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Length Functions for $G(r, p, n)$

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Abstract.

In this paper, we construct a length function $n(w)$ for the complex reflection group $W = G(r, p, n)$ by making use of certain partitions of the root system associated to $\widetilde{W} = G(r, 1, n)$. We show that the function $n(w)$ yields the Poincaré polynomial $P_W(q)$. We give some characterization of this function in a way independent of the choice of the root system.

§1. Introduction

Let $\widetilde{W} = G(r, 1, n)$ be an imprimitive complex reflection group. In [BM1], K. Bremke and G. Malle introduced a certain type of root system (and its partition into positive and negative roots) associated to $\widetilde{W}$, and defined a length function n_1 on $\widetilde{W}$ by making use of the root system. They showed that this function satisfies some good properties as a generalization of the length function of finite Coxeter groups. In particular, the polynomial $\sum_{w \in \widetilde{W}} q^{n_1(w)}$ coincides with the Poincaré polynomial $P_{\widetilde{W}}(q)$ of $\widetilde{W}$. In [RS], we studied further properties of n_1, and gave some characterization of it in a way independent of the choice of the root system, in connection with the usual length function defined by standard generators of $\widetilde{W}$.

In [BM2], a similar problem was studied for the reflection subgroup $G(r, r, n)$ of $\widetilde{W}$. They defined a length function $\tilde{n}_2$ on $\widetilde{W}$ by using a similar root system as above, but by using completely different partition into positive and negative roots. They defined a length function n_2 on $G(r, r, n)$ as the restriction of $\tilde{n}_2$, and showed that n_2 yields the Poincaré polynomial $P_{G(r,r,n)}(q)$.

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In this paper, we consider a more general group $W = G(r,p,n)$. The group W is a reflection subgroup of $\widetilde{W}$ containing $G(r,r,n)$. We construct some partitions of the root system, (in fact, we need two kinds of such partitions) and define a length function $\tilde{n}$ on $\widetilde{W}$ associated to the root system. We also define a function n on W as the restriction of $\tilde{n}$ on W. We then show that our length functions satisfy the property that

$$\frac{1}{p}\sum_{w\in\widetilde{W}} q^{\tilde{n}(w)} = \sum_{w\in W} q^{n(w)} = P_W(q),$$

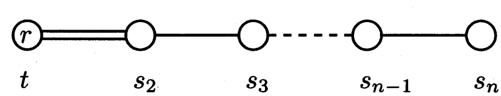
where $P_W(q)$ is the Poincaré polynomial associated to W. Our function $n(w)$ is much more complicated than the previous cases. But in some sense, it is the mixture of the functions n_1 and n_2. In fact, if $p = 1$, $n(w)$ coincides with $n_1(w)$, while if $p = r$, $n(w)$ coincides with $n_2(w)$.

We give a characterization of the function $\tilde{n}$ on $\widetilde{W}$ in a similar way as in [RS], in an independent way of the choice of the root system. This is done by making use of a certain length function on $\widetilde{W}$ defined without using the root data. However, in contrast to the case treated in [RS], it is not the function defined by generators of $\widetilde{W}$ or W.

§2. Length functions associated to a root system

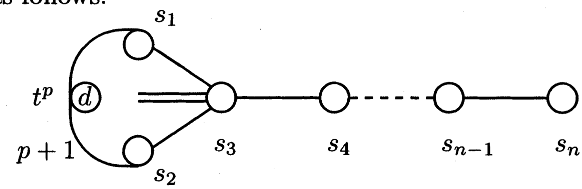
2.1 Let V be the unitary space $\mathbf{C}^n$ with the standard basis vectors $e_1,\dots,e_n$. We denote by $\widetilde{W} = G(r,1,n)$ the imprimitive complex reflection group generated by reflections $t, s_2,\dots,s_n$. Here s_i is the permutation of e_i and e_{i-1} for $i = 2,\dots,n$, and t is the complex reflection of order r defined by $te_1 = \zeta e_1$ and $te_i = e_i$ for $i \neq 1$, where ζ is a fixed primitive r-th root of unity. The group $\widetilde{W}$ has a Coxeter-like diagram with respect to the set $\widetilde{S} = \{t, s_2,\dots,s_n\}$ of generators as follows;

$B_n^{(r)}$:



For each factor p of r, we denote by $W = G(r,p,n)$ the reflection subgroup of $\widetilde{W}$ of index p generated by $S = \{t^p, s_1 = t^{-1}s_2t, s_2,\dots,s_n\}$. The special case where $p = r$, the group $W' = G(r,r,n)$ is generated by $S' = \{s_1,\dots,s_n\}$. We have $W' \subset W \subset \widetilde{W}$. We put $r = pd$. The presentation of the group W in terms of the set S is determined by

[BMR]. In particular, if $p \geq 3, d \neq 1$, the Coxeter-like diagram of W is given as follows.



2.2 Let Φ be a root system associated to $\widetilde{W}$ defined in [BM1]. Here we follow the description of Φ given in [RS]. Hence we consider a set $X = \{e_i^{(a)} \mid 1 \leq i \leq n, a \in \mathbf{Z}/r\mathbf{Z}\}$, and we express an element $(e_i^{(a)}, e_j^{(b)}) \in X \times X$ as $e_i^{(a)} - e_j^{(b)}$ whenever $i \neq j$. The root system Φ is defined by

$$
\begin{aligned}
&\Phi = \Phi_l \coprod \Phi_s \quad \text{with} \\
&\Phi_l = \{e_i^{(a)} - e_j^{(b)} \mid 1 \leq i, j \leq n, i \neq j, a, b \in \mathbf{Z}/r\mathbf{Z}\}, \\
&\Phi_s = X = \{e_i^{(a)} \mid 1 \leq i \leq n, a \in \mathbf{Z}/r\mathbf{Z}\}
\end{aligned}
$$

An element in Φ_l (resp. in Φ_s) is called a long root (resp. a short root), respectively. The group $\widetilde{W}$ acts naturally on the set Φ in such a way that s_i permutes $e_i^{(a)}$ and $e_{i-1}^{(a)}$, and $te_1^{(a)} = e_1^{(a+1)}$, $te_j^{(a)} = e_j^{(a)}$ for $j \neq 1$.

