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Eventual stability criterion for periodic points of Michio Morishima's example

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Abstract.

In this paper we discuss the Morishima's example, which implies a kind of eventually asymptotical stability of solutions for a difference equation $x(n+1) = f(x(n))$ for $n = 0, 1, 2, \cdots$. We define new definitions of eventual stability of periodic points in the meaning of the large in the same way as ones of Lakshmikantham et. al. and Yoshizawa. By applying the Lyapunov's second method we give eventual stability criteria in the large of the difference equation. In order to illustrait our main results on eventual stability an example of a set of 2-periodic points for eventual stability is given with an analytical estimation.

§1. Introduction

In 1977 Morishima[3] gave results on the stability, oscillation and chaos of periodic points concerning the following difference equation.

$$x(n+1) = \frac{A(n)}{A(n)+B(n)} \qquad \text{for } n = 0, 1, \cdots \tag{E}$$

and

$$\begin{aligned} A(n) &= \max[\frac{a}{b}x(n) + \{1-(1+a)x(n)\}, 0], \\ B(n) &= \max[(1-x(n))\{\frac{a}{b} - \frac{x(n)(1-(1+a)x(n))}{(1-x(n))^2}\}, 0] \end{aligned}$$

Here a, b are positive parameters. His results[3] with $a = 0.6, b = 1$ were studied concerning the chaos of Eq.(E) independently with Li-Yorke[2] in 1975.

Morishima[4] studied the chaotic behavior and the stability of orbits of

$$x(n+1) = f(x(n)), \tag{1}$$

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where $f : [0,1] \to [0,1]$ is continuous, $x : \mathbf{Z}_+ = \{0,1,2,\cdots\} \to [0,1]$ is the price of the commodity and also he discussed some type of stability of periodic points, where the stability is not globally uniformly asymptotically stable but every orbit of Eq.(1) has unstable properties in the beginning and the stable behavior from some iterations.

In this paper we show results on the globally asymptotical stability for periodic points of Eq.(1) as well as we discuss the globally eventually asymptotical stability. See Lakshikantham-Leela[1], Yoshizawa[5] concerning the eventual stability for the case of ordinary differential equations.

§2. Notations

Consider difference equation (1) in $I^m \subset \mathbf{R}^m$ with $I = [0,1]$ and positive integer m. Denote $x(n) = (x_1(n), x_2(n), \cdots, x_m(n))^T$, where T means the transpose, is a relative price vector of $m-$commodities, where $0 \leq x_j(n) \leq 1$ for $j = 1,2,\cdots,m$ and $\sum_{j=1}^{m} x_j(n) = 1$ for $n \in \mathbf{Z}_+$. See [3, 4] in detail. A function $f : I^m \to I^m$ is continuous.

Let k be a positive integer. Denote a set of $k-$peridic points by $P(k) = \{x^* \in I^m\}$. $x^* \in P(k)$ if and only if $f^i(x^*) \neq f^j(x^*)$ for $1 \leq i \neq j \leq k$ and $f^k(x^*) = x^*$. Denote by $x(n; n_0, x_0)$ a solution of Eq.(1) for $n \geq n_0$ with $x(n_0; n_0, x_0) = x_0$ satisfying the initial condition $(n_0, x_0) \in \mathbf{Z}_+ \times I^m$. Denote by $\| x \|$ a norm of $x \in \mathbf{R}^m$. For $r > 0$ we denote the following neighborhoods: when a point $x_0 \in I^m, B(x_0, r) = \{x \in I^m : \| x - x_0 \| < r\}$; when a subset $P \subset I^m, S(P, r) = \cup_{x \in P} B(x, r)$.

