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Relative adjoint transcendental classes and Albanese map of compact Kähler manifolds with nef Ricci curvature

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Dedicated to Professor Yujiro Kawamata at the occasion of his birthday

Abstract.

We establish here the surjectivity of the Albanese map corresponding to compact Kähler manifold whose anti-canonical bundle is nef. This answers a question raised by J.-P. Demailly, Th. Peternell and M. Schneider. One of the main techniques we are using is the positivity of the relative canonical bundle associated to Kähler fiber spaces, obtained via the "psh variation" of fiber-wise twisted Kähler-Einstein metrics.

§1. Introduction

Let $p : X \to Y$ be a holomorphic surjective map, where X and Y are compact Kähler manifolds. We denote by $W \subset Y$ an analytic set containing the singular values of p, and let $X_0 := p^{-1}(Y \setminus W)$. Let $\{\beta\} \in H^{1,1}(X, \mathbb{R})$ be a real cohomology class of $(1,1)$-type, which contains a non-singular, semi-positive definite representative β.

Our primary goal in this note is to investigate the positivity properties of the class

$$c_1(K_{X/Y}) + \{\beta\},$$

which are inherited from similar *fiberwise* properties.

In this perspective, the main statement we obtain here is as follows.

Theorem 1.1. *Let $p : X \to Y$ be a surjective map. We consider a semi-positive class $\{\beta\} \in H^{1,1}(X, \mathbb{R})$, such that the adjoint class*

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$c_1(K_{X_y}) + \{\beta\}|_{X_y}$ is Kähler for any $y \in Y \setminus W$. Then the relative adjoint class

$$c_1(K_{X/Y}) + \{\beta\}$$

contains a closed positive current Θ, which equals a (non-singular) semi-positive definite form on X_0.

As a consequence of the proof of the previous result, the current Θ will be greater than a Kähler metric when restricted to any relatively compact open subset of X_0, provided that β is a Kähler metric. Also, if $\beta \geq p^\star(\gamma)$ for some (1,1)-form γ on Y, then we have

$$\Theta \geq p^\star(\gamma).$$

We remark that if the class β is the first Chern class of a holomorphic $\mathbb{Q}$-line bundle L, that is to say, if

$$\{\beta\} \in H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Q}),$$

then there are many results concerning the positivity of the twisted relative canonical bundle, cf. [2], [3], [4], [6], [14], [17], [18], [21], [22], [23], [24], [25], [28], [39], [40], [41] to quote only a few.

The case $\beta = 0$ is due to G. Schumacher in [34], [35], (see also [1]); actually, a large part of the arguments presented in [34], [36] will be used in our proof, as it can be seen in the section 3.2.

Before stating a few consequences of our main result, we recall the following metric version of the usual notion of *nef* line bundle in algebraic geometry, as it was introduced in [10].

Definition 1.2. *Let (X, ω) be a compact complex manifold endowed with a hermitian metric, and let $\{\rho\}$ be a real (1,1) class on X. We say that $\{\rho\}$ is nef (in metric sense) if for every $\varepsilon > 0$ there exists a function $f_\varepsilon \in \mathcal{C}^\infty(X)$ such that*

$$\rho + \sqrt{-1}\partial\overline{\partial} f_\varepsilon \geq -\varepsilon\omega. \tag{1}$$

Thus the class $\{\rho\}$ is nef if it admits non-singular representatives with arbitrary small negative part. It was established in [11] that if X is projective and if $\{\rho\}$ is the first Chern class of a line bundle L, then L is nef in the algebro-geometric sense if and only if L is nef in metric sense.

Let $\mathcal{X} \to \mathbb{D}$ be a non-singular Kähler family over the unit disk. Then we have the following (direct) consequence of Theorem 1.1.

Corollary 1.3. *We assume that the bundle $K_{\mathcal{X}_t}$ is nef, for any $t \in \mathbb{D}$. Then $K_{\mathcal{X}/\mathbb{D}}$ is nef.*

We remark that in the context of the previous corollary, much more is expected to be true. For example, if the Kähler version of the *invariance of plurigenera* is correct, then it would be enough to assume in Corollary 1.3 that $K_{\mathcal{X}_0}$ is pseudo-effective in order to derive the conclusion that $K_{\mathcal{X}/\mathbb{D}}$ is pseudo-effective.

The second application of Theorem 1.1 concerns the Albanese morphism associated to a compact Kähler manifold X. We denote by $q := h^0(X, \mathcal{O}_X)$ the irregularity of X, and let

$$\mathrm{Alb}(X) := H^0(X, T_X^\star)^\star / H_1(X, \mathbb{Z})$$

be the Albanese torus of associated to X. We recall that the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is defined as follows

$$\alpha_X(p)(\gamma) := \int_{p_0}^{p} \gamma$$

modulo the group $H_1(X, \mathbb{Z})$, i.e. modulo the integral of γ along loops at p_0.

We assume that $-K_X$ is nef, in the sense of the definition above. It was conjectured by J.-P. Demailly, Th. Peternell and M. Schneider in [12] that α_X is surjective; some particular cases of this problem are established in [12], [32], [7]. If X is assumed to be *projective*, then the surjectivity of the Albanese map was established by Q. Zhang in [44], by using in an essential manner the *char p* methods. More recently, in the article [45], the same author provides an alternative proof of this result, based on the semi-positivity of direct images.

We settle here the conjecture in full generality.

Theorem 1.4. *Let X be a compact Kähler manifold such that $-K_X$ is nef. Then its Albanese morphism $\alpha_X : X \to \mathrm{Alb}(X)$ is surjective.*

Besides Theorem 1.1, our proof is using some ideas from [12] and [5]; we refer to the first paragraph for a detailed discussion about the connections with these articles.

If we do not assume the nefness of the anti-canonical class, then we recall that the surjectivity of the Albanese map α_X was established by Y. Kawamata under the hypothesis that the Kodaira dimension of X is equal to zero, cf. [21].

Our paper is organized as follows. In the first paragraph we review the proof of Theorem 1.4 under the additional assumption that X is projective. As we have already mentioned, in this case the result is known, but the proof we will present is different from the arguments in [44]: actually, it can be seen as a simplified version of some of the arguments invoked in [45] (see also [9]). It is based upon a version of Theorem 1.1 under the hypothesis that the class $\{\beta\}$ corresponds to a line bundle (this result is completely covered by the literature on the subject, cf. [4], [37]).