For $\alpha = e_i^{(a)} - e_j^{(b)} \in \Phi_l$, we define $-\alpha \in \Phi_l$ by $-\alpha = e_j^{(b)} - e_i^{(a)}$. We shall define two types of partitions, $\Phi_l = \Phi_l^+ \cup \Phi_l^- = \Phi_l^{++} \cup \Phi_l^{--}$ such that $\Phi_l^- = -\Phi_l^+, \Phi_l^{--} = -\Phi_l^{++}$. In the following formulae, long roots $\alpha \in \Phi_l$ are always denoted as $\alpha = e_i^{(a)} - e_j^{(b)}$. Also for each $a \in \mathbf{Z}$, let $\bar{a}$ be the integer determined by the condition that $\bar{a} \equiv a \pmod p$ and that $-p/2 < \bar{a} \leq p/2$. The partition of the first type is given as follows.

$$
\begin{aligned}
(2.2.1) \qquad \Phi_l^+ &= \{\alpha \mid -p/2 < a \leq 0,\ i > j\} \\
&\cup \{\alpha \mid 0 < \bar{a} \leq p/2,\ p/2 < b \leq r - p/2,\ i > j\} \\
&\cup \{\alpha \mid -p/2 < \bar{b} \leq 0,\ 0 < b \leq r - p/2,\ i < j\} \\
&\cup \{\alpha \mid 0 < \bar{b} \leq p/2,\ -p/2 < a \leq p/2,\ i < j\}, \\
\Phi_l^- &= \{\alpha \mid -p/2 < b \leq 0,\ i < j\} \\
&\cup \{\alpha \mid 0 < \bar{b} \leq p/2,\ p/2 < a \leq r - p/2,\ i < j\} \\
&\cup \{\alpha \mid -p/2 \leq \bar{a} \leq 0,\ 0 < a \leq r - p/2,\ i > j\} \\
&\cup \{\alpha \mid 0 < \bar{a} \leq p/2,\ -p/2 < b \leq p/2,\ i > j\}.
\end{aligned}
$$

The fact that $\Phi_l^- = -\Phi_l^+$, and that Φ_l is a disjoint union of Φ_l^+ and Φ_l^- is verified as follows. Set

$$
\begin{aligned}
A &= \{\alpha \mid -p/2 < a \le 0,\ i > j\},\\
B &= \{\alpha \mid 0 < \bar{a} \le p/2,\ p/2 < b \le r - p/2,\ i > j\},\\
C &= \{\alpha \mid -p/2 < \bar{a} \le 0,\ 0 < a \le r - p/2,\ i > j\},\\
D &= \{\alpha \mid 0 < \bar{a} \le p/2,\ -p/2 < b \le p/2,\ i > j\}.
\end{aligned}
$$

Then, it is easy to see that A, B, C and D are mutually disjoint, and $A \cup B \cup C \cup D$ coincides with the set $\{\alpha \in \Phi_l \mid i > j\}$. Moreover, we have

$$\Phi_l^+ = A \cup B \cup -C \cup -D, \qquad \Phi_l^- = -A \cup -B \cup C \cup D.$$

This shows the required property.

The partition of the second type is given as follows.

$$
\begin{aligned}
(2.2.2)\quad \Phi_l^{++} &= \{\alpha \mid -p/2 < \bar{a} \le 0,\ i > j\} \cup \{\alpha \mid 0 < \bar{b} \le p/2,\ i < j\},\\
\Phi_l^{--} &= \{\alpha \mid 0 < \bar{a} \le p/2,\ i > j\} \cup \{\alpha \mid -p/2 < \bar{b} \le 0,\ i < j\}.
\end{aligned}
$$

We also define a grading of Φ_s by modifying the grading of Φ_s given in [RS] as follows. Let $\Phi_s = \Phi_{s,0} \cup \Phi_{s,1} \cup \cdots \cup \Phi_{s,d-1}$, where

(2.2.3)
$$\Phi_{s,m} = \{e_i^{(a)} \mid mp - p/2 < a \le mp + p/2,\ 1 \le i \le n\} \quad (0 \le m < d).$$

Next we define a subset $\Omega = \Omega_l' \cup \Omega_l'' \cup \Omega_s$ of Φ as follows.

$$
\begin{aligned}
\Omega_s &= \{e_i^{(0)} \mid 1 \le i \le n\},\\
\Omega_l' &= \{e_i^{(0)} - e_j^{(b)} \mid b \equiv 0 \pmod{p},\ i > j\},
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_l'' = \{e_i^{(a)} - e_j^{(mp-a)} &\mid -p/2 < a < 0,\ 0 \le m < d,\ i > j\}\\
\cup\{e_i^{(mp-b+\delta)} - e_j^{(b)} &\mid 0 < b \le p/2,\ 0 \le m < d,\ i < j\},
\end{aligned}
$$

where

$$\delta = \begin{cases} 1 & \text{if } p \text{ is even},\\ 0 & \text{if } p \text{ is odd}.\end{cases}$$

We define functions $\tilde{n}'_l, \tilde{n}''_l, \tilde{n}_s : \widetilde{W} \to \mathbf{N}$ by

$$\tilde{n}'_l(w) = |w\Omega'_l \cap \Phi_l^-|, \qquad \tilde{n}''_l(w) = |w\Omega''_l \cap \Phi_l^{--}|,$$

and by

$$\tilde{n}_s(w) = \sum_{\alpha \in \Omega_s} \nu(w(\alpha)),$$

where for each $\alpha \in \Phi_s$, we put $\nu(\alpha) = k$ if $\alpha \in \Phi_{s,k}$. We define a length function $\tilde{n} : \widetilde{W} \to \mathbf{N}$ by $\tilde{n} = \tilde{n}'_l + \tilde{n}''_l + \tilde{n}_s$. We consider the restriction of these functions to W, and define n'_l, n''_l and n_s as the restriction of $\tilde{n}'_l, \tilde{n}''_l$, and $\tilde{n}_s$, respectively. Then we define a length function n of W by $n = n'_l + n''_l + n_s$.

Remark 2.3. In the case where $p = 1$, we have $\Omega''_l = \emptyset$. Moreover, $\Phi_l^+ = \{\alpha \mid a = 0\} \cup \{\alpha \mid b \neq 0\}$, and $\Phi_l^- = -\Phi_l^+$. This partition together with the set $\Omega'_l \cup \Omega_s$ coincide with the set $\Omega_l \cup \Omega_s$ of Φ_l given in [RS], and the grading of Φ_s also coincides with that of Φ_s given there. Hence the function n coincides with the length function of $G(r, 1, n)$ defined in [BM1].

While in the case where $p = r$, we have $\Phi_s = \Phi_{s,0}$. Moreover $\Phi_l^+ = \Phi_l^{++}, \Phi_l^- = \Phi_l^{--}$, and this partition of Φ_l together with $\Omega'_l \cup \Omega''_l$ coincide essentially with those given in [BM2]. (Also note that Ω'_l coincides with the root system of the symmetric group S_n). Hence n agrees with the length function of $G(r, r, n)$ defined there.