A set of $k-$periodic points $P(k)$ is called eventually uniformly stable [EV-US] if for each $\varepsilon > 0$ there exist $N_0 \in \mathbf{Z}_+$ and $\delta > 0$ such that for every $x_0 \in S(P(k), \delta)$ and every $n_0 \geq N_0$, it holds that each solution $x(n; n_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0, i.e.,$

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

Here a distance between a point $x \in \mathbf{R}^m$ and a subset $S \subset \mathbf{R}^m$ is defined by $d(x, S) = \inf\{\| x - a \| : a \in S\}$. A set of $k-$periodic points $P(k)$ is called eventually uniformly attractive to finite coverings [EV-UA.FC] if each finite covering $\{C_q \subset I^m : \cup_{q=1}^{Q} C_q \supset I^m\}$ and each $\varepsilon > 0$, there exist $N_0 \in \mathbf{Z}_+$ and $T_0 \in \mathbf{Z}_+$ such that for every $1 \leq q \leq Q$, every $x_0 \in C_q$ and every $n_0 \geq N_0$, it holds that every solution $x(n; n_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0 + T_0, i.e.,$

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

The set of $k-$periodic points $P(k)$ is callled eventually uniformly asymptotically stable to finite coverings [EV-UAS.FC] if $P(k)$ is [EV-US] and [EV-UA.FC].

§3. Criterion of Evental Stability

Assume that Eq.(1) has a set of k-periodic points

$$P(k) = \{x_1, x_2, \cdots, x_k\}$$

for $k = 1, 2, \cdots$. We show two criterion for eventually uniformly asymptotically stable of $P(k)$ by applying the Lyapunov's second method. In case $k = 1, P(1)$ is a set of fixed point.

Let a set of functions denote

$$CIP = \{a : I \to \mathbf{R}_+ \text{ is continuous, strictly increasing and positively definite}\}$$

and $R_+ = [0, \infty)$. Denote $A - B = \{x \in A : x \notin B\}$ for sets $A, B \subset I^m$.

In the following theorem we prove the eventually uniformly asymptotically stable to finite coverings of $P(k)$.

Theorem. $k-$periodic points $P(k)$ is eventually uniformly asymptotically stable to finite coverigs under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying the following conditions (a)-(b).

(a) For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that

$$a_r(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k))$$

for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

(b) Let $\Delta V(n, x) = V(n+k, f^k(x)) - V(n, x)$ for $(n, x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and a function $c_r \in CIP$ such that

$$\Delta V(n, x) \leq -c_r(d(P(k), x))$$

for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

Proof. It is proved that the set $P(k)$ is [EV-US]. At first, we get the following inequalities.

$$\tilde{a_r}(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k))); \tag{2}$$

$$\Delta V(n, x) \leq -\tilde{c_r}(d(x, P(k))). \tag{3}$$

where $\tilde{a_r}(d) = \min[a_r(d), c_r(d)]$ and $\tilde{c_r}(d) = \frac{1}{2}\tilde{a_r}(d)$ for $d > 0$. For a sufficiently large $\alpha_1 > 0$ and small $\alpha_2 > 0$ and any $p_\omega \in P(k)$ it can be seen that $I^m \subset S(P(k), \alpha_1)$ and that

$$\text{(4)} \qquad \text{if } x \in B(p_\omega, \alpha_2), \text{ then } f^k(x) \in B(p_\omega, \alpha_1).$$

For any $\varepsilon > 0$ define

$$\text{(5)} \qquad \phi_\omega(\varepsilon) = \inf\{V(n,x) : \varepsilon \leq \| x - p_\omega \| \leq \alpha_1, n \geq n_0\}.$$

We get

$$\text{(6)} \qquad V(n,x) < \phi_\omega(\varepsilon) \text{ for } x \in B(p_\omega, \delta_\omega),\ n_0 \geq N_0.$$

If not so, it holds that

$$0 < \forall\delta < \varepsilon, \exists n_\delta \geq N_0 : V(n_\delta, x_\delta) \geq \phi_\omega(\varepsilon)$$

for $\exists x_\delta \in B(p_\omega, \delta)$. Inequality (2) means that for $r < \varepsilon$

$$b_r(d(x_\delta, P(k))) \geq \phi_\omega(\varepsilon) > 0.$$

Putting $\delta = 1/j$ for $j = 1, 2, \cdots$, we have a sequence $\{x_j : j = 1, 2, \cdots\} \subset cl(B(p_\omega, \delta))$, the closure of $B(p_\omega, \delta)$, and

$$\lim_{j \to +\infty} \| x_j - p_\omega \| = 0.$$

This imlpies that $0 = \phi_\omega(\varepsilon) > 0$, which leads to a contradiction.

