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Diffusion phenomenon of solutions to the Cauchy problem for the damped wave equation

Kenji Nishihara

Abstract.

We survey the recent results for the damped wave equations and obtain the critical exponent for the semilinear problem. Those results are based on the diffusion phenomenon of the damped wave equation.

§1. Introduction

In this note we consider the Cauchy problem for the damped wave equation

$$(P) \qquad \begin{cases} u_{tt} - \Delta u + b(t,x)u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

where $|f(u)| = O(|u|^\rho)$, $\rho > 1$. When $b(t,x) \equiv 1$, (P) is reduced to

$$(DW) \qquad \begin{cases} u_{tt} - \Delta u + u_t = f(u), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases}$$

which has the property of diffusion phenomenon, that is, the solution is expected to behave as $t \to \infty$ like the solution ϕ to the corresponding diffusive equation

$$(H) \qquad \begin{cases} -\Delta \phi + \phi_t = f(\phi), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ \phi(0,x) = \phi_0(x), & x \in \mathbf{R}^N. \end{cases}$$

When $f(\phi) = |\phi|^{\rho-1}\phi$, the equation in (H) is called the Fujita equation, named after his pioneering work [2] and, for non-trivial data $\phi_0(x) \geq 0$,

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there is a critical exponent, the Fujita exponent,

$$\rho_F(N) = 1 + \frac{2}{N} \tag{1.1}$$

in the following sense: if $\rho > \rho_F(N)$, then there exists a unique and global-in-time solution $\phi(t,x)$ for suitably small data $\phi_0(x)$, and if $\rho \le \rho_F(N)$, then there is no global solution $\phi(t,x)$ for any small data $\phi_0(x)$. By the diffusion phenomenon it is expected that (DW) has the same critical exponent $\rho_F(N)$. In fact, these have been investigated by many authors [3]–[6], [9]–[11], [15], [17]–[18], [23], [28] etc. See also the references therein.

Our aims are to survey the related topics for (P) and to determine the critical exponent $\rho_c(N,\beta)$ for (P) with $f(u) = |u|^\rho$ and $b(t,x) = (t+1)^{-\beta}$, $-1 < \beta < 1$. Our final result is to show

$$\rho_c(N,\beta) = \rho_F(N) \left(= 1 + \frac{2}{N} \right) \tag{1.2}$$

even for time-dependent damping.

Our plan of this note is as follows. In Section 2 we show the diffusion phenomenon of the linear wave equation with constant coefficient damping, using the explicit formula of solutions. In Section 3 we survey the properties of solutions for (DW) and (P) dependent on the exponent ρ. In the final section we treat (P) with $b(t,x) =: b(t) = (t+1)^{-\beta}$.

§2. Linear damped wave equation

We first consider the linear wave equation with constant coefficient damping

$$(LDW) \qquad \begin{cases} u_{tt} - \Delta u + u_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0,x) = (u_0, u_1)(x), & x \in \mathbf{R}^N. \end{cases}$$

The solution to (LDW) is given explicitly, which is found in Courant and Hilbert [1]. By $v(t,x) = [S_N(t)g](x)$ we denote the solution to

$$\begin{cases} v_{tt} - \Delta v + v_t = 0, & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (v, v_t)(0,x) = (0,g)(x), & x \in \mathbf{R}^N, \end{cases}$$

then it is given, for example when $N = 3$, by

$$S_3(t)g = \frac{e^{-\frac{t}{2}}}{4\pi t} \partial_t \int_{|z| \le t} I_0\Big(\frac{\sqrt{t^2 - |z|^2}}{2}\Big) g(x+z)\,dz,$$

where I_ν is the modified Bessel function given by the series

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!(m+\nu)!} \left(\frac{y}{2}\right)^{2m+\nu}, \quad \nu = 0, 1, 2, \cdots .$$

The solution $u(t,x)$ to (LDW) is given by

$$u(t,x) = S_N(t)(u_0 + u_1) + \partial_t(S_N(t)u_0). \tag{2.1}$$

Hence we analyse $S_N(t)g$ and $\partial_t(S_N(t)g)$. Here we only show the case $N = 3$. To do so, we need the properties of the modified Bessel function.

