t * \Phi,$$

where the heat kernel G_m^t is defined by

$$G_m^t := F^{-1}(e^{-(2\pi|\cdot|)^m t}),$$

F^{-1} means the Fourier inverse transformation. From the Young inequality, we obtain the L^p-L^q estimate of the heat kernel as follows.

Proposition 4.5. *For every* $0 < m < \infty$, *there exists a constant* C *depending only on* n *and* m *such that*

$$\|G_m^t * u\|_{L^q} \leqq C\, t^{-(n/m)(1/p-1/q)} \|u\|_{L^p}$$

holds for all $u \in L^p$ *and for all* $t > 0$, *where* p *and* q *are satisfying* $1 \leqq p \leqq q \leqq \infty$.

Since L^∞ is the dual space of L^1, we can show Theorem 3.5 by applying Corollary 3.2 and Proposition 4.5.

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