We shall prove that there exist $1 \leq k(1), k(2) \leq k$ and $\delta > 0$ as follows:

$$\text{(7)} \qquad \begin{aligned} &\exists p_{k(1)} \in P(k), 0 < \exists\delta < \delta_\omega : \forall y \in B(p_{k(1)}, \delta), \forall n_0 \geq N_0;\\ &\forall \ell = 1, 2, \cdots, \exists p_{k(2)} \in P(k) : x(n_0 + \ell k; n_0, y)) \in B(p_{k(2)}, \varepsilon). \end{aligned}$$

If not so, then

$$\begin{aligned} &\forall p_\omega \in P(k), 0 < \forall\delta \leq \delta_\omega, \exists y_1 \in B(p_\omega, \delta) : \exists \overline{n_0} \geq N_0,\\ &\quad \exists \ell_1 \geq 1, \forall p_\omega \in P(k), x(\overline{n_0} + \ell_1 k; \overline{n_0}, y_1) \notin B(p_\omega, \varepsilon), \end{aligned}$$

which means that $V(\overline{n_0}, y_1) < \phi_\omega(\varepsilon)$ Dnote $x(\cdot) = x(\cdot; \overline{n_0}, y_1)$. Considering $S(P(k), \alpha_1) \supset I^m$, we get

$$\varepsilon < \| x(\overline{n_0} + \ell_1 k) - p_\omega \| < \alpha_1$$

and $\phi_\omega(\varepsilon) \le V(\overline{n_0}+\ell_1 k, x(\overline{n_0}+\ell_1 k))$. By Condition(b), it holds that

$$\begin{aligned}\phi_\omega(\varepsilon) &\le V(\overline{n_0}+\ell_1 k, x(\overline{n_0}+\ell_1 k))\\ &= V(\overline{n_0}+\ell_1 k, f^{\ell_1 k}(y_1))\\ &< V(\overline{n_0}+\ell_1 k-k, f^{(\ell_1-1)k}(y_1)))\\ &< \cdots\\ &< V(\overline{n_0}, y_1) < \phi_\omega(\varepsilon).\end{aligned}$$

This leads to a contracition. Hence, $P(k)$ is [EV-US], because for any $0<\varepsilon<\alpha_2$ there exist a positive $\delta < \min\{\delta_\omega : 1\le \omega \le k\}$ and an integer $N_0 \ge 0$ such that for any $n_0 \ge N_0$ and any $n \ge n_0$ if $x_0 \in S(P(k),\delta)$, then $x(n;n_0,x_0) \in S(P(k),\varepsilon)$.

It can be seen that Eq.(1) is uniformly bounded as follows:

(8) $$\forall \alpha > 0,\ \exists \beta(\alpha) > 0 : \forall n_0 \ge 0,$$
$$\| x(n;n_0,x) \| < \beta(\alpha) \text{ for } \| x \| < \alpha, n \ge n_0.$$

If Eq.(1) is not [EV-UA.FC], then there exist a real number $\varepsilon_1 > 0$ and a finite covering $\{C_q \subset I^m \text{ such that } \cup_{q=1}^{Q} C_q \supset I^m\}$ such that for some $1 \le \tilde{q} \le Q$ and any integers $N, T \in \mathbf{Z}_+$ there exist an initial point $x_0 \in C_{\tilde{q}}$ and integers $n_0 = n_0(N,T) \ge N, n_1 = n_1(N,T) \ge n_0 + T$ such that

(9) $$d(x(n_1;n_0,x_0),P(k)) \ge \varepsilon_1.$$

Then we have a sequence $\{z_j : \| z_j \| \le \alpha\}$ and $z = \lim_{j\to\infty} z_j \notin P(k)$. For a sufficiently small $\eta > 0$ we get a neighborhood $O(z,\eta)$ of z such that