2.4. Let W_I be the reflection subgroup of W generated by $I = \{t^p, s_1, s_2, \dots, s_m\}$ for some $m \leq n$. Then W_I is isomorphic to $G(r, p, m)$. It is clear from the definition that the restriction of n on W_I coincides with the function n_I defined similarly for $G(r, p, m)$. On the other hand, let $J = \{t^p, s_2, \dots, s_n\}$ be a subset of S, and W_J the subgroup of W generated by J. If $d > 1$, then W_J is isomorphic to $G(d, 1, n)$, and J coincides with the standard set of generators of $G(d, 1, n)$. While if $d = 1$, W_J is isomorphic to S_n. Let n_J be the length function of W_J as given in [RS]. In the case where $d > 1$, we denote by $n_{J,l}$ and $n_{J,s}$ the functions associated to long roots and short roots, respectively.

Lemma 2.5. *The restriction of n on W_J coincides with n_J.*

Proof. The case where $d = 1$ is easy. So, we assume that $d > 1$. Let $\Phi_{l,J}$ be the subset of Φ_l consisting of roots of the form $e_i^{(a)} - e_j^{(b)}$ with $p \mid a$, $p \mid b$. Then $\Phi_{l,J}$ is in a natural correspondence, via the map $e_i^{(a)} - e_j^{(b)} \mapsto e_i^{(a')} - e_j^{(b')}$ with $a' = a/p, b' = b/p$, with the set of long roots for $G(d, 1, n)$, where $\Phi_{l,J} \cap \Phi_l^+$ (resp. $\Phi_{l,J} \cap \Phi_l^-$) corresponds to the set of positive (resp. negative) roots, respectively. Similarly, let $\Phi_{s,J}$ be the subset of Φ_s consisting of $e_i^{(a)}$ with $p \mid a$. Then $\Phi_{s,J}$ corresponds naturally to the set of short roots for $G(d,1,n)$, and the restriction of the grading of Φ_s to $\Phi_{s,J}$ coincides with the grading of the set of short roots. Note that the above correspondence is compatible with the actions of W_J. Under this correspondence, the sets Ω_l' and Ω_s are mapped to the sets Ω_l and Ω_s in the root system for $G(d,1,n)$. Since $w(\Omega_s) \subset \Phi_{s,J}$ (resp. $w(\Omega_l') \subset \Phi_{l,J}$) for each $w \in W_J$, we see that the restriction of n_s (resp. n_l') on W_J coincides with $n_{J,s}$ (resp. $n_{J,l}$), respectively. Hence in order to prove the lemma, it suffices to show that $n_l''(w) = 0$, i.e., $w(\Omega_l'') \subset \Phi_l^{++}$ for $w \in W_J$. Take an element $\alpha = e_i^{(a)} - e_j^{(b)} \in w(\Omega_l'')$. Then either $-p/2 < \bar{a} < 0$ and $\bar{b} = -\bar{a}$, or $0 < \bar{b} \le p/2$ and $\bar{a} = -\bar{b} + \delta$. This implies that $\alpha \in \Phi_l^{++}$ and the lemma follows. Q.E.D.

2.6. By applying Lemma 2.5, we can determine the values $n(s)$ for $s \in S$ as follows.

$$
(2.6.1) \qquad n(s) = \begin{cases} 1 & \text{if } s \in \{s_2, \dots, s_n\}, \\ 1 & \text{if } s = t^p \text{ with } d > 1, \\ d & \text{if } s = s_1 \text{ with } p \ge 3 \text{ or } d = 1, \\ 3d - 1 & \text{if } s = s_1 \text{ with } p = 2, d > 1. \end{cases}
$$

In fact, the first two case follow from the lemma. We consider the remaining cases. We have $s_1(\Omega_l') \subset \Phi_l^+$ if $p \ge 3$ or $d = 1$. While if $p = 2$, and $d > 1$, then $s_1(e_2^{(0)} - e_1^{(b)}) < 0$ for $b \equiv 0 \ (p)$. On the other hand, $s_1(e_1^{(0)}) = e_2^{(1)}$ and $s_1(e_2^{(0)}) = e_1^{(-1)}$, and s_1 leaves other short roots fixed. Hence by (2.2.3), $s_1(\Omega_s) \subset \Phi_{s,0}$ if $p \ge 3$. While if $p = 2$, we have $s_1(e_2^{(0)}) \in \Phi_{l,d-1}$, and s_1 maps all other elements in Ω_s to $\Phi_{s,0}$. Moreover we have

$$
\Omega_l'' \cap s_1(\Phi_l^{--}) = \begin{cases} \{e_1^{(mp)} - e_2^{(1)} \mid 0 \le m < d\} & \text{if } p \text{ is even}, \\ \{e_2^{(-f)} - e_1^{(mp+f)} \mid 0 \le m < d\} & \text{if } p \text{ is odd}, \end{cases}
$$

where $p = 2f+1$. This implies that $n_l'(s_1) = 0$, $n_l''(s_1) = d$ and $n_s(s_1) = 0$ if $p \ge 3$ or $d = 1$, and $n_l'(s_1) = d, n_l''(s_1) = d$ and $n_s(s_1) = d - 1$ otherwise. So we have $n(s_1) = d$ or $3d - 1$ and (2.6.1) follows.

Let $\Phi_{l,J}$ be the subset of Φ_l defined in the beginning of the proof of Lemma 2.5. Set $\Phi_{l,J}^+ = \Phi_{l,J} \cap \Phi_l^+$. We define a subset $\widetilde{W}^J$ of $\widetilde{W}$ by

$$
(2.6.2) \qquad \widetilde{W}^J = \{w \in \widetilde{W} \mid w(\Phi_{l,J}^+) \subset \Phi_l^+, w(\Omega_s) \subset \Phi_{s,0}\}.
$$

Then the following lemma holds.

Lemma 2.7. *Let $w \in \widetilde{W}^J, w' \in W_J$. Then we have*

$$\begin{aligned} \tilde{n}'_l(ww') &= \tilde{n}'_l(w'), \\ \tilde{n}''_l(ww') &= \tilde{n}''_l(w), \\ \tilde{n}_s(ww') &= \tilde{n}_s(w'). \end{aligned} \tag{2.7.1}$$

In particular, $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$.

