

Toward motivic integration over wild Deligne-Mumford stacks

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Abstract.

We discuss how the motivic integration will be generalized to wild Deligne-Mumford stacks, that is, stabilizers may have order divisible by the characteristic of the base or residue field. We pose several conjectures on this topic. We also present some possible applications concerning stringy invariants, resolution of singularities, and weighted counts of extensions of local fields.

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§1. Introduction

The aim of this paper is to present an attempt to generalize the motivic integration to wild Deligne-Mumford stacks. In relation to the

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McKay correspondence, Denef and Loeser [9] developed the motivic integration applicable to the quotient map $\mathbb{A}_k^d \rightarrow \mathbb{A}_k^d/G$ for a finite group $G \subset GL_d(k)$ with k a field of characteristic zero. Motivated by this, the author [31, 30] developed the motivic integration over Deligne-Mumford stacks under some tameness condition. In this paper, we will make conjectures on how the theory will be generalized by dropping the tameness condition. The easiest wild case where G has order equal to the characteristic has been already studied in [32].

Let A be a complete discrete valuation ring with algebraically closed residue field and $D := \text{Spec } A$. For a Deligne-Mumford stack \mathcal{X} of finite type over D , we will define a *twisted arc* of \mathcal{X} as a representable D -morphism $\mathcal{E} \rightarrow \mathcal{X}$, where \mathcal{E} is the quotient stack associated to some Galois cover of D . We expect that we can develop the motivic integration on the space of twisted arcs of \mathcal{X} , which we will denote by $\mathcal{J}_\infty \mathcal{X}$. Our main conjecture is the following:

Conjecture 1.1 (Conjecture 5.5). *For a proper birational morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of pure-dimensional Deligne-Mumford stacks of finite type over D , we have a natural map $f_\infty : \mathcal{J}_\infty \mathcal{Y} \rightarrow \mathcal{J}_\infty \mathcal{X}$. Then, for a measurable function F on a subset $C \subset \mathcal{J}_\infty \mathcal{X}$, we have the change of variables formula*

$$\int_C \mathbb{L}^{F+w_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int_{f_\infty^{-1}(C)} \mathbb{L}^{F \circ f_\infty - \text{ord Jac}_f + w_{\mathcal{Y}}} d\mu_{\mathcal{Y}}.$$

Here ord Jac_f is the Jacobian order function of f and $w_{\mathcal{X}}$ and $w_{\mathcal{Y}}$ are canonically defined weight functions on $\mathcal{J}_\infty \mathcal{X}$ and $\mathcal{J}_\infty \mathcal{Y}$ respectively.

In this paper, we will try to justify the conjecture. To do so, our main tool is what we call *untwisting*, a technique reducing twisted arcs to non-twisted ones. We will study the formula in more detail when \mathcal{Y} is the quotient stack $[\mathbb{A}_D^d/G]$ and \mathcal{X} is the quotient variety \mathbb{A}_D^d/G for some linear action $G \curvearrowright \mathbb{A}_D^d$ of a finite group G . In this case, we will make a more explicit and conjectural expression of the weight function.

As an application, we will reach another conjecture which ties stringy invariants of quotient singularities with mass formulae for extensions of local fields. For a local field K with residue field having q elements, Serre [23] proved a mass formula

$$\sum_L \frac{1}{\#\text{Aut}(L/K)} \cdot q^{-d(L)} = q^{1-n},$$

where L runs over isomorphism classes of totally ramified field extensions L/K of degree n and $d(L)$ the discriminant exponent of L/K . Bhargava

[5] proved an analogous mass formula for all étale extensions of fixed degree. Also we may consider the motivic version of such counts, replacing q with \mathbb{L} and summation with motivic integration over the space of extensions. From our conjectural change of variables formula, we will derive an equality between a stringy invariant of the associated quotient singularity and some motivic count of extensions of the local field, the fraction field of A . We may regard this as a version of the McKay correspondence. Kedlaya [13] and then Wood [25] worked on this kind of counting problems in terms of local Galois representations. Their works might have a more direct relation with our motivic count.¹

Another possible application concerns resolution of singularities. Stringy invariants have a lot of information on resolution of singularities. Our change of variables formula would be helpful in reducing stringy invariants for varieties with quotient singularities to those for smooth Deligne-Mumford stacks, which might be easier to compute. An existence of resolution impose some constraints on stringy invariants. Thus, if we find a singular variety with stringy invariant violating one of them, then we can prove the non-existence of resolution of singularities.² Similar arguments can apply to the problem on the non-existence of crepant resolution.

We will also discuss the problem when a family of quotient singularities has a constant stringy invariant, especially in the case where the family contains both tame and wild ones.

There have been considerable developments [27, 26, 28, 29] on the subject after the acceptance of the paper for publication until the final proof. I add several footnotes to mention them. All footnotes in the paper are added at the final proof.

1.1. Acknowledgements

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¹Afterwards such a relation was studied in [27, 26].

²Afterwards this viewpoint was adopted in [26].

1.2. Conventions

We fix an algebraically closed field k , a complete discrete valuation ring A with residue field k . We denote the fraction field of A by K . We put $D := \text{Spec } A$ and $D^* := \text{Spec } K$: we call them a *formal disk* and *punctured formal disk* respectively. Always G will denote a finite group.

As many of our statements will be conjectural, we will often make heuristic arguments rather than rigorous ones. For this reason, we often identify a k -scheme with its k -point set.

§2. The ring of values

Motivic measures and integrals usually take values in some extension of the Grothendieck ring of varieties. There are several versions of such extensions which are slightly different one another. Throughout the paper, we fix an appropriate one and denote it by $\hat{\mathcal{M}} = \hat{\mathcal{M}}_k$. The following are some of its properties which will be necessary in subsequent sections:

- (1) $\hat{\mathcal{M}}$ has an element associated to each scheme X of finite type over k , denoted $[X]$. We write $\mathbb{L} := [\mathbb{A}_k^1]$.
- (2) $\hat{\mathcal{M}}$ contains the fractional powers \mathbb{L}^a , $a \in \mathbb{Q}$ of \mathbb{L} .
- (3) For a bijective morphism $Y \rightarrow X$, $[Y] = [X]$. In particular, $[X] = [X_{\text{red}}]$. Also, if $Y \subset X$ is a closed subset, then $[X] = [Y] + [X \setminus Y]$. This enables us to determine $[C] \in \hat{\mathcal{M}}$ for a constructible subset C of a variety.
- (4) Let $f : Y \rightarrow X$ be a morphism such that every fiber $f^{-1}(x)$ admits a bijective morphism from or to the quotient variety \mathbb{A}_k^n/H for some fixed n and for some linear action $H \curvearrowright \mathbb{A}_k^n$ of a finite group H . (We will call f an \mathbb{L}^n -fibration.) Then $[Y] = [X]\mathbb{L}^n$.
- (5) For $r \in \mathbb{Z}_{>0}$ and a countable index set I , an infinite sum of the form

$$\sum_{i \in I} [X_i] \mathbb{L}^{a_i}, \quad a_i \in \frac{1}{r} \mathbb{Z}$$

converges if and only if for every $a \in \mathbb{R}$, there exists at most finitely many i such that $\dim X_i + a_i \geq a$.

- (6) There exists a ring homomorphism

$$P : \hat{\mathcal{M}} \rightarrow \bigcup_{r=1}^{\infty} \mathbb{Z}((T^{-1/r}))$$

called the *virtual Poincaré realization*. For a variety X , $P([X])$ equals the virtual Poincaré polynomial $P(X)$ of X . If X is

smooth and proper, then $P(X) = \sum_i (-1)^i b_i(X) T^i$, where $b_i(X)$ is the i -th Betti number for l -adic cohomology. Also we have $P(\mathbb{L}) = T^2$.

Concerning the construction of $\hat{\mathcal{M}}$, Properties (1) and (2) determine generators, (3) and (4) relations and (5) how to complete the ring. Property (6) is rather a consequence of the construction. For instance, see [32] for details.

§3. Motivic integration over varieties

In this section, we will briefly recall the motivic integration invented by Kontsevich, developed by Denef and Loeser [8, 9] in characteristic zero, and Sebag [22] (see also [18]) in positive or mixed characteristic.

3.1. Varieties over k

Let X be a variety of pure dimension d over k . For a non-negative integer n , the n -jet scheme of X , denoted $J_n X$, is the fine moduli scheme parameterizing n -jets:

$$J_n X = \{ \text{Spec } k[t]/(t^{n+1}) \rightarrow X \}.$$

They are schemes of finite type over k . For $n \geq m$, we have a natural morphism $J_n X \rightarrow J_m X$, called the *truncation*. The *arc space* of X , denoted $J_\infty X$, is the projective limit of $J_n X$, $n \in \mathbb{Z}_{\geq 0}$, which parameterizes *arcs*:

$$J_\infty X = \{ \text{Spec } k[[t]] \rightarrow X \}.$$

The natural maps $\pi_n : J_\infty X \rightarrow J_n X$ are also called *truncations*.