This serves us as a motivation for the second paragraph, where we prove Theorem 1.1. In a word, we show that the so-called *fiberwise twisted Kähler-Einstein* metric endows the bundle $K_{X/Y}|_{X_0}$ with a metric whose curvature is bounded from below by $-\beta$. Thus, the twisted version of the psh variation of the Kähler-Einstein metric established in [34] holds true. Finally, we show that the local weights of the metric constructed in this way are bounded near the analytic set $X \setminus X_0$. This is by no means automatic, given the tools which are involved in the proof (the approximation theorem in [13], together with a precise version of the Ohsawa-Takegoshi extension theorem, [4]). The difficulty steams from the fact that in order to establish the estimates for the said weights we cannot rely on the geometry of the manifold X_y, as y is approaching a singular value of the map p.

Finally, a complete proof of the Corollary 1.3 and Theorem 1.4 is provided, together with a few questions/comments.

Acknowledgments. *It is my privilege and pleasure to thank S. Boucksom, J.-P. Demailly, A. Höring and H. Tsuji for enlightening discussions about many topics here. This article was first presented at the beautiful conference "Higher Dimensional Complex Geometry", in Tokyo* (2012) *celebrating Y. Kawamata* 60^{th} *birthday. I would like to express here my admiration for his deep contributions in algebraic geometry, whose ramifications, direct images and derived products are at the heart of current developments of the field.*

§2. Surjectivity of the Albanese map: review of the projective case

As we have already mentioned in the introduction, our proof for the subjectivity of the Albanese map corresponding to the compact Kähler manifolds with nef anti-canonical class relies heavily on Theorem 1.1. However, if the manifold X is projective, then the following stronger version of Theorem 1.1 was obtained in [4], [37].

Theorem 2.1. *Let (F, h_F) be a line bundle on X, endowed with a metric with semi-positive definite curvature form. We assume moreover that for some generic $y \in Y$ the bundle*

$$k(mK_{X_y} + F)$$

admits a section which is $L^{\frac{2}{km}}$–integrable with respect to $h_F^{1/m}$, where m is a positive integer. Then the bundle $mK_{X/Y} + F$ is pseudo-effective.

As a consequence, we infer the following statement.

Corollary 2.2. *Let $p : X \to Y$ be a surjective map between non-singular projective manifolds. We consider $L \to X$ a nef line bundle, such that $H^0\big(X_y, K_{X_y} + L|_{X_y}\big) \neq 0$. Then the bundle $K_{X/Y} + L$ is pseudo-effective.*

Proof. Let $A \to X$ be a very ample line bundle. Then for each positive integer m we define the bundle

$$L_m := mL + A;$$

it is ample, hence it can be endowed with a metric h_m with positive definite curvature. We consider the bundle $mK_{X/Y} + L_m$; by hypothesis, there exists a section $u \in H^0\big(X_y, K_{X_y} + L|_{X_y}\big)$, so the bundle

$$mK_{X/Y} + L_m|_{X_y}$$

admits a non-trivial section, e.g. $u^{\otimes m} \otimes s_A$ where s_A is a non-zero section of A. By Theorem 2.1 the bundle $mK_{X/Y} + L_m$ is pseudo-effective; as $m \to \infty$, we infer that $K_{X/Y} + L$ is pseudo-effective. Q.E.D.

We will explain next the relevance of the previous result in the proof of Theorem 1.4 under the assumption that X is projective; we first recall a few notions.

Let X be a non-singular manifold such that $-K_X$ is nef, and let

$$\alpha_X : X \to \mathrm{Alb}(X)$$

be its Albanese morphism. We assume that α_X is not surjective, and let $Y \subset \mathrm{Alb}(X)$ be the image of α_X.

We denote by $\pi_Y : \widehat{Y} \to Y$ the desingularization of Y, and let $p : \widehat{X} \to \widehat{Y}$ be the map obtained by resolving the indeterminacy of the rational map $X \dashrightarrow \widehat{Y}$.

We apply Corollary 2.2 with the following data

$$X := \widehat{X}, \quad Y := \widehat{Y}$$

and $L := \pi_X^\star(-K_X)$; here we denote by $\pi_X : \widehat{X} \to X$ the modification of X, so that we have

$$\pi_Y \circ p = \alpha_X \circ \pi_X.$$

The hypothesis required by Corollary 2.2 are quickly seen to be verified: indeed, the nefness of the bundle L is due to the fact that $-K_X$ is nef, and if we denote by E the effective divisor such that

$$K_{\widehat{X}} = \pi^\star(K_X) + E \tag{2}$$

then we see that $K_{\widehat{X}_y} + L$ is simply equal to $E|_{\widehat{X}_y}$. This bundle is clearly effective.

Hence we infer that

$$K_{\widehat{X}/\widehat{Y}} + \pi_X^\star(-K_X) \tag{3}$$

is pseudo-effective. But this bundle equals $E - p^\star(K_{\widehat{Y}})$; let Λ be a closed positive current in the class corresponding to $E - p^\star(K_{\widehat{Y}})$. Since the Kodaira dimension of $K_{\widehat{Y}}$ is at least 1 (we refer to [20] for a justification of this property), we obtain two $\mathbb{Q}$-effective divisors say $W_1 \neq W_2$ linearly equivalent with $K_{\widehat{Y}}$.

As a conclusion, we obtain two different closed positive currents belonging to the class of the exceptional divisor E, namely $\Lambda + p^\star(W_j)$ for $j = 1, 2$. This gives a contradiction, since any closed positive current linearly equivalent to E must be π_X-contractible, so its support is contained in the support of E. The existence of two closed positive currents having the support contained in E whose cohomology classes coincide shows that one of the irreducible components of the support of E must be equal to a linear combination of the other components in $H^{1,1}(X, \mathbb{R})$. This is of course absurd. Q.E.D.

Remark 2.3. The proof above shows that Theorem 1.4 still holds if X is projective, and if we replace the hypothesis $-K_X$ *nef* with the hypothesis $-K_X$ *pseudo-effective, and the multiplier ideal sheaf associated to some of its positively curved metrics is equal to the structural sheaf*. The arguments are absolutely similar.

Remark 2.4. Let X be a Fano manifold, and let $p : X \to Y$ be a submersion onto a non-singular manifold Y. Then this implies that Y is also Fano (see [26], [16]). This result can be obtained *via* the following elegant argument, very recently found and explained to us by S. Boucksom. By the results e.g. in [2], the direct image of the bundle

$$K_{X/Y} + L$$

is positive provided that L is an ample line bundle. We take $L = -K_X$ and we are done. A similar idea, ε-close to our arguments in this section can be found in the article [12] by J.-P. Demailly, Th. Peternell and M.Schneider (cf. the proof of their Theorem 2.4).