Proof. Since $\Omega'_l \subset \Phi^+_{l,J}$, it follows from (2.6.2) that $\tilde{n}'_l(w) = 0$. (2.6.2) implies also $\tilde{n}_s(w) = 0$. On the other hand, we know that $\tilde{n}''_l(w') = n''_l(w') = 0$ from the proof of Lemma 2.4. Hence the last formula follows from (2.7.1). We show (2.7.1). Since $w(\Phi^-_{l,J}) \subset \Phi^-_l$, $w'(\alpha)$ and $ww'(\alpha)$ have the same sign for $\alpha \in \Omega'_l$. This implies the first assertion of (2.7.1). Let

$$\begin{aligned} \widetilde{\Omega}''_l = \{e_i^{(a)} - e_j^{(b)} \mid -p/2 < \bar{a} < 0,\ \bar{a} + \bar{b} = 0,\ i > j\} \\ \cup\{e_i^{(a)} - e_j^{(b)} \mid 0 < \bar{b} \le p/2,\ \bar{a} + \bar{b} = \delta,\ i < j\}. \end{aligned}$$

Since $w'(\Omega''_l) \subset \Phi^{++}_l$, we see that w' stabilizes $\widetilde{\Omega}''_l$. The second assertion follows from this if we notice that the definition of the sets Φ^{++}_l or Φ^{--}_l depends only on $\bar{a}$ and $\bar{b}$ for $\alpha = e_i^{(a)} - e_j^{(b)}$, and that $\widetilde{\Omega}''_l$ has the same pattern as Ω''_l for the action of w'. The last assertion is also immediate from (2.2.3). This proves the lemma. Q.E.D.

2.8. By modifying the definition in [BM2], we define an element $w(a, m) \in \widetilde{W}$ for $-p/2 < a \le p/2, 1 \le m \le n$ as follows.

$$w(a, m) = \begin{cases} s_m \cdots s_2 t^a & \text{if } 0 < a \le p/2, \\ s_m \cdots s_2 t^a s_2 \cdots s_m & \text{if } -p/2 < a \le 0. \end{cases} \tag{2.8.1}$$

Let us define a subset $\mathcal{N}$ of $\widetilde{W}$ by

$$\mathcal{N} = \{w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \mid -p/2 < a_i \le p/2\}.$$

We set $\mathcal{N}' = \mathcal{N} \cap W$. Then $\mathcal{N}'$ can be written as

(2.8.2)
$$\mathcal{N}' = \{w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \in \mathcal{N} \mid \textstyle\sum a_i \equiv 0 \pmod{p}\}.$$

Also we set $W^J = \widetilde{W}^J \cap W$. We have the following proposition.

Proposition 2.9. *The set $\mathcal{N}$ (resp. $\mathcal{N}'$) coincides with the set $\widetilde{W}^J$ (resp. W^J). Moreover, $\mathcal{N}$ (resp. $\mathcal{N}'$) gives rise to a system of complete representatives of left cosets $\widetilde{W}/W_J$ (resp. W/W_J), respectively.*

Proof. First we show that $\mathcal{N}$ is contained in $\widetilde{W}^J$. Take $\alpha = e_i^{(mp)} - e_j^{(m'p)} \in \Phi_{l,J}$. Then for $w \in \mathcal{N}$, $w(\alpha)$ is expressed as $w(\alpha) = e_k^{(mp+a_k)} - e_l^{(m'p+a_l)}$, where a_k and a_l satisfy the following condition;

$$\begin{aligned}
-p/2 < &a_k \le p/2, & 0 < &a_l \le p/2 & &\text{if } i > j, k < l,\\
-p/2 < &a_k \le 0, & -p/2 < &a_l \le p/2 & &\text{if } i > j, k > l,\\
-p/2 < &a_k \le p/2, & -p/2 < &a_l \le 0 & &\text{if } i < j, k < l,\\
0 < &a_k \le p/2, & -p/2 < &a_l \le p/2 & &\text{if } i < j, k > l.
\end{aligned}$$

Then it is easy to see that $w(\alpha) \in \Phi_l^+$ exactly when $m = 0$ if $i > j$, and when $m' \ne 0$ if $i < j$. But this condition is equivalent to the condition that $\alpha \in \Phi_{l,J}^+$. It follows that $w(\Phi_{l,J}^+) \subset \Phi_l^+$. Next take $e_i^{(0)} \in \Omega_s$. Then we have $w(e_i^{(0)}) = e_j^{(a_j)}$ for some j with $-p/2 < a_j \le p/2$. This implies that $w(\Omega_s) \subset \Phi_{s,0}$. Hence we have $\mathcal{N} \subset W^J$.

Next we note that W^J is a subset of the set of left coset representatives of W by W_J. In fact assume that there exist $w_1, w_2 \in W^J$ such that $w_1 = w_2 x$ with $x \in W_J$. Then by (2.7.1) in the proof of Lemma 2.7, we have $n_l'(w_2 x) = n_l'(x)$ and $n_l'(w_1) = 0$. Hence $n_l'(x) = 0$. Since the restriction of n_l' on W_J is the length function on $W_J = G(d,1,n)$, we have $x = 1$. So $w_1 = w_2$.

It follows from the above remark that $|\widetilde{W}^J| \le |\widetilde{W}/W_J| = p^n$. On the other hand, we have $|\mathcal{N}| = p^n$. (In fact, if $w = w(a_1,1)\cdots w(a_n,n) \in \mathcal{N}$, then there exists $e_i^{(0)}$ such that $w(e_i^{(0)}) = e_n^{(a_n)}$. Hence the elements in $\mathcal{N}$ are parametrized by n-tuples $(a_1,\dots,a_n)$ with $-p/2 < a_i \le p/2$). This shows that $\mathcal{N} = \widetilde{W}^J$ gives a complete set of representatives for $\widetilde{W}/W_J$.

The statement for W follows from this by noticing that $|\mathcal{N}'| = |W/W_J| = p^{n-1}$. Q.E.D.

Remark 2.10. The above proposition shows that any element $w \in \widetilde{W}$ (resp. $w \in W$) can be expressed in a unique way as

$$w = w(a_1,1)w(a_2,2)\cdots w_n(a_n,n)w', \tag{2.10.1}$$

where $w' \in W_J$ (resp. and $\sum_i a_i \equiv 0 \pmod p$). The numbers $a_1,\dots,a_n$ occuring in the decomposition (2.10.1) can be interpreted directly as follows; since $\widetilde{W} \simeq S_n \ltimes (\mathbf{Z}/r\mathbf{Z})^n$, an element w in $\widetilde{W}$ can be written in a form $w = \sigma z$, with $\sigma \in S_n$ and $z \in (\mathbf{Z}/r\mathbf{Z})^n$. Note that z can be written uniquely as $z = (z_1, \dots, z_n)$ with $z_i \in \mathbf{Z}$ such that $-r/2 < z_i \le r/2$ for $i = 1, \dots, n$. Each z_i determines an integer $\bar{z}_i$ such that $-p/2 < \bar{z}_i \le p/2$, and that $\bar{z}_i \equiv z_i \pmod p$ as in 2.2. Under these notations, we have $a_i = \bar{z}_i$ for $i = 1, \dots, n$. See also 3.2 for more details.