The arc space has a measure taking values in $\hat{\mathcal{M}}$, which is called the *motivic measure*. We denote it by μ_X and define it as follows: a subset $C \subset J_\infty X$ is said to be *stable* if for some n ,

- (1) $\pi_n(C)$ is a constructible subset,
- (2) $C = \pi_n^{-1}(\pi_n(C))$, and
- (3) for every $n' \geq n$, $\pi_{n'+1}(C) \rightarrow \pi_n(C)$ is an \mathbb{L}^d -fibration.

For a stable subset C , we put

$$\mu_X(C) := [\pi_n(C)] \mathbb{L}^{-nd} \in \hat{\mathcal{M}}, \quad (n \gg 0).$$

This defines the motivic measure on stable subsets. We can extend this to a larger class of subsets called *measurable subsets*. Roughly, a measurable subset is a subset of $J_\infty X$ which can be approximated by a series of stable subsets. Let

$$F : J_\infty X \supset C \rightarrow \frac{1}{r} \mathbb{Z} \cup \{\infty\}$$

be a measurable function on a subset C for some $r \in \mathbb{Z}_{>0}$, that is, every fiber of it is a measurable subset and $F^{-1}(\infty)$ has measure zero. Then we define the *motivic integral* of \mathbb{L}^F by

$$\int_C \mathbb{L}^F d\mu_X := \sum_{n \in \frac{1}{r}\mathbb{Z}} \mu_X(F^{-1}(n))\mathbb{L}^n \in \hat{\mathcal{M}} \cup \{\infty\}.$$

Here, if the infinite sum diverges, then we put the integral to be ∞ .

3.2. Varieties over D

All these about the motivic integration over k -varieties can be generalized to D -varieties. Let X be a D -variety, that is, an integral scheme of finite type over D . For $n \in \mathbb{Z}_{\geq 0}$, we put $A_n := A/\mathfrak{m}^{n+1}$, $D_n := \text{Spec } A_n$. In this setting, we put

$$J_n X = \{D\text{-morphisms } D_n \rightarrow X\},$$

which is a scheme of finite type over k (not D). (The functor represented by this scheme is called a *Greenberg functor*.) Then $J_\infty X$ is again the projective limit of $J_n X$, $n \geq 0$ and

$$J_\infty X = \{D\text{-morphisms } D \rightarrow X\}.$$

Let us suppose also that X is flat and of relative dimension d over D . Then, in the same way as in the case of k -varieties, we can define the motivic measure on $J_\infty X$ and the motivic integral $\int_C \mathbb{L}^F d\mu_X$ for a measurable function F on a subset C of $J_\infty X$. The motivic integration for a k -variety X is equivalent to the one for the induced D -variety $X \times_k D$ with $D = \text{Spec } k[[t]]$.

§4. Motivic integration over Deligne-Mumford stacks

4.1. Spaces of G -covers of a formal disk

We mean by a G -cover of D^* an étale G -torsor $E^* \rightarrow D^*$. Then a G -cover of D is the finite cover $E \rightarrow D$ associated to a G -cover $E^* \rightarrow D^*$ of D^* . Here E is the unique normal scheme finite over D containing E^* as an open dense subscheme. Let \overline{K} be the algebraic closure of K . A G -cover $E^{(*)} \rightarrow D^{(*)}$ is called *pointed* if a lift of the natural \overline{K} -point of $D^{(*)}$ to $E^{(*)}$ is prescribed.

Conjecture 4.1. *There exist the moduli spaces parameterizing isomorphism classes of the pointed/unpointed G -covers of D : we will denote them by $G\text{-Cov}^{\text{pt}}(D)$ and $G\text{-Cov}(D)$ respectively.*

The group G acts on $G\text{-Cov}^{\text{pt}}(D)$ by changing the given pointing or equivalently by replacing the given G -action on E with conjugation. Then we should have

$$G\text{-Cov}^{\text{pt}}(D)/G = G\text{-Cov}(D).$$

The conjecture holds if $\text{char}(k) \nmid \#G$. Indeed the moduli spaces are just finite points in this case. Moreover $G\text{-Cov}(D) \cong G\text{-Cov}(D^*)$ has exactly $\#\text{Conj}(G)$ points. When A has equal characteristic $p > 0$ and G is a p -group, Harbater [10] constructed the (coarse) moduli spaces of G -covers of D^* . If G_K denotes the absolute Galois group of K , then there exists a one-to-one correspondence between $G\text{-Cov}^{\text{pt}}(D)$ and the set of continuous homomorphisms $G_K \rightarrow G$. Actually the conjecture above seems to be one of the technical keys of the whole story in this paper. For instance, the existence of moduli spaces of G -jets/arcs and twisted jets/arcs discussed below would easily follow from the conjecture along the same line as in [32]. A recent result by Tonini [24] may be helpful in addressing this problem.

Besides Conjecture 4.1, we also expect that $G\text{-Cov}(D)$ is the inductive limit of some series

$$V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \dots$$

such that V_i are schemes of finite type and $V_i \rightarrow V_{i+1}$ are injective morphisms. (We expect them to be not immersions but rather something like immersions followed by Frobenius morphisms, as such a phenomenon appears in [10].) We will say that a subset of $G\text{-Cov}(D)$ is *constructible* if it is the image of a constructible subset of some V_i . To a constructible subset C of $G\text{-Cov}(D)$, we associate $[C] \in \hat{\mathcal{M}}$ in the obvious way. We obtain the *tautological motivic measure* on $G\text{-Cov}(D)$:

$$\begin{aligned} \tau : \{\text{constructible subsets of } G\text{-Cov}(D)\} &\rightarrow \hat{\mathcal{M}} \\ C &\mapsto \tau(C) = [C] \end{aligned}$$

4.2. G -arcs and jets

Let M be a D -variety endowed with a G -action.

Definition 4.2. We define a G -arc of M as a G -equivariant D -morphism $E \rightarrow M$ for some G -cover $E \rightarrow D$. Two G -arcs $E, E' \rightarrow M$ are *isomorphic* if there exists an isomorphism $E \rightarrow E'$ of G -covers compatible with the given morphisms $E, E' \rightarrow M$.

For a connected Galois cover $E \rightarrow D$ of degree e , we define a closed subscheme E_n of E to be the one having length $1 + ne$. In particular,

$E_0 \cong \text{Spec } k$. For a non-connected Galois cover $E = \bigsqcup_i E^i \rightarrow D$ with E^i connected components, we put $E_n := \bigsqcup_i E_n^i$.

Definition 4.3. We define a G - n -jet of M as a pair $(E, E_n \rightarrow M)$ of a G -cover $E \rightarrow D$ and a G -equivariant D -morphism $E_n \rightarrow M$. Two G - n -jets $(E, E_n \rightarrow M)$ and $(E', E'_n \rightarrow M)$ are *isomorphic* if there exists an isomorphism $E \rightarrow E'$ of G -covers such that the induced isomorphism $E_n \rightarrow E'_n$ is compatible with the given morphisms to M .

Conjecture 4.4. *There exist moduli spaces parameterizing isomorphism classes of G -arcs and G - n -jets of M for every n : we will denote them by $J_\infty^G M$ and $J_n^G M$ respectively.*

If this is the case, we would have natural morphisms

$$J_\infty^G M \rightarrow \cdots \rightarrow J_{n+1}^G M \rightarrow J_n^G M \rightarrow \cdots \rightarrow J_0^G M \rightarrow G\text{-Cov}(D).$$

For $E \in G\text{-Cov}(D)$ and for each n , we put $J_n^{G,E} M \subset J_n^G M$ to be the fiber over E .

Lemma 4.5. *Let $E \in G\text{-Cov}(D)$ and $G' \subset G$ the stabilizer of a connected component E' of E , that is, $G' := \{g \in G \mid g(E') = E'\}$. Let $M_0 := M \times_D D_0$. (Recall $D_0 = \text{Spec } k$.) Then*

$$J_0^{G,E} M \cong M_0^{G'} / N_G(G').$$

Here $N_G(G')$ is the normalizer of G' in G .

Proof. A pointed G -0-jet $\alpha : E_0 \rightarrow M_0$ is uniquely determined by $\alpha(E'_0) \in M_0$, which is invariant under the G' -action. Changing the connected component, $\alpha(E'_0)$ will be changed to a point of M_0 invariant under a subgroup conjugate to G' . Thus we have

$$J_0^{G,E} M \cong \left(\bigcup_{H: \text{conjugate to } G'} M_0^H \right) / G \cong M_0^{G'} / N_G(G').$$

Q.E.D.

4.3. Twisted arcs and jets

Let \mathcal{X} be a DM stack of finite type over D .