Lemma 2.1. *The modified Bessel function I_ν ($\nu = 0, 1, 2, \cdots$) satisfies*

$$I_0(0) = 1, \; I_1(y)/y\Big|_{y=0} = 1/2, \; (I_0(y) - \frac{2}{y} I_1(y))/y^2\Big|_{y=0} = 1/8, \tag{2.2}$$

$$I_0'(y) = I_1(y), \quad I_1'(y) = I_0(y) - I_1(y)/y \tag{2.3}$$

and, moreover, the following expansion formula as $y \to \infty$:

$$\begin{aligned} I_\nu(y) = \tfrac{e^y}{\sqrt{2\pi y}} \Big(1 &- \tfrac{(\nu-1/2)(\nu+1/2)}{2y} + \tfrac{(\nu-1/2)(\nu-3/2)(\nu+3/2)(\nu+1/2)}{2!2^2y^2} \\ &- \cdots + (-1)^k \tfrac{(\nu-1/2)\cdots(\nu-(k-1/2))(\nu+(k-1/2))\cdots(\nu+1/2)}{k!2^ky^k} + O(y^{-k-1})\Big). \end{aligned}$$

Differentiating in t in $S_3(t)g$ and using (2.2), we obtain

$$\begin{aligned} S_3(t)g &= e^{-t/2} \cdot \tfrac{t}{4\pi} \int_{S^2} g(x + t\omega)\, d\omega + \tfrac{e^{-t/2}}{8\pi} \int_{|z|\le t} I_1\big(\tfrac{\sqrt{t^2-|z|^2}}{2}\big) \tfrac{g(x+z)\, dz}{\sqrt{t^2-|z|^2}} \\ &=: e^{-t/2} W(t)g + J_0(t)g \quad (W(t)g: \text{ Kirchhoff formula}). \end{aligned}$$

Again, using (2.2)–(2.3), we get

$$\begin{aligned} \partial_t(S(t)g) &= e^{-t/2} \cdot \{(-\tfrac{1}{2} + \tfrac{t}{8})W(t)g + \partial_t(W(t)g)\} \\ &\quad + \int_{|z|\le t} \partial_t \Big[\tfrac{e^{-t/2} I_1(\frac{\sqrt{t^2-|z|^2}}{2})}{8\pi\sqrt{t^2-|z|^2}} \Big] g(x+z)\, dz \\ &=: e^{-t/2} \tilde{W}(t)g + J_1(t)g. \end{aligned}$$

Hence, by (2.1) the solution $u(t,x)$ to (LDW) is decomposed as

$$u(t,x) = e^{-t/2}\{W(t)(u_0 + u_1) + \tilde{W}(t)u_0\} + J_0(t)(u_0 + u_1) + J_1(t)u_0.$$

By the expansion in Lemma 2.1 each $J_i(t)g$ is estimated as follows.

Lemma 2.2 ([18], L^p-L^q estimate in $N=3$). *For p, q with $1 \le q \le p \le \infty$, it holds that*

$$\begin{aligned}
\|J_0(t)g\|_{L^p} &\le C\|g\|_{L^q}(t+1)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}, && t \ge 0,\\
\|(J_0(t)-e^{t\Delta})g\|_{L^p} &\le C\|g\|_{L^q}t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-1}, && t > 0,\\
\|J_1(t)g\|_{L^p} &\le C\|g\|_{L^q}(t+1)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-1}, && t \ge 0,
\end{aligned}$$

where $e^{t\Delta}g = \int_{\mathbf{R}^N} G(t, x-y)g(y)\,dy$ *with* $G(t,x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$.

Therefore, the decomposition of solution $u(t,x)$ to (LDW) is symbolically seen as

$$u(t,x) = \underbrace{e^{-t/2}\{W(t)(u_0+u_1) + \tilde{W}(t)u_0\}}_{\textit{wave part decaying fast}} + \underbrace{J_0(t)(u_0+u_1) + J_1(t)u_0}_{\textit{diffusive part}}.$$

Thus, we can say about the solution to (LDW):

(i) $u(t,x)$ may have the singularity,
(ii) $u(t,x)$ has the finite propagation property,
(iii) $u(t,x)$ has not the smoothing effect nor the maximum principle,
(iv) $u(t,x) \sim \phi(t,x)$ as $t \to \infty$, if $\phi_0 = u_0 + u_1$.