We shall compute th values $\tilde{n}''_l(w)$ for $w \in \mathcal{N}$, and $n''_l(w)$ for $w \in \mathcal{N}'$.

Lemma 2.11. *The following formulae hold.*

(i) $$\tilde{n}''_l(w(a,m)) = \begin{cases} d(m-1)(2a-1) & \text{if } 0 < a \le p/2, \\ d(m-1)(-2a) & \text{if } -p/2 < a \le 0. \end{cases}$$

(ii) *For* $w = w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \in \mathcal{N}$ *we have,*

$$\tilde{n}''_l(w) = \sum_{i=1}^{n} \tilde{n}''_l(w(a_i, i)). \tag{2.11.1}$$

Moreover, the function $\tilde{n}''_l$ *coincides with* $\tilde{n}$ *on* $\mathcal{N}$. *In particular, if* $w \in \mathcal{N}'$, *the value* $n(w)$ *is given by the right hand side of (2.11.1).*

Proof. First we show (i). Let $w = w(a,m)$. Assume that $0 < a \le p/2$. Then $w = s_m s_{m-1} \cdots s_2 t^a$. Take $\alpha = e_i^{(b)} - e_j^{(kp-b)} \in \Omega''_l$, where $i > j$ and $-p/2 < b < 0$. Then $w(\alpha)$ becomes positive unless $j = 1, i \le m$. In that case we have $w(\alpha) = e_{i-1}^{(b)} - e_m^{(kp-b+a)}$, and $w(\alpha) < 0$ if and only if $-p/2 < \overline{-b+a} \le 0$. This condition is equivalent to $p/2 < a - b < p$, and we have

$$\begin{aligned} \sharp\{\alpha = e_i^{(b)} - e_j^{(kp-b)} \in \Omega''_l \mid w(\alpha) < 0\} \\ &= \sharp\{\alpha \mid p/2 < a-b < p,\ 0 \le k < d,\ 2 \le i \le m\} \\ &= \begin{cases} d(m-1)(a-1) & \text{if } p \text{ is even}, \\ d(m-1)a & \text{if } p \text{ is odd}. \end{cases} \end{aligned}$$

Next take $\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} \in \Omega''_l$, where $i < j$ and $0 < b \le p/2$. A similar consideration as above shows that $w(\alpha) < 0$ if and only if $i = 1$ and $0 < \overline{a-b+\delta} \le p/2$. Then we have

$$\begin{aligned} \sharp\{\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} \in \Omega''_l \mid w(\alpha) < 0\} \\ &= \sharp\{\alpha \mid 0 < a-b+\delta \le p/2,\ 0 \le k < d,\ 2 \le j \le m\}, \\ &= \begin{cases} d(m-1)a & \text{if } p \text{ is even}, \\ d(m-1)(a-1) & \text{if } p \text{ is odd}. \end{cases} \end{aligned}$$

It follows that $\tilde{n}''_l(w) = d(m-1)(2a-1)$.

Next assume that $-p/2 < a \le 0$. Then $w = s_m \cdots s_2 t^a s_2 \cdots s_m$. First take $\alpha = e_i^{(b)} - e_j^{(kp-b)}$, where $i > j$ and $-p/2 < b < 0$. Then $w(\alpha)$ is positive unless $i = m$. In that case, $w(\alpha) = e_m^{(a+b)} - e_j^{(kp-b)}$ and $w(\alpha) < 0$ if and only if $0 < \overline{a+b} \le p/2$. This implies that $-p < a+b \le -p/2$, and we have

$$\begin{aligned}\sharp\{\alpha = e_i^{(b)} - e_j^{(kp-a)} \in \Omega''_l \mid w(\alpha) < 0\}\\ &= \sharp\{\alpha \mid -p < a+b \le -p/2,\ 0 \le k < d,\ 1 \le j < m\}\\ &= d(m-1)(-a).\end{aligned}$$

Next take $\alpha = e_i^{(kp-b+\delta)} - e_j^{(b)}$, where $i < j$ and $0 < b \le p/2$. Then $w(\alpha)$ is positive unless $j = m$. In that case $w(\alpha) = e_i^{(kp-b+\delta)} - e_m^{(a+b)}$, and $w(\alpha) < 0$ if and only if $-p/2 < \overline{a+b} \le 0$. Hence we have

$$\begin{aligned}\sharp\{\alpha = e_i^{(kp-b+\delta)} - e_j^{(b)} \in \Omega''_l \mid w(\alpha) < 0\}\\ &= \{\alpha \mid -p/2 < a+b \le 0, 0 \le k < d, 1 \le i < m\}\\ &= d(m-1)(-a)\end{aligned}$$

It follows that $\tilde{n}''_l(w) = (m-1)d(-2a)$, and we get (i).

Next we show (ii). Take $\alpha = e_i^{(b)} - e_j^{(mp-b)} \in \Omega''_l$, with $i > j$, and assume that $w(\alpha) < 0$. Now $w(\alpha)$ can be written as $w(\alpha) = e_k^{(b+a_k)} - e_l^{(mp-b+a_l)}$ for some k, l. First consider the case where $k > l$. Let $w' = w(a_{k+1}, k+1) \cdots w(a_n, n)$. Then $w'(\alpha)$ can be written as $w'(\alpha) = e_k^{(b)} - e_{j'}^{(mp-b)}$ for some $j' < k$. It follows that $\beta = w'(\alpha) \in \Omega''_l$ and $w(a_k, k)\beta < 0$. If $k < l$, we consider $w'' = w(a_{l+1}, l+1) \cdots w(a_n, n)$ instead of w'. Then $w''(\alpha)$ can be written as $w''(\alpha) = e_{i'}^{(b)} - e_1^{(mp-b)}$ for some $i' > 1$. Hence $\beta = w''(\alpha) \in \Omega''_l$ and $w(a_l, l)\beta < 0$. Conversely, we take $\beta = e_{i'}^{(b)} - e_{j'}^{(mp-b)} \in \Omega''_l$ with $i' > j'$, and assume that $w(a_k, k)\beta < 0$. Then $i' = k$ or $j' = 1$. If we set $\alpha = w'^{-1}(\beta)$, then we see that $\alpha = e_i^{(b)} - e_j^{(mp-b)} \in \Omega''_l$ with $i > j$, and that $w(\alpha) < 0$.