Definition 4.6. A *stacky formal disk* is a connected and normal DM stack \mathcal{E} which is birational and finite over D . A *twisted arc* of \mathcal{X} is a representable D -morphism $\mathcal{E} \rightarrow \mathcal{X}$ with \mathcal{E} a stacky formal disk. (Note that $\mathcal{E} \rightarrow \mathcal{X}$ is representable if and only if the induced map of stabilizers

of closed points is injective.) Two twisted arcs $\mathcal{E} \rightarrow \mathcal{X}$ and $\mathcal{E}' \rightarrow \mathcal{X}$ are *isomorphic* if there exists an isomorphism $\mathcal{E} \rightarrow \mathcal{E}'$ making the diagram

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}' \\ & \searrow & \downarrow \\ & & \mathcal{X} \end{array}$$

2-commutative.

Conjecture 4.7. *There exists a moduli space parameterizing isomorphism classes of twisted arcs of \mathcal{X} : we will denote it by $\mathcal{J}_\infty \mathcal{X}$.*

Let us suppose that this conjecture holds.

Lemma 4.8. *Suppose that \mathcal{X} is a quotient stack $[M/G]$ for M and G as above. Then we have*

$$\mathcal{J}_\infty \mathcal{X} \cong J_\infty^G M.$$

Proof. Let $E \rightarrow M$ be a G -arc. Taking the quotient stacks, we obtain a twisted arc

$$[E/G] \rightarrow \mathcal{X}.$$

This defines a map $J_\infty^G M \rightarrow \mathcal{J}_\infty \mathcal{X}$. Conversely, for a twisted arc $\mathcal{E} \rightarrow \mathcal{X}$, we put $E := M \times_{\mathcal{X}} \mathcal{E}$. Then the map $E \rightarrow M$ is a G -arc. Q.E.D.

Although not defining *twisted jets*, we expect:

Conjecture 4.9. *For each $n \in \mathbb{Z}_{\geq 0}$, there exists a moduli space $\mathcal{J}_n \mathcal{X}$ of twisted n -jets of \mathcal{X} satisfying:*

- (1) *If $\mathcal{X} = [M/G]$, then $\mathcal{J}_n \mathcal{X} = J_n^G M$.*
- (2) *There exist truncation maps $\mathcal{J}_{n+1} \mathcal{X} \rightarrow \mathcal{J}_n \mathcal{X}$ making $\mathcal{J}_\infty \mathcal{X}$ the projective limit of $\mathcal{J}_n \mathcal{X}$, $n \in \mathbb{Z}_{\geq 0}$.*

Assuming these conjectures, when \mathcal{X} is of pure dimension d (relatively over D), we can define the motivic measure $\mu_{\mathcal{X}}$ on $\mathcal{J}_\infty \mathcal{X}$ in a similar way as the one on the arc space of a variety. The untwisting technique (Section 7) would provide an evidence that there are enough stable or measurable subsets of $\mathcal{J}_\infty \mathcal{X}$.

§5. Formulating the change of variables formula for DM stacks

5.1. Maps of twisted arc spaces

Given a morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of DM stacks of finite type over D , we can construct a map

$$f_\infty : \mathcal{J}_\infty \mathcal{Y} \rightarrow \mathcal{J}_\infty \mathcal{X}$$

as follows: let $\gamma : [E/G] \rightarrow \mathcal{Y}$ be a twisted arc with E a connected G -cover of D . Let $y \in \mathcal{Y}$ be the image of the closed point by γ and $x \in \mathcal{X}$ the image of y by f . Set

$$N := \text{Ker}(G \xrightarrow{\gamma} \text{Aut}(y) \xrightarrow{f} \text{Aut}(x)).$$

Then $f \circ \gamma$ factors as

$$f \circ \gamma : \mathcal{E} = [E/G] \rightarrow \mathcal{E}' = [(E/N)/(G/N)] \rightarrow \mathcal{X}.$$

Here $[(E/N)/(G/N)]$ is the quotient stack associated to the induced action of the quotient group G/N on the quotient scheme E/N . We define $f_\infty(\gamma)$ to be the induced morphism $\mathcal{E}' \rightarrow \mathcal{X}$.

Lemma 5.1 (Almost bijectivity lemma). *Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of DM stacks of finite type over D . Suppose that for closed substacks $\mathcal{Y}' \subset \mathcal{Y}$ and $\mathcal{X}' \subset \mathcal{X}$, f induces an isomorphism $\mathcal{Y} \setminus \mathcal{Y}' \xrightarrow{\sim} \mathcal{X} \setminus \mathcal{X}'$. Then f_∞ restricts to a bijection*

$$\mathcal{J}_\infty \mathcal{Y} \setminus \mathcal{J}_\infty \mathcal{Y}' \rightarrow \mathcal{J}_\infty \mathcal{X} \setminus \mathcal{J}_\infty \mathcal{X}'.$$

Proof. Let $(\gamma : \mathcal{E} \rightarrow \mathcal{X}) \in \mathcal{J}_\infty \mathcal{X} \setminus \mathcal{J}_\infty \mathcal{X}'$. Then we put $\tilde{\mathcal{E}}$ to be the normalization of the irreducible component of $\mathcal{E} \times_{\mathcal{X}} \mathcal{Y}$ dominating \mathcal{E} . The induced morphism $\tilde{\gamma} : \tilde{\mathcal{E}} \rightarrow \mathcal{Y}$ is the unique twisted arc with $f_\infty(\tilde{\gamma}) = \gamma$. This shows the lemma. Q.E.D.

If \mathcal{Y} and \mathcal{X} have pure dimension d , and if $\dim \mathcal{X}'$, $\dim \mathcal{Y}' < d$, then we expect that $\mathcal{J}_\infty \mathcal{X}'$ and $\mathcal{J}_\infty \mathcal{Y}'$ have measure zero as subsets of $\mathcal{J}_\infty \mathcal{X}$ and $\mathcal{J}_\infty \mathcal{Y}$ respectively. Then the lemma says that $f_\infty : \mathcal{J}_\infty \mathcal{Y} \rightarrow \mathcal{J}_\infty \mathcal{X}$ is almost bijective. Thanks to this, we can expect that motivic integrals on $\mathcal{J}_\infty \mathcal{X}$ are transformed into ones on $\mathcal{J}_\infty \mathcal{Y}$ and vice versa. The formula describing the transform will be called the *change of variables formula*.

5.2. Order functions associated to submodules

To formulate the change of variables formula, we need to introduce order functions on the space of twisted arcs.

Definition 5.2. Let \mathcal{X} be a DM stack of pure dimension d , let $N \subset M$ be coherent $\mathcal{O}_{\mathcal{X}}$ -modules which are generically of rank one. (In practice, we often take $\mathcal{O}_{\mathcal{X}}$, $\Omega_{\mathcal{X}/D}^d$ or $\omega_{\mathcal{X}/D}$ as M .) Then we define the *order function* of the submodule N

$$\text{ord } N : \mathcal{J}_\infty \mathcal{X} \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}$$

as follows: let $\gamma : [E/G] \rightarrow \mathcal{X}$ be a twisted arc with E a connected G -cover of D . If N_E and M_E denote the pull-backs of N and M to E , and if M_E/tors is free of rank one, then

$$\text{Im}(N_E \rightarrow M_E/\text{tors}) = s^n(M_E/\text{tors})$$

for some $n \in \mathbb{Z}_{\geq 0}$ with s a uniformizer of the function field of E . Then we define

$$(\text{ord } N)(\gamma) := \frac{n}{\#G}.$$

If M_E/tors is not free of rank one, then we put $(\text{ord } N)(\gamma) = \infty$, expecting that this does not happen for almost all γ .

Remark 5.3. Generalizing this, we can also define the order function of a fractional submodule N of M , that is, an $\mathcal{O}_{\mathcal{X}}$ -submodule of $M \otimes_{\mathcal{O}_{\mathcal{X}}} Q$ with Q the sheaf of total quotient rings. Then the function may take negative values.

Definition 5.4. For a generically étale morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of DM stacks, we define its *Jacobian order function*, ord Jac_f , as the order function of the submodule $\text{Im}(f^* \Omega_{\mathcal{X}/D}^d \rightarrow \Omega_{\mathcal{Y}/D}^d) \subset \Omega_{\mathcal{Y}/D}^d$.

5.3. The change of variables formula

The following is the main conjecture of this paper:

Conjecture 5.5 (Main Conjecture). *For each DM stack \mathcal{X} of finite type and pure dimension over D , there exists a canonically defined function*

$$w_{\mathcal{X}} : \mathcal{J}_{\infty} \mathcal{X} \rightarrow \left(\bigcup_{x \in \mathcal{X}} \frac{1}{\#\text{Aut}(x)} \mathbb{Z} \right)$$

such that if \mathcal{X} is a scheme, then $w_{\mathcal{X}} \equiv 0$. (We call this the weight function on $\mathcal{J}_{\infty} \mathcal{X}$.) Moreover, for a proper birational morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of such stacks, and for a measurable function $F : \mathcal{J}_{\infty} \mathcal{X} \supset C \rightarrow \frac{1}{r} \mathbb{Z} \cup \{\infty\}$, we have

$$\int_C \mathbb{L}^{F+w_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int_{f^{-1}(C)} \mathbb{L}^{F \circ f_{\infty} - \text{ord Jac}_f + w_{\mathcal{Y}}} d\mu_{\mathcal{Y}}.$$

This is basically of the same form as the formula in the tame setting proved in [31]. However differences lie for instance in the facts that in the wild case, the moduli spaces of twisted arcs are much larger and that weight functions may take negative and unbounded values. The conjecture has been proved in [32] for the morphism from the quotient stack $[\mathbb{A}_k^d/G]$ to the quotient variety \mathbb{A}_k^d/G associated to a linear action of $G = \mathbb{Z}/p\mathbb{Z}$ with $p = \text{char}(k)$.