(10) $$S(P(k),\varepsilon_1) \cap cl(O(z,\eta)) = \emptyset.$$

From $V(n,x) \ne 0$ on $O(z,\eta)$, we can define $h(n,x) = V(n+k, f^k(x))/V(n,x)$ on $O(z,\eta)$, which is continuous and $h(n,x) < 1$ by Condition (3). By $\tilde{c_r}(d)/\tilde{a_r}(d) = 1/2$, it can be seen that

(11) $$h(n,x) \le 1 - [\tilde{c_r}(d(x,P(k)))/\tilde{a_r}(d(x,P(k)))] = 1/2.$$

For $\delta_1 > 0$ we have the closure of δ_1-neighborhood at z such that $cl(B(z,\delta_1)) \subset O(z,\eta)$ and also if $x \in cl(B(z,\delta_1))$ then $h(n,x) \le 1/2$. Denote $U = \{u_{i+1} = f^k(u_i) : u_1 = z, i = 1,2,\cdots\} \in B(z,\delta_1)$.

$$\begin{aligned}V(ik+k, f^k(u_i)) &= h(ik,u_i)V(ik,u_i) \le \frac{1}{2}V(ik,u_i)\\ &\le \frac{1}{2^2}V(ik-k,u_{i-1}) \le \cdots \le \frac{1}{2^i}V(0,z)\end{aligned}$$

Then we have

$$\tilde{a_r}(d(f^k(u_i), P(k))) \leq V(ik, f^k(u_i)) \leq \frac{1}{2^i} b_r(d(z, P(k))).$$

As $r \to 0+, i \to +\infty$ it can be seen that

$$0 \leq \tilde{a_r}(d(f^k(u_i), P(k))) \leq \lim_{i \to \infty} \frac{1}{2^i} b_r(d(z, P(k))) = 0.$$

Hence there exist a subsequence $\{f^k(u_{j_q}) : q = 1, 2, \cdots\} \subset U \subset cl(B(z, \delta_1))$ and a point $v \in I^m$ such that

$$v \in cl(B(z, \delta_1)), \quad f^k(v) \in O(z, \eta), \text{ and } \lim_{q \to \infty} f^k(u_{j_q}) = f^k(v) \in P(k).$$

This implies that $f(v) \in P(k) \cap O(z, \eta)$, which contradicts with the assumtion(9). Therefore $P(k)$ is [EV-UA.FC]. Q.E.D.

In case where $k = 1$ the above theorem leads to an eventual stability theorem of fixed point for Eq.(1).

Corollary. Eq.(1) has a fixed point x^*. The point x^* is eventually uniformly asymptotically stable to finite coverings under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying Condition (a)-(b).

(a) For any $r > 0$ there exist an integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that

$$a_r(\| x - x^* \|) \leq V(n, x) \leq b_r(\| x - x^* \|)$$

for any integers $n \geq N_0$ and any initial points $x \in I^m - \{x^*\}$.

(b) Let $\Delta V(n, x) = V(n+1, f(x)) - V(n, x)$ for $(n, x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist an intger $N_0 \geq 0$ and a function $c_r \in CIP$ such that

$$\Delta V(n, x) \leq -c_r(\| x - x^* \|)$$

for any $n \geq N_0$ and any $x \in I^m - \{x^*\}$.

§4. Illustration of Theorem

We illustrait Theorem in the case $k = 2$ and $P(2) = \{0.5, 0.7\}$ in the space $\mathbf{R}$ with an analytical result. Consider the Morishima's example as follows.

$$x(n+1) = f(x(n)) = \frac{A(n)}{A(n) + B(n)}.$$

Here $A(n) = \max[x + bE_1(x(n)), 0], B(n) = \max[1 - x + bE_2(x(n)), 0]$ and $a = 0.6$ and $E_1(x) = -x + \frac{1-x}{a}, E_2(x) = -\frac{xE_1(x)}{1-x}$. See [3] in detail. Then, in $b = 0.6$, we get

$$f(x) = \frac{1.8x^2 - 4.8x + 3}{9.6x^2 - 13.8x + 6}, \quad f'(x) = \frac{21.24x^2 - 36x + 12.6}{(9.6x^2 - 13.8x + 6)^2}.$$