Here $f \sim g$ as $t \to \infty$ means $(f-g)/g \to 0$ as $t \to \infty$. The final relation (iv) means the diffusion phenomenon.

§3. Semilinear damped wave equation

We now consider the Cauchy problem (P) for the semilinear wave equation with time- or space-dependent damping. Here we assume that the data $(u_0, u_1) \in H^1 \times L^2$ have the compact supports and the semilinear term $f(u)$ satisfies $|f(u)| \le C|u|^\rho$ $(1 < \rho < \frac{N+2}{[N-2]_+} := \infty$ $(N=1,2)$ and $:= (N+2)/(N-2)$ $(N \ge 3))$. Then there exists a local-in-time solution $u \in C([0,T]; H^1) \cap C^1([0,T]; L^2)$ with compact support.

Our main concern is in the large time behavior of the solution, which corresponds to that for the corresponding diffusive equation.

First state the case $b(t,x) \equiv 1$, which is again reduced to

$$(DW) \qquad u_{tt} - \Delta u + u_t = f(u), \quad (u, u_t)(0,x) = (u_0, u_1)(x).$$

By the diffusion phenomenon, we expect the solution $u(t,x)$ behaves like the solution $\phi(t,x)$ to

$$(H) \qquad \phi_t - \Delta\phi = f(\phi), \quad \phi(0,x) = \phi_0(x).$$

We have several known results depending on the semilinear absorbing term $f(u) = -|u|^{\rho-1}u$ or semilinear source term $f(u) = +|u|^{\rho-1}u,\ |u|^\rho$. Roughly speaking, we can sum them up as follows.

(Absorbing semilinear term) If $f(u) = -|u|^{\rho-1}u$, then there exists a global-in-time solution $u(t)$ to (DW) for any large data (u_0, u_1), which satisfies

(i) when $\rho > \rho_F(N) = 1 + \frac{2}{N}$, $u(t) \sim M_0 G(t,x)$ as $t \to \infty$, where $M_0 = \int_{\mathbf{R}^N}(u_0 + u_1)(x)\,dx - \int_0^\infty \int_{\mathbf{R}^N} |u|^{\rho-1}u(t,x)dxdt$.

(ii) when $\rho = \rho_F(N)$, $\|u(t)\|_{L^2} = O(t^{-\frac{N}{4}}(\log t)^{-\frac{N}{2}})$ as $t \to \infty$.

(iii) when $1 < \rho < \rho_F(N)$, $\|u(t)\|_{L^2} = O(t^{-\frac{1}{\rho-1}+\frac{N}{4}})$ as $t \to \infty$.

Here we note that, when $1 < \rho < \rho_F(N)$, there exists a similarity solution $w_0(t,x)$ of the form $w_0 = (t+1)^{-\frac{1}{\rho-1}} f_0(\frac{|x|}{\sqrt{t+1}})$, whose decay rate is the same as u in (iii), to the corresponding semilinear heat equation $-\Delta\phi + \phi_t + |\phi|^{\rho-1}\phi = 0$, where f_0 is a solution of

$$-f_0'' - (\frac{r}{2} + \frac{N-1}{r})f_0' + |f_0|^{\rho-1}f_0 = \frac{1}{\rho-1}f_0, \quad \lim_{r\to\infty} r^{\frac{2}{\rho-1}} f_0(r) = 0.$$

(Source semilinear term) If $f(u) = +|u|^{\rho-1}u,\ |u|^\rho$, then

(i) when $\rho > \rho_F(N)$, there exists a global-in-time solution $u(t)$ for small data (u_0, u_1), and $u(t) \sim M_0 G(t,x)$ as $t \to \infty$ with $M_0 = \int_{\mathbf{R}^N}(u_0 + u_1)(x)\,dx + \int_0^\infty \int_{\mathbf{R}^N} f(u)(t,x)dxdt$.