§6. More details on the linear case

Let $G \subset GL_d(A)$ be a finite subgroup, which acts on $V := \mathbb{A}_D^d$: we call V a G -representation over D . Then we put $\mathcal{X} := [V/G]$ and $X = V/G$, the quotient stack and variety. In this section, we will study the detailed structure of twisted arc and jet spaces and obtain a conjectural “explicit” expression of the weight function $w_{\mathcal{X}}$.

6.1. Spaces of arcs and jets

Let

$$E = \text{Spec } B \rightarrow D = \text{Spec } A$$

be a G -cover. We will study the structure of $J_{\infty}^{G,E}V$ and $\pi_n(J_{\infty}^{G,E}V)$, $n \in \mathbb{Z}_{\geq 0}$, where π_n are the truncation maps. Let $A[x_1, \dots, x_d]$ be the coordinate ring of V . We have the induced A -linear action of G on $A^d = \bigoplus_{i=1}^d A \cdot x_i$. This uniquely extends to a B -linear action on B^d . Let

$$\rho : G \rightarrow \text{Aut}_B(B^d)$$

be the corresponding map. On the other hand, G acts on B and diagonally on B^d , which induces a map

$$\delta : G \rightarrow \text{Aut}_A(B^d).$$

Since $\rho(g), g \in G$ are represented by $d \times d$ matrices with entries from A , for every $g, g' \in G$, $\rho(g)$ and $\delta(g')$ are commutative. Hence the map

$$\rho\delta^{-1} : G \rightarrow \text{Aut}_A(B^d), g \mapsto \rho(g) \circ \delta(g)^{-1}$$

is a group homomorphism.

Definition 6.1. We define $\Xi_{B/A}^V \subset B^d$ to be the G -invariant subset for $\rho\delta^{-1}$, which is the locus where the two G -actions ρ and δ coincide. For $n \in \mathbb{Z}_{\geq 0}$, writing $E_n = \text{Spec } B_n$, we put $\Xi_{B/A,n}^V$ to be the image of $\Xi_{B/A}^V$ in B_n^d .

By definition, these modules $\Xi_{B/A}^V, \Xi_{B/A,n}^V$ have the unique natural G -action.

Lemma 6.2. We can identify $J_{\infty}^{G,E}V$ with $\Xi_{B/A}^V/G$ and $\pi_n(J_{\infty}^{G,E}V)$ with $\Xi_{B/A,n}^V/G$.

Proof. A G -arc $E \rightarrow V$ corresponds to an equivariant homomorphism

$$A[x_1, \dots, x_d] \rightarrow B.$$

In turn, this corresponds to an element of $\Xi_{B/A}^V$. The set $J_\infty^{G,E}V$ is identified with the set of such equivariant homomorphisms modulo the G -action, in turn with $\Xi_{B/A}^V/G$. The assertion on jets follows from this. Q.E.D.

Proposition 6.3. $\Xi_{B/A}^V$ is a free A -module of rank d . Moreover it is saturated, that is, if $ax \in \Xi_{B/A}^V$ with $a \in A$ and $x \in B^d$, then $x \in \Xi_{B/A}^V$.

Proof. For $a \in A$, the multiplication map, $B^d \rightarrow B^d$, $x \mapsto ax$, commutes with all $\rho(g)$ and $\sigma(g)$. This shows the saturatedness. The freeness follows from the facts that it is torsion-free and that A is a PID. It remains to show that the rank is d .

The G -action $\rho\delta^{-1}$ makes B^d a G -equivariant B -module. Let L be the fraction field of B and $p : \text{Spec } L \rightarrow \text{Spec } K$ the associated G -torsor. Then $\Xi_{B/A}^V \otimes_A K$ is identified with $(p_*(B^d \otimes_B L))^G$. We can see that this has rank d , for instance, by trivializing the G -torsor with a base change. Q.E.D.

From the proposition, we can identify $\Xi_{B/A,n}^V$ with affine spaces over k , using Witt vectors in the mixed characteristic case (for instance, see [6, page 276]). Now we can easily deduce the following:

Corollary 6.4. For every $n \in \mathbb{Z}_{\geq 0}$, $\pi_{n+1}(\mathcal{J}_\infty \mathcal{X}) \rightarrow \pi_n(\mathcal{J}_\infty \mathcal{X})$ is an \mathbb{L}^d -fibration, where $\pi_n : \mathcal{J}_\infty \mathcal{X} \rightarrow \mathcal{J}_n \mathcal{X}$ is the truncation map.

6.2. Weight functions in the linear case

Definition 6.5. Let $\alpha_1 = (\alpha_{1j})_{1 \leq j \leq d}, \dots, \alpha_d = (\alpha_{dj})_{1 \leq j \leq d} \in B^d$ be an A -basis of $\Xi_{B/A}^V$ and put $Q := (\alpha_{ij}) \in M_d(B)$. Let $G' \subset G$ be the stabilizer of some connected component of E . Let $V_0 := V \times_D D_0$. Then we define³ a weight function $w = w_V : G\text{-Cov}(D) \rightarrow \mathbb{Q}$ by

$$\begin{aligned} w(E) &:= d - \frac{1}{\sharp G} \text{length} \frac{B}{(\det Q)} - \dim V_0^{G'} \\ &= d - \frac{1}{\sharp G} \text{length} \frac{B^d}{B \cdot \Xi_{B/A}^V} - \dim V_0^{G'}. \end{aligned}$$

Note that if B is a domain, then the middle terms are written as $v_B(\det Q)/\sharp G$ with v_B the normalized valuation of B .

³Afterwards it turned out that this definition is not quite correct; the term $\dim V_0^{G'}$ needs to be modified, although they coincide in some cases; see [28].

Conjecture 6.6. *For the quotient stack $\mathcal{X} = [V/G]$ associated to a G -representation V over D , the weight function $w_{\mathcal{X}}$ from Conjecture 5.5 factors as*

$$w_{\mathcal{X}} : \mathcal{J}_{\infty} \mathcal{X} = J_{\infty}^G M \rightarrow G\text{-Cov}(D) \xrightarrow{wv} \mathbb{Q}.$$

We will show an evidence for the conjecture in the next section.

Example 6.7. Let $\zeta \in k$ be a primitive m -th root of 1 with $\text{char}(k) \nmid m$ and $G := \langle \text{diag}(\zeta^{a_1}, \dots, \zeta^{a_d}) \rangle \subset GL_d(k)$ with $0 \leq a_i < m$ and $\text{gcd}(a_1, \dots, a_d)$ prime to m . Then G is cyclic of order m . Let

$$E = \text{Spec } k[[t^{1/m}]] \rightarrow D = \text{Spec } k[[t]]$$

be a G -cover with $\sigma(t^{1/m}) = \zeta t^{1/m}$. Then for a natural basis of Ξ , the matrix in Definition 6.5 becomes

$$Q = \text{diag}(t^{a_1/m}, \dots, t^{a_d/m}).$$

Hence

$$\begin{aligned} w(E) &= d - \frac{1}{m} \sum_{i=1}^d a_i - \#\{i \mid a_i = 0\} \\ &= d - \frac{1}{m} \sum_{i=1}^d a'_i, \end{aligned}$$

where a'_i is an integer such that $a'_i \equiv a_i \pmod{m}$ and $0 < a'_i \leq m$. This agrees with weight functions in [9, 30, 31].

Example 6.8. Suppose that $\text{char}(k) = p > 0$ and $G = \langle \sigma \rangle$ is cyclic of order p . Also suppose that G acts on $k[x_1, \dots, x_d]$, $d \leq p$ by

$$\sigma(x_i) = \begin{cases} x_i + x_{i+1} & (i < d) \\ x_d & (i = d). \end{cases}$$

Let

$$E = \text{Spec } B \rightarrow D = \text{Spec } k[[t]]$$

be a G -cover with ramification jump $j > 0$. Namely, if $s \in B$ is a uniformizer, then $v_B(\sigma(s) - s) = j + 1$. (Note that j is necessarily prime to p .) Then there exist $f_0, \dots, f_{d-1} \in B$ such that

$$(1) \quad v_B(f_i) \equiv -ij \pmod{p} \text{ and } 0 \leq v_B(f_i) < p,$$

(2) If we put $\delta = \sigma^* - \text{id}_B$, then

$$v_B(\delta^n(f_i)) = \begin{cases} v_B(f_i) + nj & (0 \leq n \leq i) \\ \infty & (n > i), \end{cases}$$

(3) $(\delta^i(f_0))_{1 \leq i \leq d}, \dots, (\delta^i(f_{d-1}))_{1 \leq i \leq d} \in B^d$ form a basis of Ξ_B .