Let

$$V(x) = d(x, P(2)) = \min[|x - 0.5|, |x - 0.7|]$$

for $x \in I$. Let $a_r(d) = b_r(d) = d$ $(d > 0)$ to any $r > 0$. Then $a_r, b_r \in CIP$ and it holds that Condition(a) of Theorem is satisfied. It can be seen that

$$\begin{aligned}
\Delta V(x) &= \min(|f^2(x) - 0.5|, |f^2(x) - 0.7|) - d(x, P(2)) \\
&= \min(|f^2(x) - f^2(0.5)|, |f^2(x) - f^2(0.7)|) - d(x, P(2)) \\
&= \min_{x^*=0.5,0.7} \left| \int_0^1 \frac{df^2}{dx}(x^* + \theta(x - x^*))(x - x^*)d\theta \right| - d(x, P(2)) \\
&\leq \min_{x^*=0.5,0.7} \max_{x \in I} \left| \frac{df^2}{dx}(x) \right| |x - x^*| - d(x, P(2)) \\
&= \max_{x \in I} \left| \frac{df^2}{dx}(x) \right| d(x, P(2)) - d(x, P(2)) \\
&= (\max_{x \in I} |f'(f(x))f'(x)| - 1)d(x, P(2)).
\end{aligned}$$

It holds that $\Delta V(x) \leq \max_{x \in I}(f'(f(x))f'(x) - 1)V(x)$.

We shall show that $\Delta V(x) \leq -cV(x)$ for $x \in I$ with a real number $c > 0$. Putting $y(x) = f'(f(x))f'(x) - 1$, when $y(x) < 0$, then there exists a positive number c such that

$$\Delta V(x) \leq -cV(x). \tag{12}$$

Putting $C(d) = cd$, we have

$$C \in CIP : \Delta V(x) \leq -C(V(x)).$$

Therefore it holds that Condition (b) of Theorem is satisfied.

Denote

$$p = 21.24x^2 - 36x + 12.6, \quad q = (9.6x^2 - 13.8x + 6)^2,$$

then we have $f' = p/q$ and $\max_{x\in I} |p/q| < 1$. In fact

$$\begin{aligned}
&\frac{p^2 - q^2}{q^2} \\
&= \frac{(p-q)(p+q)}{q^2} \\
&= [(21.24x^2 - 36x + 12.6) - (9.6x^2 - 13.8x + 6)^2][21.24x^2 - 36x \\
&\quad +12.6 + (9.6x^2 - 13.8x + 6)^2]/q^2 \\
&= [-92.16x^4 + 264.96x^3 - 284.4x^2 - 118.8x - 23.4] \\
&\times[21.24(x - (1.18)^{-1})^2 + 12.6 - (9/5.09) + (9.6x^2 - 13.8x + 6)^2]/q^2
\end{aligned}$$

and $12.6-(9/5.09) > 0$, $264.96x^3 - 284.4x^2 = 264.96x^2(x - 284.4/264.96) < 0$ for $0 \le x \le 1$, then we have $|f'(x)| \le \max_{x\in I} \frac{|p|}{|q|} < 1$. Hence it holds that on $x \in [0, 1]$

$$y(x) = f'(f(x))f'(x) - 1 \le (\max_{x\in I} \frac{|p|}{|q|})^2 - 1 < 0.$$

Since y is continuous and $[0, 1]$ is compact, then there exists a positive number c such that $y(x) \le -c < 0$ on $[0, 1]$.

§5. Conclusions

We considered a definition of [EV-UAS.FC] (eventually uniformly asymptotic stability to finite coverings) in the same way as theory of ordinary differential equations.

We proved a theorem for [EV-UAS.FC] of difference equation $x(n+1) = f(x(n))$ by the Lyapunov's second method including an analytical estimation of ΔV.

We illustrated the eventual stability theorem by applying it to the Morishima's example.

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