(ii),(iii) when $\rho \le \rho_F(N)$, $f(u) = |u|^\rho$ ($f(u) = +|u|^{\rho-1}u$ in some cases), there is no existence of global-in-time solution $u(t)$ for some data (u_0, u_1).

Thus the critical exponent is the Fujita exponent $\rho_F(N)$. We want to know the critical exponent of our problem (P) for the general time- or space-dependent damping case.

In the case $b(t,x) = \langle x\rangle^{-\alpha}$, $\langle x\rangle = \sqrt{1+|x|^2}$, our problem is

$$(P)_x \qquad u_{tt} - \Delta u + \langle x\rangle^{-\alpha}u_t = f(u), \quad (u, u_t)(0,x) = (u_0, u_1)(x).$$

For this we have the following results.

- If $\alpha > 1$, then the damping is weak and non-effective, so that the equation has the wave structure and the total energy not necessarily decays (Mochizuki [16]).
- If $0 \le \alpha < 1$, then the damping is effective and the equation will have the diffusive structure.
- When $\alpha = 1$, the situation is very delicate (Ikehata, Todorova and Yordanov [8]).

The results on $(P)_x$ are the followings.

Theorem 1 ([19], Absorbing semilinear term). *In case of* $0 \le \alpha < 1$ *the solution of* $(P)_x$ *satisfies*

$$\|u(t)\|_{L^2} = \begin{cases} O(t^{-\frac{N-2\alpha}{2(2-\alpha)}+\delta}) & 1+\frac{2}{N-\alpha} \le \rho < \frac{N+2}{[N-2]_+} \\ O(t^{-\frac{2}{2-\alpha}(\frac{1}{\rho-1}-\frac{N}{4})}) & 1+\frac{2\alpha}{N-\alpha} < \rho \le 1+\frac{2}{N-\alpha} \\ O(t^{-\frac{2}{2-\alpha}(\frac{1}{\rho-1}-\frac{N}{4})}(\log t)^{\frac{1}{2}}) & \rho = 1+\frac{2\alpha}{N-\alpha} \\ O(t^{-\frac{1}{\rho-1}+\frac{\alpha}{2(2-\alpha)}}) & 1 < \rho < 1+\frac{2\alpha}{N-\alpha}. \end{cases}$$

By this, we conjecture $\rho_c(N,\alpha) = 1 + \frac{2}{N-\alpha}$ is critical for $(P)_x$. In fact, Ikehata, Todorova and Yordanov [7] showed the following theorem.

Theorem 2 ([7], Source semilinear term). *Assume* $0 \le \alpha < 1$. *When* $\rho_c(N,\alpha) < \rho < \frac{N+2}{[N-2]_+}$, *a small data global existence of solutions holds for* $(P)_x$. *On the other hand, when* $1 < \rho \le \rho_c(N,\alpha)$, *the solution does not exist globally for any positive in average initial data.*

Thus, $\rho_c(N,\alpha)$ is critical for $(P)_x$, $0 \le \alpha < 1$ with source.

In the case $b(t,x) =: b(t) = (1+t)^{-\beta}$, our problem is

$$(P)_t \qquad u_{tt} - \Delta u + b(t)u_t = f(u), \quad (u,u_t)(0,x) = (u_0,u_1)(x).$$

For the linear equation

$$(LP)_t \qquad v_{tt} - \Delta v + b(t)v_t = 0, \quad (u,u_t)(0,x) = (v_0,v_1)(x),$$

Wirth ([25], [26]) showed the followings by the Fourier transformation.

- If $\beta > 1$, the damping is non-effective and the solution of $(LP)_t$ has the wave property.
- If $-1 < \beta < 1$, the damping is effective and the decay rate is the same as that of solutions of the corresponding diffusive equation (See also Yamazaki [27] for $0 \le \beta < 1$).
- The rest case $\beta < -1$ is classified as the over-damping.
- When $\beta = \pm 1$, the situations are delicate.