The matrix $Q = (\delta^i(f_j))_{0 \leq i, j \leq d-1}$ corresponding to the basis is triangular and its determinant is $\prod_{i=0}^{d-1} \delta^i(f_i)$. Since

$$v_B(\delta^i(f_i)) = v_B(f_i) + ij = p \cdot \left\lfloor \frac{ij}{p} \right\rfloor,$$

we have

$$\begin{aligned} w(E) &= d - \sum_{i=0}^{d-1} \left\lfloor \frac{ij}{p} \right\rfloor - 1 \\ &= - \sum_{i=1}^{d-1} \left\lfloor \frac{ij}{p} \right\rfloor. \end{aligned}$$

This agrees with the weight function in [32].

These examples would suggest that the tame part positively contributes to the weight and the purely wild part negatively does,⁴ and that the weight function measures how tame/wild a G -cover $E \rightarrow D$ is. The formula defining w_V contains $\det Q$ and it is not clear how to compute it in general. Therefore we would like to ask:

Problem 6.9. Does $w(E)$ depend only on numerical invariants of E and V as in the above examples? If it is the case, find a formula.⁵

To attack the above problem, knowing properties of the weight function would be helpful. For instance, it is natural to expect the following properties:

- (1) Let $E \rightarrow D$ be a G -cover, $E' \subset E$ a connected component, $G' \subset G$ its stabilizer and V' the same scheme as V endowed with the induced G' action. Then $w_V(E) = w_{V'}(E')$.
- (2) For a trivial G -cover $E = \bigsqcup_{g \in G} D \rightarrow D$, $w_V(E) = 0$.

⁴Afterwards this phenomenon was also confirmed in the case of permutation representations in [27].

⁵Afterwards it turned out that $w(E)$ is described by discriminants for permutation representations [27] and for little more complicated ones [29]. However the problem is still open for the general case.

- (3) For G -representations V and W over D , let G act on $V \times_D W$ diagonally. Then $w_{V \times_D W}(E) = w_V(E) + w_W(E)$.

§7. Untwisting: a justification of conjectures

In this section, we will try to justify Conjectures 5.5 and 6.6. Here we will use a technique which we call *untwisting*. This would reduce the change of variables formula for stacks to the one for varieties, which has been already mostly established. Such an argument was already used by Denef and Loeser [9]. Our argument will be somehow more involved than theirs. One of the reasons is that we have to untwist in the opposite direction. It is inevitable because the weight function may take negative values in the wild case.⁶

We will only consider the proper birational morphism

$$\phi : \mathcal{X} = [M/G] \rightarrow X = M/G,$$

associated to a G -variety M , as all keys seem to appear already in this situation. Then, like the change of variables formula in known cases, Conjecture 5.5 would basically follow from:

Conjecture 7.1 (Key lemma). *Let $\gamma : \mathcal{E} \rightarrow \mathcal{X}$ be a twisted arc of \mathcal{X} which sends the generic point of \mathcal{E} into the isomorphism locus of ϕ . Let $\phi_n : \pi_n(\mathcal{J}_\infty \mathcal{X}) \rightarrow J_n X$ be the natural map. Then for $n \gg 0$,*

$$[\phi_n^{-1}(\phi_n(\pi_n(\gamma)))] = \mathbb{L}^{\text{ord Jac}_\phi(\gamma) - w_{\mathcal{X}}}.$$

7.1. Fixing a G -cover of D

Lemma 7.2. *Let $\gamma, \gamma' \in J_\infty^G M$ be G -arcs sending the generic points into the étale locus of the quotient map $M \rightarrow X$. If they have the same image in $J_n X$ for $n \gg 0$, then they have the same image in $G\text{-Cov}(D)$.*

Proof. Let $E \rightarrow D$ be the G -cover associated to γ . We consider only the case where E is connected. We may suppose that $M = \text{Spec } S$ and $X = \text{Spec } R$ with $R = S^G$. Let $f \in R$ be such that $\gamma(E^*) \subset \text{Spec } S_f$ and we can write

$$S_f = R_f[x]/(h(x)).$$

Let $\bar{h}(x) \in K[x]$ to be the image of $h(x)$ by $R_f[x] \rightarrow K[x]$ derived from γ . Then the function field of E is $L = K[x]/(\bar{h}(x))$. For $n \gg 0$, the

⁶These two sentences were wrong. This misunderstanding caused the complicated presentation of the untwisting. However this was the way how the author reached the definition of weight functions and Conjecture 7.1 below. In [29] he revisited this technique in a more intrinsic way.

induced n -jet $f_n(\pi_n(\gamma))$ of X determines the coefficients of $\bar{h}(x)$ up to sufficiently high order. Hence it determines L and E . Q.E.D.

7.2. The linear case

Now we consider the case where $M = V = \text{Spec } A[x_1, \dots, x_d]$ with a linear G -action. Fix $E = \text{Spec } B \in G\text{-Cov}(D)$. As in Definition 6.5, we take an A -basis of $\Xi_{B/A}^V$,

$$\alpha_1 = (\alpha_{1j})_{1 \leq j \leq d}, \dots, \alpha_d = (\alpha_{dj})_{1 \leq j \leq d} \in B^d,$$

which we now think of row vectors, and put

$$Q := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} = (\alpha_{ij})_{1 \leq i, j \leq d} \in M_d(B),$$

which is invertible over $B \otimes_A K$.

Definition 7.3. Let t be a uniformizer of A and let $l \in \mathbb{Z}_{\geq 0}$ be such that $t^l Q^{-1} \in M_d(B)$. Then $t^l Q^{-1}$ defines a B -linear transform on $B^d \cong \bigoplus B \cdot x_j$. Extending it, we obtain a B -algebra endomorphism u^* of $B[x_1, \dots, x_d]$. Putting $V_E := \text{Spec } B[x_1, \dots, x_d] = V \times_D E$, we define a B -linear map

$$u : V_E \rightarrow V_E$$

to be the one corresponding to u^* . We call u an *untwisting map*.

Suppose that G acts on $V_E = V \times_D E$ diagonally. A G -arc $E \rightarrow V$ corresponds to a G -equivariant E -morphism $E \rightarrow V_E$. For $1 \leq i \leq d$, let $\gamma_i : E \rightarrow V_E$ be the G -equivariant E -morphism such that

$$\begin{aligned} \gamma_i^* |_{\bigoplus B \cdot x_j} : \bigoplus B \cdot x_j &\rightarrow B \\ \sum a_j x_j &\mapsto \alpha_i \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} (u \circ \gamma_i)^* |_{\bigoplus B \cdot x_j} : \bigoplus B \cdot x_j &\rightarrow B \\ \sum a_j x_j &\mapsto \alpha_i t^l Q^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = t^l a_i. \end{aligned}$$

For any $\gamma \in J_\infty^{G,E}V$, $\gamma^*|_{\bigoplus B \cdot x_j}$ is an A -linear combination of $\gamma_i^*|_{\bigoplus B \cdot x_j}$, and $(u \circ \gamma)^*|_{\bigoplus B \cdot x_j}$ is an A -linear combination of $(u \circ \gamma_i)^*|_{\bigoplus B \cdot x_j}$. For $0 \leq n \leq \infty$, we denote the set of G -equivariant morphisms $E_n \rightarrow V$ by $J_n^{G,E}V$, distinguishing it from $J_n^{G,E}V = (J_n^{G,E}V)/G$. We have the identification $J_n^{G,E}V = \Xi_{B/A,n}^V$ (see Lemma 6.2 and its proof).

Proposition 7.4. *For $l \leq n \leq \infty$, let $J_n^{\geq l}V$ be the subset of n -jets $\gamma : D_n \rightarrow V$ of V with $\gamma^*(x_i) \in \mathfrak{m}_{A_n}^l$, where \mathfrak{m}_{A_n} is the maximal ideal of A_n . We have a natural bijection*

$$(7.1) \quad u_\infty : J_\infty^{G,E}V \rightarrow J_\infty^{\geq l}V.$$

Moreover, for $l \leq n < \infty$, the induced map

$$u_n : \pi_n(J_\infty^{G,E}V) \rightarrow J_n^{\geq l}V$$

is a trivial \mathbb{A}_k^c -fibration with $c = \dim V_0^{G'} + d(l - 1)$.

Proof. The first assertion is now obvious. The maps u_n are linear surjections. The dimensions of their kernels are independent of n and equal to

$$\begin{aligned} c &= \dim J_l^{G,E}V - \dim J_l^{\geq l}V \\ &= \dim J_0^{G,E}V + d(l - 1) \\ &= \dim V_0^{G'} + d(l - 1). \end{aligned}$$

Here $G' \subset G$ is the stabilizer of some connected component of E .
 Q.E.D.