§3. Twisted Kahler-Einstein metrics and their variation

In this paragraph we are going to prove Theorem 1.1. We start (cf. section 3.1) with a few notations/remarks concerning the metrics induced on $K_{X/Y}$ by a $(1,1)$-form on X which is positive definite along the fibers of $p : X \to Y$. The next section 3.2 is the main part of our paper: we show that the fiberwise twisted Kähler-Einstein metric (which exists and it is unique, thanks to the fact that the class

$$c_1\big(K_{X_y}\big) + \{\beta\}|_{X_y}$$

is Kähler, for each $y \in Y_0$) endow the bundle $K_{X/Y}|_{X_0}$ with a metric whose curvature is strictly greater than $-\beta$. As we have already mentioned, at this point we will adapt to our setting the computations in [34], [36]. In the last subsection of this paragraph, we show that this metric extends across the singularities of the map p.

3.1. Metrics for the relative canonical bundle of a fibration

Let $p : X \to Y$ be a surjective map; here X and Y are assumed to be compact Kähler manifolds. We are using the notations/conventions in the introduction, so that the restriction $p : X_0 \to Y \setminus W$ becomes a surjective, smooth, proper map between two complex manifolds.

We consider a smooth (1,1)-form ρ on X, whose restriction to the fibers of p is positive definite. Then ρ induces a metric on the bundle $K_{X/Y}$ as follows.

Let $x \in X$ be a point, and let U be a coordinate set of X centered at x. We denote by $z^1, ..., z^{n+d}$ a coordinate system on U, and we equally introduce $t^1, ..., t^d$ coordinates near the point $y = p(x)$. This data induces a trivialization of the relative canonical bundle, with respect to which the weight of the metric we want to introduce is given by the function Ψ_U, defined by the equality

$$\rho^n \wedge \prod_{j=1}^{d} \sqrt{-1} p^\star(dt^j \wedge dt^{\overline{j}}) = e^{\Psi_U} \prod_{i=1}^{n+d} \sqrt{-1} dz^i \wedge dz^{\overline{i}}. \tag{4}$$

Here the dimension of X is assumed to be $n + d$, and the dimension of Y equals d. The functions (Ψ_U) glue together as weights of a globally defined metric denoted by $h^\rho_{X/Y}$ on the relative canonical bundle; the corresponding curvature form is simply $\sqrt{-1}\partial\overline{\partial}\Psi_U$.

We remark that we can very well define the function Ψ_U even if the point x projects into a singular value of p; however, the resulting metric $h^\rho_{X/Y}$ will be identically ∞ along the zero set of the Jacobian of the map p (in other words, the weight Ψ_U acquires a log pole). Thus, the metric $h^\rho_{X/Y}$ will be singular in general, even if to start with we are using a non-singular metric ρ.

A simple example is provided by the map $p : \widehat{X} \to X$, the blow-up of a manifold X along a subset W. As one can see right away from the formula (4), the metric $h^\rho_{X/Y}$ is equal to the singular metric associated to the exceptional divisor (so it is independent of ρ).

In the next section, we will evaluate the positivity of the curvature of the metric $h^\rho_{X/Y}$; for this purpose, we have to find a lower bound for the quantity

$$\sum_{\alpha,\beta} \frac{\partial^2 \Psi_U}{\partial z^\alpha \partial z^{\overline{\beta}}}(x) v^\alpha v^{\overline{\beta}} \tag{5}$$

where

$$v := \sum_\alpha v^\alpha \frac{\partial}{\partial z^\alpha}$$

is a tangent vector at X in x. In order to obtain a lower bound for the quantity in (2), it is enough to consider a well-chosen restriction of our initial map p, namely

$$\widetilde{p} : X_{\mathbb{D}} \to \mathbb{D}$$

where $\mathbb{D} \subset Y$ is a disk containing the point $p(x)$, such that the vector v belongs to the tangent space at x to the complex manifold $X_{\mathbb{D}} := p^{-1}(\mathbb{D})$. Such a choice is clearly possible, and we formulate next our conclusion as follows.

Remark 3.1. *Let γ be a real (1,1)-form on X. We assume that for each generic enough disk $\mathbb{D} \subset Y$ so that the analytic subset $X_{\mathbb{D}} := p^{-1}(\mathbb{D})$ of X is non-singular we have*

$$\Theta_{h^\rho_{X_{\mathbb{D}}/\mathbb{D}}}(K_{X_{\mathbb{D}}/\mathbb{D}}) \geq \gamma|_{X_{\mathbb{D}}}.$$

Then we have

$$\Theta_{h^\rho_{X/Y}}(K_{X/Y}) \geq \gamma$$

on X.

This is absolutely clear, given the formula (4). Therefore, we will restrict our attention to families over 1-dimensional bases, as long as we are only interested in the curvature properties of the metric $h^{\rho}_{X/Y}$.

After these general considerations, we show here that in the context of Theorem 1.1 a very special $(1,1)$-form ρ can be obtained as follows.

Let ω be a Kähler form on X; as explained at the beginning of the current section, we can construct a metric $h^{\omega}_{X/Y}$ on the bundle $K_{X/Y}$ induced by it. We recall that we denote by β a semi-positive definite form on X given by hypothesis of Theorem 1.1.

Moreover, we know that for each $y \in Y \setminus W$, the cohomology class

$$\{\Theta_{h^{\omega}_{X/Y}}(K_{X/Y}) + \beta\}|_{X_y} \tag{6}$$

is Kähler, again by hypothesis of 1.1. Therefore, by the main result of S.-T. Yau in [43] we have.