We now consider the corresponding parabolic problem when $-1 < \beta < 1$ from another point of view. The linear equation is

$$-\Delta\phi + b(t)\phi_t = 0 \quad \text{or} \quad \phi_t = \frac{1}{b(t)}\Delta\phi, \quad \text{with} \quad \phi(0,x) = \phi_0(x).$$

The explicit formula of solution is

$$\phi(t,x) = \int_{\mathbf{R}^N} (4\pi B(t))^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4B(t)}} \phi_0(y)dy, \quad B(t) = \int_0^t \frac{d\tau}{b(\tau)} \sim t^{1+\beta}.$$

Hence, for $-1 < \beta < 1$ we have the L^p-L^q estimate

$$\|\phi(t)\|_{L^p} \le C\|\phi_0\|_{L^q}\, t^{-\frac{(1+\beta)N}{2}(\frac{1}{q}-\frac{1}{p})} \quad (1 \le q \le p \le \infty),$$

in particular,

$$\|\phi(t)\|_{L^2} = O(t^{-\frac{N}{4}(1+\beta)}). \tag{3.1}$$

On the other hand, the corresponding nonlinear equation is

$$b(t)\phi_t - \Delta\phi + |\phi|^{\rho-1}\phi = 0 \quad (b(t) = (1+t)^{-\beta}).$$

For this equation we have the self-similar solution of the form

$$w_0(t,x) = (c+ct)^{-\frac{1+\beta}{\rho-1}} f\Big(\frac{|x|}{(c+ct)^{\frac{1+\beta}{2}}}\Big) \quad \text{if} \quad \rho < 1 + \frac{2}{N}$$

with $c^{1+\beta}(1+\beta) = 1$ ([22]), which satisfies the decay rate

$$\|w_0(t,\cdot)\|_{L^2} = O(t^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)}). \tag{3.2}$$

Comparing the decay rates of (3.1) and (3.2), we can expect that $\rho_F(N)$ is still critical.

In the absorbing semilinear term we have the following theorem.

Theorem 3. *When $f(u) = -|u|^{\rho-1}u$, $1 < \rho < \frac{N+2}{[N-2]_+}$, the global-in-time solution u to $(P)_t$ with $-1 < \beta < 1$ satisfies*

(i) $\|u(t,\cdot)\|_{L^2} = O(t^{-(\frac{1}{\rho-1}-\frac{N}{4})(1+\beta)})$
provided that $1 < \rho \le \rho_F(N)$ (Nishihara and Zhai [22]).

(ii) $\|u(t,\cdot)\|_{L^2} = O(t^{-\frac{N}{4}(1+\beta)+\varepsilon})$ $(0 < \varepsilon \ll 1)$
provided that $\rho_F(N) \le \rho < \frac{N+2}{[N-2]_+}$ (Nishihara [20]).

Moreover, when $N = 1$ with $(u_0, u_1) \in H^2 \times H^1$ additionally,

$$\|u(t,\cdot) - \theta_0 G_B(t,\cdot)\|_{L^2} = o(t^{-\frac{1}{4}(1+\beta)}) \quad \text{as} \quad t \to \infty \tag{3.3}$$

holds (Nishihara [21]), where

$$\begin{aligned}\theta_0 \;=\; & \int_{\mathbf{R}^1}(u_1 + (1-\beta)u_0)(x)dx \\ & + \int_0^\infty \Big[\frac{\beta(1-\beta)}{(\tau+1)^{(2-\beta)}}\int_{\mathbf{R}^1} u\,dx - (\tau+1)^\beta \int_{\mathbf{R}^1} |u|^{\rho-1}u\,dx\Big]\,d\tau.\end{aligned}$$

Remark on the absorbing semilinear term. From the viewpoint of the diffusion phenomenon, the decay rate (i) in Theorem 3 is optimal in the subcritical exponent, and (ii) is almost optimal in the supercritical one. When $N = 1$, we can conclude that the Fujita exponent $\rho_F(N)$ is completely critical by (3.3). We conjecture that $\rho_F(N)$ is critical even when $N \ge 2$. In the critical exponent, we will have the slightly faster decay rate than $G_B(t,x)$, which is remained open.