Since u is an isomorphism outside the special fiber of $V_E \rightarrow D$, there exists a rational map $\psi \circ u^{-1} : V_E \dashrightarrow X_E$.

Lemma 7.5. *The rational map $\psi \circ u^{-1}$ is defined over D .*

Proof. Suppose that this is not true. Then there exists $f \in \text{frac}(A[x_1, \dots, x_d]^G)$ with $f \notin u^*(\text{frac}(A[x_1, \dots, x_d]))$. Then, for general $\gamma \in J_\infty^{\leq l}V$, if $\gamma_E : E \rightarrow V_E$ is the morphism induced from γ by extending scalars, then $\gamma_E^*(f) \notin K$. This contradicts the fact that $u_\infty^{-1}(\gamma) : E \rightarrow V$ induces an arc $D \rightarrow X$.
 Q.E.D.

Let $\psi : V \rightarrow X$ be the quotient map and $\psi' = \psi \circ u^{-1} : V \dashrightarrow X$ the above rational map over D .

$$(7.2) \quad \begin{array}{ccc} & V & \\ \psi \swarrow & & \searrow u \text{ (defined after extending scalars to } B) \\ X & \dashleftarrow \psi' \text{ (dashed line)} & V \end{array}$$

This diagram induces the commutative diagram of jet spaces:

$$(7.3) \quad \begin{array}{ccc} & \pi_n(j_{\infty}^{G,E}V) & \\ \psi_n \swarrow & & \searrow u_n \\ \pi_n(J_{\infty}X) & \dashleftarrow \psi'_n \dashrightarrow & \pi_n(J_{\infty}^{\geq l}V) \end{array}$$

Here ψ'_n should be understood as the restriction of a “correspondence”

$$\begin{array}{ccc} & \pi_n(j_{\infty}^{G,E}W) & \\ \swarrow & & \searrow \\ \pi_n(J_{\infty}\bar{X}) & & \pi_n(J_{\infty}^{\geq l}\bar{V}) \end{array}$$

where \bar{X} and \bar{V} are compactifications of X and V respectively, and W is a resolution of indeterminacies of $\psi' : \bar{V} \dashrightarrow \bar{X}$.

We now fix a G -arc $\gamma \in j_{\infty}^{G,E}E$ not contained in a bad locus. We denote the relative dimensions of morphisms ψ_n and u_n at jets derived from γ by $\dim \psi_n$ and $\dim u_n$ respectively. Also we write the “relative dimension of ψ'_n ” as $\dim \psi'_n$. Then for $n \gg 0$,

$$\dim \psi_n = \dim \psi'_n + \dim u_n = \dim \psi'_n + c,$$

with c as above. On the other hand, we would have the associativity of Jacobian orders,

$$\text{ord Jac}_{\psi'} = \text{ord Jac}_{\psi} + \text{ord Jac}_{u^{-1}},$$

where ord Jac_{γ} denotes the Jacobian order of γ at the arc derived from γ . Since $\text{Jac}_{u^{-1}} = \det(t^{-l}Q) = t^{-dl} \cdot \det Q$ and $\det Q \in B$, we would have

$$\text{ord Jac}_{u^{-1}} \equiv \frac{1}{\#G} \text{length} \frac{B}{(\det Q)} - dl.$$

On the other hand, suitably generalizing the change of variables formula to our rational map, we would have

$$\text{ord Jac}_{\psi'} = \dim \psi'_n.$$

Hence

$$\begin{aligned} \dim \psi_n &= \dim \psi'_n + c \\ &= \text{ord Jac}_{\psi} - w_{\mathcal{X}}(\gamma). \end{aligned}$$

These arguments would justify Conjecture 7.1 and hence Conjectures 5.5 and 6.6 in this situation.

7.3. Untwisting: the non-linear or singular case

Next we consider the case where the G -variety M is just an affine (possibly singular) D -variety.⁷ There exists an equivariant closed embedding $M \hookrightarrow V = \mathbb{A}_D^d$ with G linearly acting on V . From V , we construct the same diagram as (7.2). Let $Y := M/G$ and $\bar{M} \subset V$ the closure of $(\psi')^{-1}(Y)$. The bijection (7.1) restricts to a bijection

$$J_\infty^{G,E} M \rightarrow J_\infty^{\geq l} \bar{M} := J_\infty^{\geq l} V \cap J_\infty \bar{M}.$$

Diagram (7.3) restricts to:

$$\begin{array}{ccc} & \pi_n(J_\infty^{G,E} M) & \\ \overline{\psi}_n \swarrow & & \searrow \overline{u}_n \\ \pi_n(J_\infty Y) & \leftarrow \overline{\psi}'_n \text{ (dashed)} \rightarrow & \pi_n(J_\infty^{\geq l} \bar{M}) \end{array}$$

Here the overlines mean restrictions of maps. With similar notation as above, we would have

$$\begin{aligned} \dim \overline{\psi}_n &= \dim \overline{\psi}'_n + \dim \overline{u}_n \\ &= \text{ord Jac}_{\overline{\psi}} + \dim \overline{u}_n + \text{ord Jac}_{\overline{u}^{-1}}. \end{aligned}$$

Therefore the weight of the relevant arc, $w_{[M/G]}(\gamma)$, would be

$$- \dim \overline{u}_n - \text{ord Jac}_{\overline{u}^{-1}}.$$

Unlike the linear case, the two terms are not probably be constant even if we fix a G -cover E .

Remark 7.6. Another possible (and more direct) approach to the conjectures is to generalize Looijenga’s argument [15]: we identify fibers of ψ_n with suitable submodules of

$$\text{Hom}_B(\tilde{\gamma}^* \Omega_{\mathcal{Y}/\mathcal{X}}, \mathfrak{m}_B^a / \mathfrak{m}_B^b)$$

and compute their dimensions.

⁷Afterwards this case was studied in [29] in more details.

§8. Stringy motifs vs. motivic masses of G -covers

8.1. Stringy motifs

Let \mathcal{X} be a normal \mathbb{Q} -Gorenstein DM stack over D . Namely for some $r > 0$, the double dual $\omega_{\mathcal{X}/D}^{[r]} := (\omega_{\mathcal{X}/D}^{\otimes r})^{\vee\vee}$ of $\omega_{\mathcal{X}/D}^{\otimes r}$ is invertible. Then we define a function $F_{\mathcal{X}}$ on $\mathcal{J}_{\infty}\mathcal{X}$ as the order function of the submodule

$$(\Omega_{\mathcal{X}/D}^d)^{\otimes r} / \text{tors} \subset \omega_{\mathcal{X}/D}^{[r]}$$

divided by r .

Definition 8.1. Let $W \subset \mathcal{X} \times_D D_0$ be a closed subset and $(\mathcal{J}_{\infty}\mathcal{X})_W$ the preimage of W by the natural map $\mathcal{J}_{\infty}\mathcal{X} \rightarrow \mathcal{X} \times_D D_0$. We define the *stringy motif* of \mathcal{X} along W as the integral

$$M_{\text{st}}(\mathcal{X})_W := \int_{(\mathcal{J}_{\infty}\mathcal{X})_W} \mathbb{L}^{F_{\mathcal{X}} + w_{\mathcal{X}}} d\mu_{\mathcal{X}} \in \hat{\mathcal{M}} \cup \{\infty\}.$$

We say that \mathcal{X} is *stringily log terminal along W* if $M_{\text{st}}(\mathcal{X})_W \neq \infty$. If $W = \mathcal{X}$, we just write $M_{\text{st}}(\mathcal{X})$. For a stack \mathcal{X} defined over k and a closed subset $W \subset \mathcal{X}$, we define $M_{\text{st}}(\mathcal{X})_W$ as $M_{\text{st}}(\mathcal{X} \otimes_k k[[t]])_W$

Since $F_{\mathcal{X}}$ comes from singularities of \mathcal{X} and $\omega_{\mathcal{X}}$ from local actions of stabilizers, roughly $M_{\text{st}}(\mathcal{X})_W$ measures how \mathcal{X} is far from a smooth variety along W .

Remark 8.2. We may generalize the above definition to log stacks, that is, pairs of a DM stack and a \mathbb{Q} -divisor on it. We will not enter this issue in this paper.

Remark 8.3. A \mathbb{Q} -Gorenstein variety X is log terminal in the usual sense if it is stringily log terminal. The converse holds if the variety admits a log resolution. If \mathcal{X} is tame and smooth, then \mathcal{X} is stringily log terminal. In general, a smooth DM stack may not be stringily log terminal (see [32]).

We can compute stringy invariants using resolution data if a nice resolution exists.