Theorem 3.2 ([43]). *There exists a unique function $\varphi_y \in \mathcal{C}^{\infty}(X_y)$ such that*

$$\Theta_{h^{\omega}_{X/Y}}(K_{X/Y}) + \beta|_{X_y} + \sqrt{-1}\partial\overline{\partial}\varphi_y > 0, \tag{7}$$

and such that φ_y is the solution of the next Monge-Ampère equation

$$(\Theta_{h^{\omega}_{X/Y}}(K_{X/Y}) + \beta|_{X_y} + \sqrt{-1}\partial\overline{\partial}\varphi_y)^n = e^{\varphi_y}\omega^n. \tag{8}$$

We stress on the fact that the differential form

$$\Theta_{h^{\omega}_{X/Y}}(K_{X/Y}) + \beta|_{X_y} \tag{9}$$

is not necessarily positive definite, but still the equation (8) admits a solution, since the cohomology class corresponding to it (6) is Kähler. Hence the function φ_y can be seen to be equal to the sum of two functions: a potential whose Hessian added to (9) makes it positive definite, and the solution of the Monge-Ampère given by the main theorem in [43]. The potential we have to add is by no means unique, but the resulting function φ_y it is. Also, an important fact is that the function

$$\varphi(x) := \varphi_y(x) \in \mathcal{C}^{\infty}(X_0), \tag{10}$$

where $y = p(x)$ is smooth. That is to say, the function obtained by piecing together the solutions φ_y of the (8) is smooth on X_0; this is a standard consequence of the usual estimates for the Monge-Ampère operator.

We consider next the (1,1)-form

$$\rho := \Theta_{h^{\omega}_{X/Y}}(K_{X/Y}) + \beta + \sqrt{-1}\partial\overline{\partial}\varphi \tag{11}$$

on the manifold X_0. A first remark is that ρ is definite positive when restricted to X_y, for any $y \in Y_0$: this is contained in Yau's result [43]. Thus, we can define a metric $h^{\rho}_{X/Y}$ on the bundle $K_{X/Y}|_{X_0}$; given the equation (8), its curvature is rapidly computed as follows.

$$\Theta_{h^{\rho}_{X/Y}}(K_{X/Y}) = \rho - \beta \tag{12}$$

(we are using the relations (8) and (4) in order to derive this).

There are two main steps in the proof of Theorem 1.1, namely.

(i) Show that the form ρ is definite positive on X_0; this will imply the positivity of the form $\Theta_{h^{\rho}_{X/Y}}(K_{X/Y}) + \beta$ on X_0.

(ii) Show that the metric $h^{\rho}_{X/Y}$ extends across the set $X \setminus X_0$. As soon as this second step is performed, we can infer the positivity of $\Theta_{h^{\rho}_{X/Y}}(K_{X/Y}) + \beta$ as current on X.

These points will be addressed in the next two paragraphs.

3.2. The computation

As we have already mentioned, in order to analyze the positivity properties of the for ρ in (11) it is enough to restrict ourselves to a family over a 1-dimensional base. Therefore we will assume that the map

$$p : X \to Y$$

is a proper fibration over a 1-dimensional manifold Y; we equally assume that X is non-singular, and so it is the generic fiber of p.

Let x be a point of X such that the fiber $X_y := p^{-1}(y)$ is non-singular; here we denote by $y := p(x)$. Let t be the coordinate on Y centered at y, and let $(z^1, ..., z^n)$ be local coordinates on the manifold X_y so that the functions $(t, z^1, ..., z^n)$ are local coordinates for X at x (we use here the notation t for the inverse image of the coordinate on Y via the map p, to avoid some notational complications).

With the notations in section 3.1, we write the form ρ in coordinates as follows

$$\rho = \sqrt{-1}g_{t\bar{t}}dt \wedge d\bar{t} + \sqrt{-1}\sum_{\alpha} g_{\alpha\bar{t}}dz^{\alpha} \wedge d\bar{t} + \sqrt{-1}\sum_{\alpha} g_{t\overline{\alpha}}dt \wedge dz^{\overline{\alpha}}$$
$$+ \sqrt{-1}\sum_{\alpha,\gamma} g_{\gamma\overline{\alpha}}dz^{\gamma} \wedge dz^{\overline{\alpha}}.$$

We already know that ρ is positive definite when restricted to X_y, hence it has at least n positive eigenvalues. In local writing as above, this implies that the matrix $(g_{\gamma\overline{\alpha}})$ is invertible; we denote the coefficients of its inverse by $(g^{\overline{\alpha}\gamma})$ (with the convention that the 1^{st} index is the line index of the associated matrix). In order to show that the $n+1^{\text{th}}$ eigenvalue (in the "base direction") is equally positive, we consider the form ρ^{n+1} on X. As it is well-known, we have

$$\rho^{n+1} = c(\rho)\rho^n \wedge \sqrt{-1}dt \wedge d\bar{t} \tag{13}$$

where the function $c(\rho)$ defined globally on X_y by the preceding equality can be expressed locally near x in coordinates as follows

$$c(\rho) = g_{t\bar{t}} - \sum_{\alpha,\gamma} g^{\overline{\alpha}\gamma} g_{t\overline{\alpha}} g_{\gamma\bar{t}}. \tag{14}$$

Our next goal will be to show that we have $c(\rho)|_{X_y} > 0$, as this is equivalent to the fact that ρ is positive definite. The method (cf. [34]) is to show that this function is the solution of a certain elliptic equation on X_y. The computations to follow are straightforward.
Let

$$\Box_y := -\sum_{i,j} g^{\overline{j}i} \frac{\partial^2}{\partial z^i \partial z^{\overline{j}}} \tag{15}$$

be the Laplace operator (with positive eigenvalues) associated to the metric $\rho|_{X_y}$. We will evaluate the expression

$$\Box_y c(\rho) \tag{16}$$

by using the local coordinates fixed above; we can (and will) assume that $(z^1, ..., z^n)$ are geodesic for the metric $\rho|_{X_y}$ at the point x_0.

We first evaluate the expression

$$-\sum_{i,j} g^{\overline{j}i} \frac{\partial^2 g_{t\bar{t}}}{\partial z^i \partial z^{\overline{j}}}; \tag{17}$$

a first observation, which will be used many times in what follows is that

$$\frac{\partial^2 g_{t\bar{t}}}{\partial z^i \partial z^{\overline{j}}} = \frac{\partial^2 g_{i\overline{j}}}{\partial t \partial \bar{t}} \tag{18}$$

since ρ is locally $\partial\overline{\partial}$ exact, given the expression (11). For any indexes (i, j) we have

$$g^{\overline{j}i} \frac{\partial^2 g_{i\overline{j}}}{\partial t \partial \bar{t}} = \frac{\partial}{\partial t}\Big(g^{\overline{j}i} \frac{\partial g_{i\overline{j}}}{\partial \bar{t}}\Big) - \frac{\partial g^{\overline{j}i}}{\partial t} \frac{\partial g_{i\overline{j}}}{\partial \bar{t}} \tag{19}$$