A similar fact as above also holds for $\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} \in \Omega''_l$. (Here, $\beta = e_{i'}^{(mp-b+\delta)} - e_k^{(b)}$ with $i' < k$, or $\beta = e_1^{(mp-b+\delta)} - e_{j'}^{(b)}$ with $1 < j'$, and so $\beta \in \Omega''_l$). This proves (2.10.1).

Finally, assume that $w \in \mathcal{N}'$. Then $n(w) = \tilde{n}''_l(w)$ by (2.7.1). Hence (2.10.1) gives the value $n(w)$. Q.E.D.

Remark 2.12. If $p \ge 3$ or $d = 1$, then $s_1 = w(-1,1)w(1,2) \in \mathcal{N}'$. While if $p = 2, d \ne 1$, we have $s_1 = ww'$ with $w = w(1,1)w(1,2) \in \mathcal{N}'$

and $w' = s_2 t^{-2} s_2 \in W_J$. Here $n(w) = d$ and $n(w') = n_J(w') = 2d - 1$. So, in this case we have $n(s_1) = 3d - 1$ by Lemma 2.7. This justifies (2.6.1).

2.13. For a complex reflection group G, we denote by $P_G(q)$ the Poincaré polynomial associated to the coinvariant algebra of G. The Poincaré polynomial $P_W(q)$ for $W = G(r, p, n)$ is given as

$$P_W(q) = \prod_{i=1}^{n-1} \frac{q^{ri} - 1}{q - 1} \cdot \frac{q^{dn} - 1}{q - 1}. \tag{2.13.1}$$

Then the following proposition holds.

Proposition 2.14. *We have*

$$\frac{1}{p} \sum_{w \in \widetilde{W}} q^{\tilde{n}(w)} = \sum_{w \in W} q^{n(w)} = P_W(q).$$

Proof. We show the second equality. By Lemma 2.7 and Proposition 2.9, we have

$$\sum_{w \in W} q^{n(w)} = \sum_{w \in \mathcal{N}'} q^{(n(w)} \sum_{w \in W_J} q^{n(w)}. \tag{2.14.1}$$

Now W_J is isomorphic to $G(d, 1, n)$ and the restriction of n on W_J coincides with n_J by Lemma 2.5. Hence by [BM1, Prop. 2.12] we have

$$\sum_{w \in W_J} q^{n(w)} = P_{G(d,1,n)}(q) = \prod_{i=1}^{n} \frac{q^{di} - 1}{q - 1}. \tag{2.14.2}$$

On the other hand, in the expression $w = \sum_i w(a_i, i) \in \mathcal{N}'$, we can choose $a_2, \dots, a_n$ freely, and a_1 is determined uniquely by $a_2, \dots, a_n$. Moreover, we have $\tilde{n}_l''(w(a, 1)) = 0$ by Lemma 2.11. Hence again by using Lemma 2.10, we have

$$\sum_{w \in \mathcal{N}'} q^{n(w)} = \prod_{i=2}^{n} \sum_{k=0}^{p-1} q^{dk(i-1)} = \prod_{i=1}^{n-1} \frac{q^{ri} - 1}{q^{di} - 1}. \tag{2.14.3}$$

Substituting (2.14.2) and (2.14.3) into (2.14.1), we get the second equality. The formula $\frac{1}{p} \sum_{w \in \widetilde{W}} q^{\tilde{n}(w)} = P_W(q)$ can be proved in a similar way if one notices that

$$\sum_{w \in \mathcal{N}} q^{\tilde{n}(w)} = \prod_{i=1}^{n} \sum_{k=0}^{p-1} q^{dk(i-1)}.$$

Q.E.D.

§3. A characterization of the function $\tilde{n}$

3.1. In this section we shall characterize the length function $\tilde{n}$ in terms of a certain length function on $\widetilde{W}$, which is defined independent of the root system. We use the same notation as in Remark 2.10.

Let $\widetilde{W}_0 = G(2,1,n)$ be the Weyl group of type B_n. We define a map $\varphi : \widetilde{W} \to \widetilde{W}_0$ by $\varphi(w) = \sigma(\varepsilon_1, \dots, \varepsilon_n)$, where $w = \sigma(z_1, \dots, z_n)$ is as above, and ε_i is determined by

$$\varepsilon_i = \begin{cases} 1 & \text{if } \bar{z}_i > 0, \\ 0 & \text{if } \bar{z}_i \le 0. \end{cases}$$

(Here we use the same notation for $\widetilde{W}_0$ as the special case $r = 2$ for $G(r,1,n)$). Let us define a length function $\ell_1 : \widetilde{W} \to \mathbf{N}$ as follows. For $w = \sigma z$, we put $\ell_1(w) = \ell_0(\varphi(w))$, where ℓ_0 is the length function on $\widetilde{W}_0$ with respect to the long roots. (More precisely, using the basis $e_1, \dots, e_n$ of V, the set of long roots $\Phi_l \subset V$ associated to $\widetilde{W}_0$ is given as $\Phi_l = \{\pm e_i \pm e_j \mid 1 \le i,j \le n, i \ne j\}$, on which $\widetilde{W}_0$ acts naturally. Now the set Φ_l^+ of positive roots is given as $\Phi_l^+ = \{e_i \pm e_j \mid i > j\}$. For $w' \in \widetilde{W}_0$, we put $\ell_0(w') = |\Phi_l^+ \cap -w'(\Phi_l^+)|$). Next we define a function $\ell_2 : \widetilde{W} \to \mathbf{N}$ by $\ell_2(w) = \sum_{i=1}^n \hat{z}_i$, where

$$\hat{z}_i = \begin{cases} 2z_i - 1 & \text{if } z_i > 0, \\ -2z_i & \text{if } z_i \le 0. \end{cases}$$

Then we define a length function ℓ by $\ell = \ell_1 + \ell_2$. It is clear from the definition that if $r = 2$, ℓ_2 coincides with the length function of W_0 with respect to short roots, and so the function ℓ coincides with the usual length function of the Weyl group of type B_n.