§4. Results on $(P)_t$ with Source

In this section we consider

$$(P)_t \qquad u_{tt} - \Delta u + (t+1)^{-\beta} u_t = |u|^\rho, \quad (u, u_t)(0,x) = (u_0, u_1)(x)$$

with $-1 < \beta < 1$. Our main theorems are as follows.

Theorem 4 (Global existence in the supercritical exponent). *Assume $\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}$. If $(u_0, u_1) \in H^1 \times L^2$ is compactly supported and, for $0 < \delta \ll 1$,*

$$I_0^2 := \int_{\mathbf{R}^N} e^{\frac{(1+\beta)|x|^2}{2(2+\delta)}} (|u_1|^2 + |\nabla u_0|^2 + |u_0|^{\rho+1}) dx \ll 1,$$

then there exists a global-in-time solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ to $(P)_t$, which satisfies

$$\|u(t)\|_{L^2} \le C_\delta I_0 (t+1)^{-\frac{N}{4}(1+\beta)+\frac{\varepsilon}{2}} \quad \textit{for} \ \ \varepsilon = \varepsilon(\delta) \searrow 0 \, (\delta \to 0)$$

provided that $\rho_F(N) < \rho < \frac{N+2}{[N-2]_+}$.

Theorem 5 (Blow-up in critical and subcritical exponents). *Suppose that $(u_0, u_1) \in H^1 \times L^2$ are compactly supported with*

$$\int_{\mathbf{R}^N} (u_1(x) + \hat{b}_1 u_0(x)) dx > 0, \quad \hat{b}_1^{-1} = \int_0^\infty e^{-\int_0^t (\tau+1)^{-\beta} d\tau} \, dt.$$

Then the global-in-time solution $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ to $(P)_t$ does not exist provided that $1 < \rho \le \rho_F(N)$.

We only state the key points of proofs of Theorems 4 and 5. Precise proofs are referred to Lin, Nishihara and Zhai [14].

Note that the case $b(t,x) = \langle x \rangle^{-\alpha}(1+t)^{-\beta}$, $\alpha, \beta > 0$, $0 \le \alpha+\beta < 1$ is referred to Lin, Nishihara and Zhai [12, 13] and Wakasugi [24]. By their results, we conjecture that $\rho_c(N,\alpha)$ is still critical. The blow-up result is remained open and the critical exponent is not determined.

Key point of the proof of Theorem 4. The proof is done by the weighted energy method, in which the key is how to determine the weight. In this theorem, as $t \to \infty$ both u_{tt} and $|u|^\rho$ become small. Therefore, $(P)_t$ is approximated by the simplest equation

$$(4.1) \qquad -\Delta u + u_t = 0, \quad u(0,x) = u_0(x).$$

Here we took $\beta = 0$ for simplicity. Since the support of u in (4.1) is compact because of the finite propagation property of (P), we apply the weighted energy method to obtain $\|u(t)\|_{L^2} \le C(t+1)^{-\frac{N}{4}}$. Choosing the weight $e^{2\psi}$ with $\psi = \frac{a|x|^2}{t+1}$ $(a > 0)$, we multiply (4.1) by $2e^{2\psi}u$ to get

$$\begin{aligned}(4.2) \qquad &(e^{2\psi}u^2)_t - 2\nabla\cdot(e^{2\psi}u\nabla u)\\ &\quad +2\Big[e^{2\psi}(-\psi_t)u^2 + e^{2\psi}2\nabla\psi\cdot u\nabla u + e^{2\psi}|\nabla u|^2\Big] = 0.\end{aligned}$$