Proposition 8.4. *Let $f : Y \rightarrow X$ be a resolution of a log terminal variety X over k such that the relative canonical divisor K_f is simple normal crossing, say written as $K_f = \sum_{i \in I} a_i E_i$. For a subset $J \subset I$, we set $E_J^{\circ} := \bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i$. Then*

$$M_{\text{st}}(X)_W = \sum_{J \subset I} [E_J^{\circ} \cap f^{-1}(W)] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j + 1} - 1}.$$

In particular, if f is crepant, that is, $K_f = 0$, then $M_{\text{st}}(X)_W = [f^{-1}(W)]$.

Remark 8.5. Originally stringy invariants were defined by Batyrev [2] using a formula as in Proposition 8.4. Denef and Loeser [9] found a way to express them as motivic integrals over the arc space of the given singular variety.

Remark 8.6. If Conjecture 5.5 holds, then we can show that the stringy motif is invariant under K -equivalences. Moreover we can generalize it to log stacks.

8.2. Motivic masses

We conjectured that $G\text{-Cov}(D)$ has the tautological motivic measure τ . To a G -representation V over D , we have associated the weight function w_V on $G\text{-Cov}(D)$.

Definition 8.7. The *motivic mass* of G -covers of D with respect to V is

$$\text{Mass}(G, D, V) := \int_{G\text{-Cov}(D)} \mathbb{L}^{w_V} d\tau = \sum_{r \in \mathbb{Q}} [w_V^{-1}(r)] \mathbb{L}^r \in \hat{\mathcal{M}} \cup \{\infty\}.$$

Remark 8.8. The motivic mass is analogous to weighted counts of extensions of a local field by Serre [23] and Bhargava [5]. Serre proved a mass formula for totally ramified field extensions of a local field with finite residue field. Bhargava proved a similar formula by allowing étale extensions. Kedlaya’s reinterpretation [13] of Bhargava’s result in terms of Galois representation, and Wood’s subsequent work [25] seem to be more directly related to our motivic invariant.

Proposition 8.9. *We denote by 0 the origin of $V \times_D D_0$, and its images in $X := V/G$ and $\mathcal{X} := [V/G]$ as well. Suppose that Conjectures 5.5 and 6.6 hold. Suppose also that G has no reflection, equivalently, the quotient map $V \rightarrow X$ is étale in codimension one. Then*

$$(8.1) \quad M_{\text{st}}(X)_0 = M_{\text{st}}(\mathcal{X})_0 = \text{Mass}(G, D, V).$$

Moreover, if there exists a crepant resolution $f : Y \rightarrow X$, then these invariants are also equal to $[f^{-1}(0)]$.

Proof. Let $\phi : \mathcal{X} \rightarrow X$ be the natural morphism. Then $F_X \circ \phi_\infty = \text{ord Jac}_\phi$. Hence our conjectures show

$$\int_{(J_\infty X)_0} \mathbb{L}^{F_X} d\mu_X = \int_{(J_\infty \mathcal{X})_0} \mathbb{L}^{w_X} d\mu_{\mathcal{X}} = \int_{G\text{-Cov}(D)} \mathbb{L}^{w_V} d\tau,$$

and the proposition.

Q.E.D.

Remark 8.10. If $\text{char}(k) \nmid \#G$, then $G\text{-Cov}(D)$ have exactly $\# \text{Conj}(G)$ points and

$$M_{\text{st}}(X)_0 = \text{Mass}(G, D, V) = \sum_{[g] \in \text{Conj}(G)} \mathbb{L}^{\text{age } g}.$$

See (9.1). Hence, if $f : Y \rightarrow X$ is a crepant resolution and if $\chi_c(-)$ denotes the topological Euler characteristic (for compactly supported cohomology), then

$$\chi_c(f^{-1}(0)) = \# \text{Conj}(G).$$

Thus we recover a version of the McKay correspondence. The basically same result was conjectured by Reid and first proved by Batyrev [3] in arbitrary dimension (for the historical account, see [19]). Our approach is essentially due to Denef and Loeser [9] at least in characteristic zero.

Remark 8.11. Without assuming any conjecture, the proposition above has been established in [32] when $G = \mathbb{Z}/p\mathbb{Z}$, $A = k[[t]]$ and V is defined over k . Also, a variant of the proposition, regarded as a point-counting realization, will be proved in a forthcoming paper [27]⁸ when G is the n -th symmetric group S_n , V is the direct sum of two copies of the standard representation and Y is the Hilbert scheme of n points on the affine plane. In the same paper, it will turn out that this case is closely related to Bhargava’s mass formula for étale extensions.

§9. Equisingular families and uniformity of motivic masses

Motivated by uniformity problems of Kedlaya [13] and Wood [25], we pose the following problem:

Problem 9.1. Given a family $V_S \rightarrow S$ of G -representations over a scheme S with fibers V_s , $s \in S$, are $\text{Mass}(G, D, V_s)$ independent of $s \in S$? What finite groups admit such a uniform family of representations?

Since $\hat{\mathcal{M}}$ depends on k , to make the problem precise, we have to take a suitable realization of motivic masses. However, in all known cases, they are rational functions in \mathbb{L} , and hence we do not have to worry about realization. We are mainly interested in the case where S surjects onto $\text{Spec } \mathbb{Z}$ and what happens around points of characteristic dividing $\#G$.

Proposition 8.9 links the problem to:

⁸This paper has been published now.

Problem 9.2. What families of quotient singularities are equisingular in the sense that they have the “same” stringy motif?

These problems are wide open. From now on, we will focus on G -representations with G of prime order. In this case, there have been already explicit formulae for motivic masses and stringy motifs at least in equal characteristic. We will now recall them.

Let G be a finite group of prime order p and V a G -representation of dimension d over k . We first suppose that $p \neq \text{char}(k)$. Fixing a primitive p -th root $\zeta \in k$ of 1, we write each element $g \in G$ as $\text{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$, $0 \leq a_i < p$ for a suitable basis of V , and put

$$(9.1) \quad \text{age } g := \frac{1}{p} \sum_{i=1}^d a_i.$$

Then

$$\text{Mass}(G, D, V) = \sum_{g \in G} \mathbb{L}^{\text{age } g}.$$

If V has no reflection, then this is equal to $M_{\text{st}}(X)_0$ for the corresponding quotient singularity $0 \in X$.

Next suppose that $p = \text{char}(k)$. Then V is the direct sum of indecomposable representations of dimensions d_1, \dots, d_l with $\sum_i d_i = d$. (Such a decomposition of V is unique up to permutation, and we have $d_i \leq p$.) We put

$$D_V := \sum_{i=1}^l \frac{d_i(d_i - 1)}{2}.$$

Then $\text{Mass}(G, D, V) \neq \infty$ if and only if $D_V \geq p$. When one of the two equivalent conditions holds, then V has no reflection and we have

$$(9.2) \quad \text{Mass}(G, D, V) = M_{\text{st}}(X)_0 = 1 + \frac{(\mathbb{L} - 1) \left(\sum_{s=1}^{p-1} \mathbb{L}^{s+w(s)} \right)}{\mathbb{L} - \mathbb{L}^{p-D_V}},$$

with

$$w(s) = - \sum_{i=1}^l \sum_{j=1}^{d_i-1} \lfloor js/p \rfloor.$$

The topological Euler characteristic realization of $\text{Mass}(G, D, V)$ equals p in the tame case and

$$1 + \frac{p - 1}{D_V - p + 1}$$

in the wild case. This shows the following:

Proposition 9.3. *If a family $V_S \rightarrow S \rightarrow \text{Spec } \mathbb{Z}$ is uniform in the sense of Problem 9.1, then $D_{V_p} = p$ for every reduction V_p of V_S to characteristic p .*

The following example is a partial converse.

Example 9.4. Let G be a finite group of prime order p and V_p a G -representation in characteristic p . Then for a suitable basis, a generator $\sigma \in G$ is represented by a Jordan normal form J with blocks

$$J_i = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

of sizes $d_i \leq p$. Thus we may suppose that V_p is defined over \mathbb{F}_p . Then we lift J to the matrix \tilde{J} over $R := \mathbb{Z}[x]/(x^p - 1)$ with blocks

$$\tilde{J}_i = \begin{pmatrix} 1 & 1 & & & \\ & x & 1 & & \\ & & \ddots & \ddots & \\ & & & x^{d_i-2} & 1 \\ & & & & x^{d_i-1} \end{pmatrix}.$$

For every i , the reduction of \tilde{J}_i to a characteristic $\neq p$ has d_i distinct eigenvalues all of which are p -th roots of 1. Hence $\tilde{J}_i^p = 1$ and $\tilde{J}^p = 1$. Therefore \tilde{J} induces a G -representation V_R lifting V_p over R .