The term $\dfrac{\partial g^{\overline{j}i}}{\partial t}$ can be written in terms of the t-derivative of $g_{i\overline{j}}$ since we have

$$\sum_s \frac{\partial g^{\overline{s}i}}{\partial t} g_{k\overline{s}} = -\sum_s g^{\overline{s}i}\frac{\partial g_{k\overline{s}}}{\partial t} \tag{20}$$

which implies that

$$\sum_{s,k} \frac{\partial g^{\overline{s}i}}{\partial t} g_{k\overline{s}} g^{\overline{j}k} = -\sum_{s,k} g^{\overline{s}i} g^{\overline{j}k}\frac{\partial g_{k\overline{s}}}{\partial t} \tag{21}$$

and thus we get

$$\frac{\partial g^{\overline{j}i}}{\partial t} = -\sum_{s,k} g^{\overline{s}i} g^{\overline{j}k}\frac{\partial g_{k\overline{s}}}{\partial t}. \tag{22}$$

We notice that we have

$$\sum_{i,j} g^{\overline{j}i}\frac{\partial g_{i\overline{j}}}{\partial \overline{t}} = \frac{\partial}{\partial \overline{t}}\log(g) \tag{23}$$

where $g := \det(g_{\alpha\overline{\beta}})$; in conclusion, we obtain the following identity

$$-\sum_{i,j} g^{\overline{j}i}\frac{\partial^2 g_{t\overline{t}}}{\partial z^i \partial z^{\overline{j}}} = -\frac{\partial^2 \log(g)}{\partial t \partial \overline{t}} - \sum_{s,k,i,j} g^{\overline{s}i} g^{\overline{j}k}\frac{\partial g_{k\overline{s}}}{\partial t}\frac{\partial g_{i\overline{j}}}{\partial \overline{t}} \tag{24}$$

In local coordinates, the equation (8) implies that we have

$$\frac{\partial^2 \log(g)}{\partial t \partial \overline{t}} = g_{t\overline{t}} - \beta_{t\overline{t}} \tag{25}$$

so we get

$$-\sum_{i,j} g^{\overline{j}i}\frac{\partial^2 g_{t\overline{t}}}{\partial z^i \partial z^{\overline{j}}} = -g_{t\overline{t}} + \beta_{t\overline{t}} - \sum_{s,k,i,j} g^{\overline{s}i} g^{\overline{j}k}\frac{\partial g_{k\overline{s}}}{\partial t}\frac{\partial g_{i\overline{j}}}{\partial \overline{t}} \tag{26}$$

We will detail next the computation for the factor

$$\sum_{i,j,\alpha,\gamma} g^{\overline{j}i}\frac{\partial^2}{\partial z^i \partial z^{\overline{j}}}\Big(g^{\overline{\alpha}\gamma} g_{t\overline{\alpha}} g_{\gamma\overline{t}}\Big); \tag{27}$$

given that the coordinates (z^α) are geodesic, the only terms we have to evaluate are the following

$$I_1 := \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g_{t\overline{\alpha}} g_{\gamma\overline{t}} \frac{\partial^2 g^{\overline{\alpha}\gamma}}{\partial z^i \partial z^{\overline{j}}}, \tag{28}$$

as well as

$$I_2 := \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} g_{t\overline{\alpha}} \frac{\partial^2 g_{\gamma\overline{t}}}{\partial z^i \partial z^{\overline{j}}}, \quad I_3 := \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} \frac{\partial g_{\gamma\overline{t}}}{\partial z^{\overline{j}}} \frac{\partial g_{t\overline{\alpha}}}{\partial z^i} \tag{29}$$

together with their conjugates, and also

$$I_4 := \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} \frac{\partial g_{\gamma\overline{t}}}{\partial z^i} \frac{\partial g_{t\overline{\alpha}}}{\partial z^{\overline{j}}}. \tag{30}$$

In order to simplify the term I_1, we observe that at x we have

$$\frac{\partial^2 g^{\overline{\alpha}\gamma}}{\partial z^i \partial z^{\overline{j}}} = R^{\gamma\overline{\alpha}}_{i\overline{j}} \tag{31}$$

hence we get

$$I_1 = \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g_{t\overline{\alpha}} g_{\gamma\overline{t}} R^{\gamma\overline{\alpha}}_{i\overline{j}}. \tag{32}$$

We observe that $\sum_{i,j} g^{\overline{j}i} R^{\gamma\overline{\alpha}}_{i\overline{j}} = \sum_{p,q} \mathrm{Ricci}_{p\overline{q}} g^{\overline{q}\gamma} g^{\overline{\alpha}p}$ where we denote by $(\mathrm{Ricci}_{p\overline{q}})$ the coefficients of the Ricci curvature of the metric ρ. By using the equation (8) we infer that

$$\begin{aligned} I_1 &= -\sum_{p,q,\alpha,\gamma} g_{t\overline{\alpha}} g_{\gamma\overline{t}} g^{\overline{q}\gamma} g^{\overline{\alpha}p} (g_{p\overline{q}} - \beta_{p\overline{q}}) \\ &= -\sum_{p,q,\gamma} g_{t\overline{q}} g_{\gamma\overline{t}} g^{\overline{q}\gamma} + \sum_{p,q,\alpha,\gamma} g_{t\overline{\alpha}} g_{\gamma\overline{t}} g^{\overline{q}\gamma} g^{\overline{\alpha}p} \beta_{p\overline{q}} \end{aligned}$$

The other terms are evaluated in a similar way; we have

$$I_2 = \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} g_{t\overline{\alpha}} \frac{\partial^2 g_{\gamma\overline{t}}}{\partial z^i \partial z^{\overline{j}}} = \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} g_{t\overline{\alpha}} \frac{\partial^2 g_{i\overline{j}}}{\partial z^\gamma \partial \overline{t}}$$
$$= \sum_{i,j,\alpha,\gamma} g_{t\overline{\alpha}} g^{\overline{\alpha}\gamma} \frac{\partial^2 \log(g)}{\partial z^\gamma \partial \overline{t}}$$
$$= \sum_{i,j,\alpha,\gamma} g_{t\overline{\alpha}} g^{\overline{\alpha}\gamma} (g_{\gamma\overline{t}} - \beta_{\gamma\overline{t}})$$
$$= \sum_{i,j,\alpha,\gamma} g_{t\overline{\alpha}} g_{\gamma\overline{t}} g^{\overline{\alpha}\gamma} - \sum_{i,j,\alpha,\gamma} g_{t\overline{\alpha}} g^{\overline{\alpha}\gamma} \beta_{\gamma\overline{t}}.$$

as well as

$$I_3 = \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} \frac{\partial g_{\gamma\overline{t}}}{\partial z^{\overline{j}}} \frac{\partial g_{t\overline{\alpha}}}{\partial z^i} = \sum_{i,j,\alpha,\gamma} g^{\overline{j}i} g^{\overline{\alpha}\gamma} \frac{\partial g_{\gamma\overline{j}}}{\partial \overline{t}} \frac{\partial g_{i\overline{\alpha}}}{\partial t}.$$