Thanks to small idea, we change $e^{2\psi}2\nabla\psi\cdot u\nabla u$ more to

$$\begin{aligned}&e^{2\psi}2\nabla\psi\cdot u\nabla u\\ &= e^{2\psi}4\nabla\psi\cdot u\nabla u - e^{2\psi}2\nabla\psi\cdot u\nabla u\\ &= e^{2\psi}4\nabla\psi\cdot u\nabla u - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + e^{2\psi}2|\nabla\psi|^2u^2 + e^{2\psi}(\Delta\psi)u^2,\end{aligned}$$

then (4.2) becomes

$$\begin{aligned}&(e^{2\psi}u^2)_t - 2\nabla\cdot(e^{2\psi}u\nabla u + e^{2\psi}u^2\nabla\psi)\\ &+2e^{2\psi}\Big[\underbrace{(-\psi_t + 2|\nabla\psi|^2)}_{(1/4a+2)|\nabla\psi|^2}u^2 + 4u\nabla\psi\cdot\nabla u + |\nabla u|^2\Big] + e^{2\psi}\underbrace{(2\Delta\psi)}_{4aN/(t+1)}u^2\\ &= 0,\end{aligned}$$

by using $-\psi_t = \frac{a|x|^2}{(t+1)^2} = \frac{1}{4a}|\nabla\psi|^2$ and $\Delta\psi = \frac{2aN}{t+1}$. Taking $a = 1/8$ and integrating this equation over $\mathbf{R}^N$ yield

$$\frac{d}{dt}\int_{\mathbf{R}^N} e^{2\psi}u^2\,dx + 2\int_{\mathbf{R}^N} e^{2\psi}|2u\nabla\psi + \nabla u|^2dx + \frac{N/2}{t+1}\int_{\mathbf{R}^N} e^{2\psi}u^2\,dx = 0.$$

This implies the desired result and the basic weight will be determined.

Key point of the proof of Theorem 5. For the proof we apply the test function method developed by Qi S. Zhang [28]. We observe it by the simpler equation with $\beta = 0$

$$(4.3) \qquad u_{tt} - \Delta u + u_t = |u|^\rho, \quad (u, u_t)(0, x) = (u_0, u_1)(x).$$

Assuming that u is a global-in-time non-trivial solution to (4.3), we derive the contradiction. For $\frac{1}{\rho} + \frac{1}{\rho'} = 1$ set

$$I_R := \int_{Q_R} |u|^\rho\cdot(\psi_R)^{\rho'}(t,x)dxdt = \int_{Q_R}(u_{tt} - \Delta u + u_t)\cdot(\psi_R)^{\rho'}dxdt,$$

where $Q_R = [0, R^2] \times B_R(0),\ B_R(0) = \{|x| \le R\}$ and the test function

$$\begin{aligned}
\psi_R(t,x) &= \eta_R(t)\phi_R(r) = \eta(\tfrac{t}{R^2})\phi(\tfrac{r}{R}),\ r = |x| \\
&\quad 0 \le \eta \le 1,\ \eta(t) = \begin{cases} 1 & t \in [0,1/4] \\ 0 & t \in [1,\infty) \end{cases},\ |\eta'(t)|, |\eta''(t)| \le C, \\
\text{with} &\quad 0 \le \phi \le 1,\ \ \phi(r) = \begin{cases} 1 & r \in [0,1/2] \\ 0 & r \in [1,\infty) \end{cases},\ |\phi'(r)|, |\phi''(r)| \le C, \\
&\quad (\eta')^2/\eta \le C\ (0 \le t \le 1),\ \ |\nabla\phi|^2/|\phi| \le C\ (0 \le r \le 1).
\end{aligned}$$

Then, by the integration by parts, for example (by abbreviating $dx\,dt$),

$$\begin{aligned}
&\textstyle\int_{Q_R} u_t (\psi_R)^{\rho'} \\
&= -\textstyle\int_{B_R} u_0 - \int_{\hat{Q}_{R,t}} u \cdot \rho' (\psi_R)^{\rho'-1} \cdot \eta'(\frac{t}{R^2})\frac{1}{R^2} \cdot \phi(\frac{|x|}{R}) \\
&\le -\textstyle\int_{B_R} u_0 + \Big(\int_{\hat{Q}_{R,t}} |u|^\rho (\psi_R)^{\rho'}\Big)^{\frac{1}{\rho}} \Big(\int_{\hat{Q}_{R,t}} \{\eta'(\frac{t}{R^2})\phi(\frac{|x|}{R})\}^{\rho'}\Big)^{\frac{1}{\rho'}} \frac{C}{R^2} \\
&\le -\textstyle\int_{B_R} u_0 dx + C(\hat{I}_{R,t})^{\frac{1}{\rho}} R^{(2+N)\frac{1}{\rho'}-2},\quad B_R := B_R(0).
\end{aligned}$$