3.2. Let $w = w(a_1, 1) \cdots w(a_n, n)$ be an element in $\mathcal{N}$. The expression $w = \sigma z$ of w as in 3.1 can be described as follows. Let $I = \{1 \le i \le n \mid a_i > 0\}$, and J the complement of I in $\{1, 2, \dots, n\}$. We write $I = \{i_1 > i_2 > \cdots > i_k\}$ with $k = |I|$, and $J = \{j_1 < j_2 < \cdots < j_l\}$ with $l = |J|$. Set

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \dots & n \\ i_1 & i_2 & \cdots & i_k & j_1 & \cdots & j_l \end{pmatrix}. \tag{3.2.1}$$

and

$$z = (a_{i_1}, \dots, a_{i_k}, a_{j_1}, \dots, a_{j_l}) \in \mathbf{Z}_{>0}^k \times \mathbf{Z}_{\le 0}^l. \tag{3.2.2}$$

Then we have $w = \sigma z$. Conversely, any element $w = \sigma z$ with σ, z defined as above in terms of I, J, together with the condition that $-p/2 < a_i \le p/2$, gives an element of $\mathcal{N}$. These facts can be checked by using the induction on n.

Now $\varphi(w) \in \widetilde{W}_0$ can be expressed as a signed permutation,

$$\varphi(w) = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \ldots & n \\ -i_1 & -i_2 & \cdots & -i_k & j_1 & \cdots & j_l \end{pmatrix}. \tag{3.2.3}$$

From this we see that the set $\{\varphi(w) \mid w \in \widetilde{W}\}$ coincides with the set of distinguished representatives for the set $\widetilde{W}_0/S_n$.

We have the following lemma.

Lemma 3.3. *Let $\mathcal{N}$ and W_J be as before. Then for each $w \in \mathcal{N}$, w is the unique minimal length element in the coset wW_J with respect to ℓ. In other words,*

$$\mathcal{N} = \{w \in \widetilde{W} \mid \ell(w) \le \ell(ww') \text{ for any } w' \in W_J\}.$$

Proof. Let $w = \sigma z \in \mathcal{N}$. To prove the lemma, it is enough to show $\ell(w) < \ell(ww')$ for any $w' \in W_J - \{1\}$. Since $w' \in W_J$, one can write $w' = \sigma' z'$ with $\sigma' \in S_n$ and $z' = (z'_1, \ldots, z'_n)$ such that $z'_i \equiv 0 \pmod p$. Here $\sigma' \ne 1$ or $z' \ne 0$. Then $ww' = \sigma\sigma'\sigma'^{-1}(z)z'$, and $\sigma'^{-1}(z)_i = z_{\sigma'(i)}$. Since $z'_i \equiv 0 \pmod p$, we have $\overline{z_{\sigma'(i)} + z'_i} = \bar{z}_{\sigma'(i)}$. Hence $\varphi(ww') = \varphi(w)\sigma'$. But since $\varphi(w)$ is a distinguished representative for the cosets $\widetilde{W}_0/S_n$, we see that $\ell_1(w) < \ell_1(ww')$ if $\sigma' \ne 1$.

Next we show that $\ell_2(w) < \ell_2(ww')$ if $z' \ne 0$. We may assume that $r \ne p$. By our assumption, we have $-p/2 < z_{\sigma'(i)} \le p/2$, and $z'_i \equiv 0 \pmod p$. If $z_{\sigma'(i)}$ and z'_i have the same sign, clearly $|z_{\sigma'(i)} + z'_i| > |z_{\sigma'(i)}|$. (In this case, if $|z_{\sigma'(i)} + z'_i| > r/2$, one has to replace $z_{\sigma'(i)} + z'_i$ by $z_{\sigma'(i)} + z'_i \pm r$. But since $r \ne p$, still the inequality holds). Now assume that $z_{\sigma'(i)}$ and z'_i have the distinct sign. Then we have $|p - z_{\sigma'(i)}| \ge |z_{\sigma'(i)}|$, and the equality holds only when $z_{\sigma'(i)} = p/2$. So the only case we have to care about is the case that $z_{\sigma'(i)} = p/2$ and $z'_i = -p$. But in this case, $(z_{\sigma'(i)} + z'_i)^\wedge = p > \hat{z}_{\sigma'(i)} = p - 1$. This shows that $\ell_2(w) < \ell_2(ww')$ if $z' \ne 0$. Hence we have $\ell(w) < \ell(ww')$ if $w' \ne 1$ as asserted. Q.E.D.

3.4. Let $I = \{t^p, s_1, s_2, \ldots, s_{n-1}\}$ be a subset of S, and we consider the subgroup $\widetilde{W}_I$ of $\widetilde{W}$ generated by I. Hence $\widetilde{W}_I$ is isomorphic to $G(r, p, n-1)$. We set $\mathcal{D} = \{w(a, n) \mid -p/2 < a \le p/2\}$. Then we have the following lemma.

Lemma 3.5. (i) *The set $\mathcal{D}$ is a set of complete representatives of the double cosets $\widetilde{W}_I\backslash\widetilde{W}/W_J$.*

(ii) *For $w = w(a_1,1)\cdots w(a_n,n) \in \mathcal{N}$, we have $\ell(w) = \sum_i \ell(w(a_i,i))$.*

(iii) *The set $\mathcal{D}$ is characterized as the set of elements $w \in \widetilde{W}$ such that w is the unique minimal length element in $\widetilde{W}_I w W_J$ with respect to ℓ.*

Proof. We know already by Remark 2.10 that $\widetilde{W} = \widetilde{W}_I \mathcal{D} W_J$. On the other hand, let $x = w(a,n) \in \mathcal{D}$. Then any element $y \in \widetilde{W}_I x W_J$ has the property that y maps some $e_i^{(0)}$ to $e_n^{(a')}$ with $a' \equiv a \pmod p$. This implies that the double cosets are disjoint for distinct elements in $\mathcal{D}$, and we get (i).

We show (ii). Let $w \in \mathcal{N}$. Then by using (3.2.3), one can check that $\varphi(w) = \varphi(w(a_1,1))\cdots\varphi(w(a_n,n))$, and that $\varphi(w(a_n,n))$ is a distinguished representatives for the cosets $(\widetilde{W}_I)_0\backslash\widetilde{W}_0$. (Here $(\widetilde{W}_I)_0$ denotes the subgroup of $\widetilde{W}_0$ of type B_{n-1} obtained from $\widetilde{W}_I$). Hence the function ℓ_0 is additive with respect to the decomposition of $\varphi(w)$, and so we have $\ell_1(w) = \sum_i \ell(w(a_i,i))$. On the other hand, if we write $w = \sigma z$ as in 3.2, z is given as in (3.2.2). This implies that $\ell_2(w) = \sum_i \ell_2(w(a_i,i))$, and the assertion follows.