We want to have an intrinsic interpretation of the factor I_4, so we consider next the vector field

$$v := \frac{\partial}{\partial t} - \sum_{i,j} g^{\overline{j}i} g_{t\overline{j}} \frac{\partial}{\partial z^i}; \tag{33}$$

as it is well-known [36], [34], v is the horizontal lift of $\dfrac{\partial}{\partial t}$ with respect to the metric ρ. Then we see that we have

$$I_4 = |\overline{\partial} v|^2, \tag{34}$$

and by combining all the equalities above, we infer that we have the compact formula

$$\Box_y c(\rho) = -c(\rho) + |\overline{\partial} v|^2 + \beta(v, \overline{v}) \tag{35}$$

The Ricci curvature of the metric $\rho|_{X_y}$ is bounded from below by -1, by the equation (8); hence precisely as in [34], we infer that we have

$$\inf_{X_y} c(\rho) \geq C \int_{X_y} (|\overline{\partial} v|^2 + |v|^2_\beta) dV_\rho \tag{36}$$

where C only depends on the diameter of the fiber (X_y, ρ). Hence, the form ρ is positively defined in the base directions as well.

In conclusion, the fiberwise twisted Kähler-Einstein metric ρ defines a Kähler metric on X_0, the pre-image of the set $Y \setminus W$. Given the equation (12), this means that the curvature of the metric $h^{\rho}_{X/Y}|_{X_0}$ is bounded by $-\beta$. In the next section, we will show that this metric extends across the singularities of p, and therefore the proof of theorem 1.1 will be complete. Q.E.D.

3.3. Extension across the singularities

We will use the same notations as in the previous section. Given the point $x_0 \in X_y$, we will derive an upper bound of the metric induced by ρ on the relative canonical bundle.

Let Ω be an open coordinate set in X centered at x_0. We consider $\Omega_y := \Omega \cap X_y$; we denote by ψ_β a local potential of the Kähler metric β on Ω. We recall that we denote by Ψ_Ω the local weight of the metric on $K_{X/Y}$ induced by the metric ω (so implicitly we assume that we have fixed a coordinate system $(z^\alpha)_{\alpha=1,\dots,n+d}$ on Ω and $(t^\gamma)_{\gamma=1,\dots,d}$ near $y := p(x_0)$). The function to be bounded from above is

$$\tau_y := \Psi_\Omega + \psi_\beta + \varphi|_{\Omega_y} \tag{37}$$

At first glance this may look very simple, since we have $\varphi|_{\Omega_y} = \varphi_y$, the solution of (8), and thus the function (37) is psh on Ω_y. But the bound one can obtain from the meanvalue inequality are not good enough for our purposes, since they depend on the geometry of the manifold X_y: this is precisely what we want to avoid, as y approaches the singular values of p.

The idea is first to approximate the function (37) with log of absolute values of holomorphic functions; then we show that the holomorphic functions involved in this process admit an extension to Ω, where the use of Cauchy inequalities is "legitimate", since the manifold X is non-singular and compact.

We recall next the following approximation result, cf. [13].

Theorem 3.3. [13] *Let $\mathcal{H}_y^{(m)}$ be the Hilbert space defined as follows*

$$\mathcal{H}_y^{(m)} := \left\{ f \in \mathcal{O}(\Omega_y) \text{ such that } \|f\|_y^2 = \int_{\Omega_y} |f|^2 e^{-m\tau_y - \|z\|^2} (dd^c\tau_y)^n < \infty \right\}. \tag{38}$$

Then we have

$$\tau_y(x) = \lim_{m\to\infty} \sup_{f\in\mathcal{H}_y^{(m)}, \|f\|_y^2 \le 1} \frac{1}{m} \log |f(x)|^2 \tag{39}$$

for every $x \in \Omega_y$.

In the statement above, the fact that the manifold Ω_y is Stein is of course crucial, since without this hypothesis we cannot approximate τ_y by using global holomorphic functions. The importance of the volume element $(dd^c\tau_y)^n$ will become clear in a moment.

Let $f \in \mathcal{H}_y^{(m)}$ be a holomorphic function, such that $\|f\|_y^2 \leq 1$. By Hölder inequality we have

$$\begin{aligned}\int_{\Omega_y} |f|^{2/m} e^{-\tau_y} (dd^c\tau_y)^n \leq& \Big(\int_{\Omega_y} (dd^c\tau_y)^n\Big)^{\frac{m-1}{m}} \\ \leq& C\end{aligned}$$

where C can be taken to be the maximum between 1 and the volume of the fiber X_y with respect to the Kähler class $c_1(K_{X_y}) + \{\beta\}$; hence, it is a constant independent of m and y.

We use now again the equation (8): in local coordinates, it can we written as

$$(dd^c\tau_y)^n = e^{\tau_y - \varphi_\beta} \Big|\frac{dz}{dt}\Big|^2 \tag{40}$$

where the notations are (hopefully...) self-explanatory. Then the estimate above implies

$$\int_{\Omega_y} |f|^{2/m} e^{-\varphi_\beta} \Big|\frac{dz}{dt}\Big|^2 \leq C \tag{41}$$

We now invoke the $L^{2/m}$ version of the Ohsawa-Takegoshi theorem obtained in [4]: it implies the existence of a holomorphic function F, such that:

(a) The restriction of F to Ω_y is equal to f.

(b) There exists a numerical constant $C_0 > 0$ *independent of* m such that

$$\int_{\Omega} |F|^{2/m} e^{-\varphi_\beta} \big|dz\big|^2 \leq C_0 \int_{\Omega_y} |f|^{2/m} e^{-\varphi_\beta} \Big|\frac{dz}{dt}\Big|^2 \tag{42}$$

In particular, this implies that the value $|F(x_0)|^{2/m}$ is bounded from above by a constant which is independent of y and on m. Hence the weight function τ_y have the same property (by the restriction statement (a) above), and this implies that the metric $h^{\rho}_{X/Y}|_{X_0}$ *extends* as a singular metric for the relative canonical bundle of the fibration p; moreover, its curvature current is greater than $-\beta$. Theorem 1.1 is therefore completely proved. Q.E.D.