Now suppose that $D_{V_p} = \sum_i \frac{d_i(d_i-1)}{2} = p$. Then

$$\text{Mass}(G, D, V_p) = 1 + \sum_{s=1}^{p-1} \mathbb{L}^{s+ w_{V_p}(s)},$$

while for a reduction V_l of V_R to characteristic $l \neq p$,

$$\text{Mass}(G, D, V_l) = 1 + \sum_{s=1}^{p-1} \mathbb{L}^{\alpha(s)}.$$

Here $\alpha(s) = \sum_{i=1}^l \sum_{j=1}^{d_i} \{js/p\}$, $\{-\}$ denoting the fractional part. We now claim that $\alpha(s) = s + w_{V_p}(s)$. Indeed,

$$\begin{aligned} s &= \frac{s \cdot D_{V_p}}{p} \\ &= \sum_{i=1}^l \sum_{j=1}^{d_i-1} \frac{sj}{p} \\ &= \sum_i \sum_j \left\lfloor \frac{sj}{p} \right\rfloor + \sum_i \sum_j \left\{ \frac{sj}{p} \right\} \\ &= -w_{V_p}(s) + \alpha(s). \end{aligned}$$

Hence, thinking of \mathbb{L} as an indeterminate, the motivic mass $\text{Mass}(G, D, V_l)$ is independent of the characteristic l .

Example 9.5. In the preceding example, suppose that $V_p = W^{\oplus p}$ with W the unique indecomposable G -representation of dimension 2 (in characteristic p). Then for any characteristic l ,

$$\text{Mass}(G, D, V_l) = \sum_{i=0}^{p-1} \mathbb{L}^i = [\mathbb{P}^{p-1}].$$

§10. Non-existence of resolution of singularities

Stringy motifs of log terminal varieties have a lot of information on their resolutions of singularities if exist. A detailed knowledge of this invariant might lead to the non-existence of resolution of singularities in positive characteristic. Conjecture 5.5 enables us to reduce the stringy invariant of a quotient variety to that of the corresponding quotient stack, which would be easier to compute in some cases. For this purpose, we have to know what constraints the existence of resolution imposes on the invariant.

10.1. Rationality and the Poincaré duality

Definition 10.1. For a \mathbb{Q} -Gorenstein k -variety X and a closed subset $W \subset X$, we put

$$P_{\text{st}}(X, \Delta)_W := P(M_{\text{st}}(X, \Delta)_W) \in \bigcup_{r>0} \mathbb{Z}((T^{-1/r})).$$

with P the virtual Poincaré realization. If $W = X$, we omit the subscript W .

Definition 10.2. A rational function $f(T^{1/r}) \in \mathbb{Z}(T^{1/r})$ in $T^{1/r}$ is said to *satisfy the d -dimensional Poincaré duality* if

$$T^{2d} \cdot f(T^{-1/r}) = f(T^{1/r}).$$

Proposition 10.3. *Let X be a \mathbb{Q} -Gorenstein k -variety which is log terminal along W . Suppose that there exists a resolution $f : Y \rightarrow X$ such that K_f is simple normal crossing. Then $P_{\text{st}}(X)_W$ is a rational function. Moreover, if X is proper and $W = X$, then $P_{\text{st}}(X)$ satisfies the d -dimensional Poincaré duality with $d = \dim X$.*

Proof. The rationality follows from Proposition 8.4. We can prove the Poincaré duality in the same way as Batyrev [2] did. Q.E.D.

10.2. Medusa singularities

Next, let us recall the following result due to Kerz and Saito:

Theorem 10.4 ([14]). *Let M be a smooth k -variety with a finite group G acting on it,*

$$\pi : M \rightarrow X = M/G$$

the quotient map. Let $S \subset X$ be a closed subset containing the singular locus such that the reduced preimage $T := \pi^{-1}(X_{\text{sing}})_{\text{red}}$ is smooth. Suppose that a functorial resolution of singularities holds over k . Let $f : Y \rightarrow X$ be a resolution such that $E = f^{-1}(S)$ (with reduced structure) is a simple normal crossing divisor and f is an isomorphism over $X \setminus X_{\text{sing}}$. (We will call such f a log resolution of X and S .) Then the dual complex $\Gamma(E)$ of E is contractible.

Putting this conversely, if either

- (1) there exists a log resolution $f : Y \rightarrow X$ of X and S with $\Gamma(f^{-1}(S))$ not contractible, or
- (2) X and S do not admit any log resolution $f : Y \rightarrow X$ of X and S with $\Gamma(f^{-1}(S))$ contractible,

then a functorial resolution over k never holds. Saito calls such singularities of X *Medusa singularities*. Since stringy motifs have a lot of information on resolution, we might use them to find Medusa singularities (if exist) and disprove the functorial resolution. For this purpose, a key result is the following:

Proposition 10.5. *Let X be a \mathbb{Q} -Gorenstein variety which is stringly log terminal along a closed subset $S \subset X$. Let $f : Y \rightarrow X$ be a log resolution of X and S such that K_f is a simple normal crossing divisor supported in $E := f^{-1}(S)$. Suppose that there exists a closed subset $R \subset S$*

such that R is proper over k and $F := f^{-1}(R)$ is a normal crossing variety having the same dual complex as E . Then

$$\chi(\Gamma(E)) = \chi(\Gamma(F)) = P_{\text{st}}(X)_R|_{T=0},$$

where χ is the Euler characteristic.

Proof. The following is a modification of Batyrev’s computation in [2]. Write $E = \bigcup_{i \in I} E_i$, $F = \bigcup_{i \in I} E_i$ and $K_{Y/X} = \sum a_i E_i$. Since

$$P(F_{J^\circ}) = \sum_{J \subset J'} (-1)^{\#J - \#J'} P(F_{J'}),$$

we have

$$\begin{aligned} P_{\text{st}}(X)_R &= \sum_{\emptyset \neq J \subset I} P(F_J^\circ) \prod_{j \in J} \frac{T^2 - 1}{T^{2(a_i+1)} - 1} \\ &= \sum_{\emptyset \neq J \subset I} \left(\sum_{J \subset J'} (-1)^{\#J - \#J'} P(F_{J'}) \right) \prod_{j \in J} \frac{T^2 - 1}{T^{2(a_i+1)} - 1} \\ &= \sum_{\emptyset \neq J \subset I} P(F_J) \left(\prod_{j \in J} \left(\frac{T^2 - 1}{T^{2(a_i+1)} - 1} - 1 \right) - (-1)^{\#J} \right). \end{aligned}$$

Substituting 0 for T , we obtain

$$\sum_{\emptyset \neq J \subset I} (-1)^{\#J-1} \#\pi_0(F_J),$$

where π_0 denotes the set of connected components. This clearly equals to the Euler characteristic of the dual complex of F and to that of E . Q.E.D.

Remark 10.6. Looking at finer realizations, we might able to capture finer invariants of $\Gamma(E)$.

Example 10.7. Let G be the cyclic group of prime order $p > 0$ and V a G -representation over k of characteristic $p > 0$. Suppose that V is isomorphic to the direct sum of l indecomposable representations and that V has no reflection. Then the invariant locus $V^G \subset V$ and the singular locus X_{sing} of the quotient variety $X := V/G$ are isomorphic to \mathbb{A}_k^l . The additive group $\mathbb{G}_a^l \cong \mathbb{A}_k^l$ acts on X and transitively on X_{sing} . Therefore, if a functorial resolution exists over k , then the resolution $Y \rightarrow X$ of X is \mathbb{G}_a^l -equivariant. Put $S = X_{\text{sing}}$ and R to be any point

of S . Suppose that $D_V \geq p$. Then the assumption of Proposition 10.5 holds, and from Equation 9.2,

$$P_{\text{st}}(X)_R|_{T=0} = 1.$$

Hence we cannot see whether $\Gamma(E)$ is contractible or not in this case.

10.3. Crepant resolutions

Problem 10.8. What quotient singularity admits a crepant resolution?

Crepant resolutions are certainly interesting, for instance, since they give a simple geometric meaning to stringy invariants and motivic masses. In characteristic zero, Ito [11, 12], Markushevich [16, 17, 4] and Roan [21, 20] proved that every three-dimensional Gorenstein quotient singularity admits a crepant resolution. On the other hand, our knowledge is quite limited in the wild situation. As far as the author knows, the following are only known cases:

- (1) Rational double points which are wild quotient singularities in Artin's classification [1]. In this case, the minimal resolution is crepant.
- (2) $G = S_n$ and V is the direct sum of two copies of the standard representation. Then the Hilbert scheme of n points on the affine plane is a crepant resolution of the quotient variety, which is isomorphic to the n -th symmetric product of the affine plane (see [7, page 229] for a historical account of this fact).
- (3) $\text{char}(k) = \#G = 3$ and $V = V_3$, the 3-dimensional indecomposable representation. See [32].

We will be able to use stringy invariants also to study the above problem. For instance, we can show that the second example with $n = 2$ and the last one are only possible cases for linear actions of a cyclic group of prime order in dimension ≤ 4 . It follows from the fact that $D_V = p$ is a necessary condition for the existence of crepant resolution. It is because otherwise the topological Euler characteristic realization is not an integer, which is not allowed if a crepant resolution exists. The same reasoning shows that if a crepant resolution exists, then $P_{\text{st}}(X)$ is a polynomial in T (not in $T^{1/r}$).

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