Here we have used the Hölder inequality with $\rho' = (\rho'-1)\rho$ and

$$\hat{I}_{R,t} := \int_{\hat{Q}_{R,t}} |u|^\rho (\psi_R)^{\rho'} dx\,dt = \int_{R^2/4}^{R^2} \int_{B_R} |u|^\rho (\psi_R)^{\rho'} dx\,dt.$$

By similar ways to the other terms in I_R, we have

$$I_R \ \le\ -\textstyle\int_{B_R} (u_0 + u_1)(x)dx + C(\hat{I}_{R,t} + \hat{I}_{R,|x|})^{\frac{1}{\rho}} R^{\frac{N+2}{\rho'}-2},$$

where $\hat{I}_{R,|x|} = \int_0^t \int_{B_R \setminus B_{R/2}} |u|^\rho (\psi_R)^{\rho'} dx\,dt$. Here $\frac{N+2}{\rho'} - 2 < 0$ is equivalent to $\rho < 1 + \frac{2}{N} = \rho_F(N)$. Since $\int_{\mathbf{R}^N} (u_0+u_1)(x)dx > 0$, if $\rho < \rho_F(N)$, then $(I_R)^{1-\frac{1}{\rho}} \le CR^{\frac{N+2}{\rho'}-2}$ and $I_R \to 0$ as $R \to \infty$, which contradicts to the non-triviality of u. If $\rho = \rho_F(N)$, then $I_R \le C$, that is, $\int_0^\infty \int_{\mathbf{R}^N} |u|^\rho dx\,dt < \infty$ by taking $R = \infty$. Hence $I_R \le -\int_{B_R} (u_0 + u_1)dx + C(\hat{I}_{R,t} + \hat{I}_{R,|x|})^{\frac{1}{\rho}}$ and both $\hat{I}_{R,t}$ and $\hat{I}_{R,|x|}$ tend to zero as $R \to \infty$. Thus we again reach to the contradiction.

By this proof we know it is a key point that the left hand side of (4.3) is the divergence form. However, our equation in $(P)_t$ is not and some idea is necessary. To overcome this, we multiply $(P)_t$ by some non-negative function $g(t)$ to get

$$(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + (-g'(t) + b(t)g(t))u_t = g(t)|u|^\rho.$$

Hence we choose $g(t)$ by the ordinary differential equation

$$-g'(t) + b(t)g(t) = 1,\quad g(0) = 1/\hat{b}_1,\quad \hat{b}_1 = \Big(\int_0^\infty e^{-\int_0^t b(s)ds} dt\Big)^{-1},$$

that is, explicitly, $g(t) = e^{\int_0^t b(s)ds}(\int_0^\infty e^{-\int_0^\tau b(s)ds}d\tau - \int_0^t e^{-\int_0^\tau b(s)ds}d\tau)$. Thus, we have the divergence form

$$(g(t)u)_{tt} - \Delta(g(t)u) - (g'(t)u)_t + u_t = g(t)|u|^\rho,$$

and, as above, we set $I_R = \int_{Q_R} g(t)|u|^\rho \cdot (\psi_R)^{\rho'} dx\, dt$ with $Q_R = [0, R^{\frac{2}{1+\beta}}] \times B_R(0)$ and $\psi_R(t,x) = \eta_R(t) \cdot \phi_R(r) = \eta(\frac{t}{R^{2/(1+\beta)}}) \cdot \phi(\frac{|x|}{R})$.

Again, assuming that u is a non-trivial global solution, we can derive the contradiction if $\rho \le \rho_F(N)$. We omit the details.

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Faculty of Political Science and Economics
Waseda University
Tokyo 169-8050, Japan
E-mail address: `kenji@waseda.jp`