Remark 3.4. As we have already mentioned, in the "linear" context, the positivity properties of the relative adjoint bundles of type $K_{X/Y}+L$ is established in a very explicit way, by showing that the fiberwise Bergman kernel has a psh variation. The only assumptions which are needed to obtain a non-trivial positively curved metric is the positivity of L, together with the existence of an L^2 section of $K_{X_y}+L|_{X_y}$.

In order to compare this theory with our previous considerations, let $p: X \to Y$ be a map such that K_{X_y} is ample, for some generic $y \in Y$. In the article [37], H. Tsuji shows that his method of iterating the Bergman kernels can be used to construct inductively a metric on the bundle $mK_{X/Y}+A$, for any $m \geq 1$ (here we denote by A some positive enough line bundle). Moreover, he shows that the metric obtained on $K_{X/Y}$ by a limiting process is precisely the fiberwise Kähler-Einstein metric considered in G. Schumacher paper [34]. As we have seen, the method in [34] has the advantage that it offers a *lower bound* for the curvature of the metric constructed on the relative adjoint class, but on the other hand, it should be further extended e.g. to encompass the case where the adjoint class has base points when restricted to fibers.

§4. Further corollaries, consequences and comments

4.1. Proof of Corollary 1.2

Let $p: \mathcal{X} \to \mathbb{D}$ be a Kähler family. We denote by β a Kähler metric on $\mathcal{X}$. For each $\varepsilon > 0$ and for each $t \in \mathbb{D}$, the class

$$c_1(K_{\mathcal{X}_t}) + \varepsilon\{\beta\} \tag{43}$$

is Kähler, since by hypothesis the canonical bundle $K_{\mathcal{X}_t}$ is nef. The family p is assumed to be non-singular, hence by the results we have obtained in the section 3.2, the class

$$c_1(K_{\mathcal{X}/\mathbb{D}}) + \varepsilon\{\beta\}$$

is Kähler. This means that $K_{\mathcal{X}/\mathbb{D}}$ is nef, so Corollary 1.2 is proved. Q.E.D.

4.2. Proof of Theorem 1.4

Let X be a compact Kähler manifold, such that $-K_X$ is nef in the sense of the definition 1.3 in the introduction. We denote by $\alpha_X : X \to \mathrm{Alb}(X)$ the Albanese morphism of X.

As we have seen in the section 1, a successful approach towards the subjectivity of the map α_X in the projective case is using in an essential manner the positivity properties of the relative canonical bundle associated to the desingularization of α_X. In the general case we will follow basically the same line of arguments, except that the Kähler version of the results 2.1, 2.1 is less general.

Indeed, as a consequence of Theorem 1.1 we infer the next statement.

Corollary 4.1. *Let $p : X \to Y$ be a surjective map between two compact Kähler manifolds, which are assumed to be non-singular. Let $L \to X$ be a nef line bundle, such that the adjoint system $K_{X_y} + L|_{X_y}$ is equally nef for any $y \in Y$ generic. Then the bundle $K_{X/Y} + L$ is pseudo-effective.*

We recall that in the projective case, instead of the nefness of the adjoint bundle $K_{X_y} + L|_{X_y}$ we have assumed that this bundle has a non-trivial section. Also, even if in the statement of the preceding corollary there is no *transcendental* class involved, its proof is using the full force of Theorem 1.1: the class $\{\beta\} := c_1(L) + \varepsilon\omega$ is Kähler, for every positive real ε. This being said, the Corollary 4.1 is a direct consequence of Theorem 1.1. Q.E.D.

In order to be able to use Corollary 4.1, we first consider a desingularization $\widehat{Y}$ of the $\alpha_X(X)$. We denote by $\overline{X}$ the fibered product of X with $\widehat{Y}$ over the base Y. This variety may be singular, but its singular loci projects into an analytic set of X whose co-dimension is at least 2. This is seen e.g. by considering the rational map $X \dashrightarrow \widehat{Y}$ obtained by composing the inverse of π_Y with the Albanese map α_X: this map is defined outside a set of co-dimension at least 2, and the set $\overline{X}$ is non-singular at each point of the pre-image of this set. Next we invoke the desingularisation result of H. Hironaka and infer the existence of a map $\widehat{X} \to \overline{X}$ which is an isomorphism outside the singular set of $\overline{X}$. We note that this procedure does not use the fact that X is projective. Let $\pi_X : \widehat{X} \to X$ be the map obtained by composing the desingularization of $\overline{X}$ with the natural map $\overline{X} \to X$; the manifolds/maps constructed above have the next important properties

(i) *The generic fiber of the map $p : \widehat{X} \to \widehat{Y}$ is disjoint from the support of the exceptional divisor associated to the map π_X defined by the relation*

$$K_{\widehat{X}} = \pi_X^\star(K_X) + E.$$

(ii) *The divisor E is π_X–contractible*

The rest of the proof of Theorem 1.4 is identical to the arguments invoked in the projective case; the hypothesis of Corollary 4.1 are verified, by the properties (i) and (ii) above.

Remark 4.2. In a forthcoming paper, we will investigate the singular version of Theorem 1.1; the precise statement can be easily guessed from the projective case ([25], [40]), as follows. Let $\{\beta\}$ be a $(1,1)$-class on X, admitting a representative $\Theta = \beta + \sqrt{-1}\partial\overline{\partial} f$ which is a closed positive current, such that

$$\int_X e^{-f} dV < \infty$$

(i.e. (X, Θ) is the analogue of a klt pair in algebraic geometry). If the class

$$c_1(K_{X_y}) + \{\beta\}|_{X_y} \tag{44}$$

contains a Kähler current for each $y \in Y$ generic, then the class $c_1(K_{X/Y}) + \{\beta\}$ should contain a Kähler current. In order to adapt the argument used in this note for the proof of such a result, it seems to us that there are serious technical difficulties to overcome. Nevertheless, if Θ is allowed to be singular only along a SNC divisor, and if the class (44) is Kähler, then a slight generalization of the results in [8], [19] are enough to conclude. However, it is highly desirable to prove the result in singular context, as the article by J. Kollár [25] shows it: one should allow base points for the positive representatives of the class (44). We refer to the work of O. Fujino [15] for new results and an overview of related topics from the algebraic geometry point of view.

References

[1] R. Berman, *Relative Kahler-Ricci flows and their quantization,* arXiv:1002.3717.