Finally we show (iii). Take $x = w(a,n) \in \mathcal{D}$. Then by Remark 2.10, any element $y \in \widetilde{W}_I x W_J$ can be written uniquely as $y = w_1 x w_2$ with $w_1 \in \mathcal{N}_I$ and $w_2 \in W_J$. (Here $\mathcal{N}_I = \mathcal{N} \cap \widetilde{W}_I$). Then by Lemma 3.3, $\ell(w_1 x) \le \ell(w_1 x w_2)$, where the equality holds only when $w_2 = 1$. On the other hand, by (ii), we have $\ell(w_1 x) = \ell(w_1) + \ell(x)$. Hence (iii) holds. Q.E.D.

Remark 3.6. The set $\mathcal{N}$ (resp. $\mathcal{D}$) is also characterized as the set of minimal length elements in each coset in $\widetilde{W}/W_J$ (resp. $\widetilde{W}_I\backslash\widetilde{W}/W_J$) by Proposition 2.9 and Lemma 2.11.

3.7. We now give a characterization of the function $\tilde{n}$ in terms of the function ℓ. In some sense this gives a characterization of the function n on W since $\tilde{n}|_W = n$. Note that by Lemma 3.3 and Lemma 3.5, the sets $\mathcal{N}$ and $\mathcal{D}$ are determined by the function ℓ independently of the choice of the root system.

Theorem 3.8. *Assume that $d \ne 1$. Then the function $\tilde{n} : \widetilde{W} \to \mathbf{N}$ is the unique function satisfying the following properties.*

(i) *The restriction of $\tilde{n}$ on W_J (resp. on $\widetilde{W}_I$) coincides with n_J (resp. $\tilde{n}_I$), where $\tilde{n}_I$ denotes the function on $\widetilde{W}_I = G(r,1,n-1)$ defined in a similar way as $\tilde{n}$ on $\widetilde{W}$.*

(ii) *For $w \in \mathcal{N}$, $w' \in W_J$, we have $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$. For $w \in \mathcal{N}_I$, $w' \in \mathcal{D}$, we have $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$.*

(iii) *Let g be an element in $\widetilde{W}$ which is conjugate to t, with $g \neq t$. Set $\alpha = p/2$ if p is even, and $\alpha = -(p-1)/2$ if p is odd. Then we have*

$$0 < \tilde{n}(g) < \tilde{n}(g^{-1}) < \tilde{n}(g^2) < \tilde{n}(g^{-2}) < \cdots < \tilde{n}(g^{\alpha}),$$

(iv) $\displaystyle \frac{1}{p} \sum_{w \in \widetilde{W}} q^{\tilde{n}(w)} = P_W(q).$

Proof. We have already seen in section 2 that $\tilde{n}$ satisfies the condition (i), (ii) and (iv). We show that $\tilde{n}$ satisfies (iii). Take $g \in \widetilde{W}$ as in (iii). Then g can be written as $g = s_i s_{i-1} \cdots s_2 t s_2 \cdots s_{i-1} s_i$ for some $i \geq 2$. Hence we have

$$g^a = \begin{cases} w(a,i) s_i \cdots s_2 & \text{if } 0 < a \leq p/2, \\ w(a,i) & \text{if } -p/2 < a \leq 0. \end{cases} \tag{3.8.1}$$

Since $s_i \cdots s_2 \in W_J$, the length $\tilde{n}(g^a)$ can be computed by Lemma 2.7 and Lemma 2.11, as follows.

$$\tilde{n}(g^a) = \begin{cases} (i-1)\{d(2a-1)+1\} & \text{if } 0 < a \leq p/2, \\ (i-1)(-2ad) & \text{if } -p/2 < a \leq 0. \end{cases}$$

Since $d \neq 1$, the condition (iii) is verified by using the above formula.

Next we show the uniqueness of $\tilde{n}$. If $n = 1$, $\widetilde{W}$ is the cyclic group generated by t and W_J is the subgroup of $\widetilde{W}$ generated by t^p. Hence it is determined by the conditions (i) and (ii). So we assume that $n > 1$. By (i) and (ii), it is enough to see that $\tilde{n}(w)$ is determined uniquely for $w \in \mathcal{D}$. Let $w = w(a,n) \in \mathcal{D}$ and set $c(a) = \tilde{n}(w)/(n-1)$. Then by (iv), we have

$$\{c(a) \mid -p/2 < a \leq p/2\} = \{0, d, 2d, \ldots, (p-1)d\}. \tag{3.8.2}$$

Since $|\mathcal{D}| = p$, $c(a)$ are all distinct. On the other hand, let $g = s_n \cdots s_2 t s_2 \cdots s_n$. Then by (3.8.1) and (ii), we have

$$\tilde{n}(g^a) = \begin{cases} (n-1)(c(a)+1) & \text{if } 0 < a \leq p/2, \\ (n-1)c(a) & \text{if } -p/2 < a \leq 0. \end{cases}$$

Hence by using (iii), we have

$$c(i) + 1 < c(-i) < c(i+1) + 1$$

for $i = 1, 2, \ldots$. Since $c(a) \equiv 0 \pmod d$, and $d \neq 1$, we have $c(i) < c(-i) < c(i+1)$. It follows, by (3.8.2), that we have

$$c(a) = \begin{cases} (2a-1)d & \text{if } a > 0, \\ (-2a)d & \text{if } a \leq 0. \end{cases}$$

The function $\tilde{n}$ is now determined on $\mathcal{D}$, and so the theorem follows.
Q.E.D.

Remark 3.9. In the case where $d = 1$, the property (iii) in the theorem does not hold. Instead, we have the following relation.

(iii') $\qquad 0 < \tilde{n}(g) = \tilde{n}(g^{-1}) < \tilde{n}(g^2) = \tilde{n}(g^{-2}) < \cdots \leq \tilde{n}(g^\alpha).$

Then the function $\tilde{n}$ is characterized by the properties (i) $\sim$ (iv), but replacing (iii) by (iii'). In fact, by a similar argument as above, we have

$$c(i) + 1 = c(-i) < c(i+1) + 1$$

for $i = 1, 2, \ldots$. Thus $c(i)$ is the smallest integer among all the $c(a)$ such that $|a| \geq i$. Since the set $\{c(a) \mid -p/2 < a \leq p/2\}$ coincides with the set $\{0, 1, \ldots, p-1\}$, this determines $c(i)$ and so $c(-i)$ successively for $i = 1, 2, \ldots$. Hence the function $\tilde{n}$ is determined uniquely.

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