[2] B. Berndtsson, *Curvature of Vector bundles associated to holomorphic fibrations,* to appear in Ann. of Maths. (2007).

[3] B. Berndtsson, M. Păun, *Bergman kernels and the pseudo-effectivity of the relative canonical bundles,* arXiv:math/0703344, Duke Math. Journal 2008.

[4] B. Berndtsson, M. Păun, *Qualitative extensions of twisted pluricanonical forms and closed positive currents,* arXiv 2009.

[5] R. Berman, S. Boucksom, P. Eyssidieux, V. Guedj, A. Zeriahi *Kähler-Ricci flow and Ricci iteration on log-Fano varieties,* arXiv:1111.7158.

[6] F. Campana, *Special Varieties and Classification Theory,* Annales de l'Institut Fourier 54, 2004.

[7] F. Campana, Th. Peternell, Q. Zhang, *On the Albanese maps of compact Kähler manifolds,* Proc. Amer. Math. Soc. 131 (2003), no. 2, 549–553.

[8] F. Campana, H. Guenancia, M. Păun *Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields,* arXiv:1104.4879.

[9] M. Chen, Q. Zhang, *On a question of Demailly-Peternell-Schneider,* arXiv 1110.1824, to appear in J. Eur. Math. Soc.

[10] J.-P. Demailly, *Singular hermitian metrics on positive line bundles,* Proc. Conf. Complex algebraic varieties (Bayreuth, April 26, 1990), edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math., Vol. 1507, Springer-Verlag, Berlin, 1992.

[11] J.-P. Demailly, Th. Peternell and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles,* J. Algebraic Geometry 3 (1994) 295- 345.

[12] J.-P. Demailly, Th. Peternell and M. Schneider, *Kähler manifolds with numerically effective Ricci class,* Compositio Math. 89 (1993) 217-240.

[13] J.-P. Demailly, *Analytic methods in algebraic geometry,* on the web page of the author, December 2009.

[14] T. Fujita, *On Kahler fiber spaces over curves,* J. Math. Soc. Japan 30 (1978), no. 4, 779–794.

[15] O. Fujino, *Fundamental theorems for the log minimal model program,* Publ. Res. Inst. Math. Sci. 47 (2011), no. 3.

[16] O. Fujino, Y. Gongyo, *On images of weak Fano manifolds I and II,* arXiv:1201.1130.

[17] P. Griffiths, *Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping,* Inst. Hautes Etudes Sci. Publ. Math. No. 38 1970.

[18] A. Höring, *Positivity of direct image sheaves - a geometric point of view,* available on the author's home page.

[19] T. Jeffres, R. Mazzeo, Y. Rubinstein, *Kähler-Einstein metrics with edge singularities,* arXiv:1105.5216.

[20] Y. Kawamata, *Characterization of abelian varieties,* Compositio Math. 43 (1981), no. 2, 253–276.

[21] Y. Kawamata, *Kodaira dimension of algebraic fiber spaces over curves,* Invent. Math. 66, 1982.

[22] Y. Kawamata, *Kodaira dimension of certain algebraic fiber spaces,* J. Fac. Sci. Univ. Tokyo Sect. Math. 30 (1983), no. 1, 1–24.

[23] Y. Kawamata, *Semipositivity theorem for reducible algebraic fiber spaces,* Pure Appl. Math. Q. 7 (2011), no. 4, Special Issue: In memory of Eckart Viehweg.

[24] J. Kollár, *Higher direct images of dualizing sheaves, I and II,* Ann. of Math. (2) 124, 1986.

[25] J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type,* Algebraic geometry, Sendai, 1985, 361–398, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[26] J. Kollár, Y. Miyaoka, S. Mori*Rational connectedness and boundedness of Fano manifolds,* J. Differential Geom. 36 (1992), no. 3, 765–779.

[27] R. Lazarsfeld, *Positivity in Algebraic Geometry,* Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete.

[28] C. Mourougane, S. Takayama, *Extension of twisted Hodge metrics for Kähler morphisms,* arXiv: 0809.3221, to appear in Ann. Sci. Ens.

[29] T. Ohsawa, K. Takegoshi, *On the extension of L^2 holomorphic functions,* Math. Z., 1987.

[30] T. Ohsawa, *On the extension of L^2 holomorphic functions. VI. A limiting case,* Contemp. Math., Amer. Math. Soc., Providence, 2003.

[31] T. Ohsawa, *Generalization of a precise L2 division theorem,* Complex analysis in several variables: Memorial Conference of Kiyoshi Oka's Centennial Birthday, 249–261, Adv. Stud. Pure Math., 42, Math. Soc. Japan, Tokyo, 2004.

[32] M. Păun, *On the Albanese map of compact Kähler manifolds with numerically effective Ricci curvature,* Comm. Anal. Geom. 9 (2001), no. 1, 35–60.

[33] G. Schumacher, S. Trapani, *Variation of cone metrics on Riemann surfaces,* J. Math. Anal. Appl. 311 (2005), no. 1, 218–230.

[34] G. Schumacher, *Curvature of higher direct images and applications,* arXiv:1002.4858.

[35] G. Schumacher, *Positivity of relative canonical bundles and applications,* arXiv:1201.2930 to appear Inventiones Math.

[36] Y-T. Siu, *Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class* Contributions to several complex variables, Hon. W. Stoll, Proc. Conf. Complex Analysis, Notre Dame/Indiana 1984, Aspects Math. E9, 261–298 (1986).

[37] H. Tsuji, *Extension of log pluricanonical forms from subvarieties,* math.CV/0511342.

[38] H. Tsuji, *Global generation of the direct images of relative pluricanonical systems,* arXiv:1012.0884.

[39] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces,* Proc. Algebraic Varieties and Analytic Varieties, Adv. Studies in Math. 1, Kinokunya–North-Holland Publ., 1983.

[40] E. Viehweg, *Weak positivity and the stability of certain Hilbert points,* Invent. Math. 96 (1989), no. 3, 639–667.

[41] E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds,* Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[42] H. Yamaguchi, *Variations of pseudoconvex domains over $\mathbb{C}^n$*, Michigan Math J., 1989.

[43] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge- Ampère equation*, Comm. Pure Appl. Math. 31 (1978).

[44] Q. Zhang, *On projective manifolds with nef anticanonical bundles,* J. Reine Angew. Math. 478 (1996), 57–60.

[45] Q. Zhang, *On projective manifolds with nef anticanonical divisors,* Math. Ann. 332 (2005), 697–703.

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