Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 386–405

# Prolongation Projection Commutativity Theorem

**Jose M. Veloso**

**Abstract.**

If the symbol $g_k$ of a SPDE $R_k$ is 2-acyclic, then the operations of prolongation and projection on $R_k$ commute

$$\rho_{k+l+1}^{k+l+2}((R_k)_{+l+2}) = \left(\rho_{k+l}^{k+l+1}((R_k)_{+l+1})\right)_{+1}.$$

We apply this to study contact of three-dimensional CR-manifolds.

## §1. Introduction

S. Chern and J. Moser [2] proved that two real hypersurfaces of $\mathbf{C}^2$ have a contact of fifth order and in the non-umbilic case of sixth order. The $G$-structure associated to a real hypersurface is of order two but their definition involves fifth order derivatives. Studying these facts through the SPDE of jets of biholomorphic functions between the real hypersurfaces, we found the following theorem:

**Theorem 1.1** (Prolongation projection commutativity theorem). *Let $R_k \subset J_k(M, N, \rho)$ be a system of partial differential equations such that*

*(i)* $\alpha : R_k \to N$ *is a submersion*
*(ii) the symbol* $g_k$ *of* $R_k$ *is 2-acyclic*
*(iii)* $g_{k+1}$ *is a vector bundle on* $(\rho_k^{k+1})^{-1}(R_k)$

*Then, for every* $l \geq 0$,

$$\rho_{k+l+1}^{k+l+2}((R_k)_{+l+2}) = \left(\rho_{k+l}^{k+l+1}((R_k)_{+l+1})\right)_{+1}.$$

**Theorem 1.2** (Formal integrability theorem [4]). *Under the hypothesis of the above theorem and the assumption that*

$$\rho_k^{k+1}((R_k)_{+1}) = R_k,$$

Received October 2, 1995.

*we get that*

$$\rho_{k+l}^{k+l+1} : (R_k)_{+l+1} \to (R_k)_{+l}$$

*is a submersion for every* $l \geq 1$.

The formal integrability theorem for linear PDE systems was first proved by Quillen [6] and with weak assumptions by Goldschmidt [3], who also proved it in the non-linear case [4]. A version of this theorem using involutivity is in Kuranishi [5]. All these publications used the set $R_k$ of integral jets of the PDE system to prove the theorem. Ruiz [7, 8, 9] utilizes the sheaf $I_k$ of functions which are null on $R_k$; this approach seems to us more natural and we follow this approach.

In Section 2 we present the basic facts following [7, 8, 9]. Section 3 contains the proof of Theorem 1.1. In Section 4 we apply the theorem to study contact of three-dimensional CR-manifolds. Corollary 4.1 shows that the $G$-structure associated to a CR-manifold $M$ is the projection in order two of fifth order jets which have fifth-order contact with the hyperquadric $\mathrm{Im} w = z\bar{z}$. Theorem4.3 relates the normal form of $M$ [2] with the invariants of Cartan [1].

I'd like to thank to A.M.Rodrigues and J.Verderesi for several discussions on this subject. I dedicate this paper to prof.M.Kuranishi, in occasion of his seventieth birthday. He was very kind to me and my contacts with him were always stimulating.

## §2. Basic definitions

Let $M, N$ be manifolds, $T = TN$ the tangent bundle of $N$, $\rho : M \to N$ a submersion, and $J_k = J_k(M, N, \rho)$ the manifold of $k$-jets of local sections of $\rho : M \to N$. Denoting by $\rho_l^k : J_k \to J_l$, $k > l$, the canonical projections, and by $\rho_0^k = \beta_k : J_k \to M$ and $\rho_{-1}^k = \alpha_k : J_k \to N$ the projections to *target* and *source* respectively, the sheaf of algebras of $C^\infty$-functions on $J_k$ will be denoted by $F_k$. If $Z_k \in J_k$, let be $Z_l = \rho_l^k(Z_k)$, for $l \leq k$. In particular, $\beta_k(Z_k) = Z$ and $\alpha_k(Z_k) = z$.

We identify $Z_k$ with the linear application (cf. [7])

$$Z_k = (Z_k)_* : T_z N \to T_{Z_{k-1}} J_{k-1}$$

given by

$$(Z_k)_* = (j^{k-1}\sigma)_* v$$

where $Z_k = j_z^k \sigma$.

If $\theta$ is a vector field on $N$, we define the *formal derivative*

$$\partial_\theta : F_k \to F_{k+1}$$

by

$$(\partial_\theta f)(Z_{k+1}) = df(Z_{k+1})_*(\theta_z))$$

where $f \in F_k$. This derivative has the properties
(i) $\partial_{a.\theta} f = a.\partial_\theta f$
(ii) $\partial_{[\theta,\eta]} = [\partial_\theta, \partial_\eta]$
where $a$ is a real function on $N$, and $\eta$ is a vector field on $N$. Let $x = (x^1, \cdots, x^n)$ be a chart on $U \subset N$, $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^m)$ a chart on $\rho^{-1}(U)$, and $(x, y^j_\alpha, 0 \leq j \leq m, 0 \leq \mid \alpha \mid \leq k)$ a chart on $(\rho^k_0)^{-1}(U)$, where

$$y^j_\alpha(j^k_z \sigma) = \frac{\partial^{|\alpha|}\sigma^j}{\partial x^\alpha}(z)$$

and $\sigma = (\sigma^1, \cdots, \sigma^m)$ is a section of $\rho$ on $U$.

In this coordinate system

$$(Z_{k+1})_*(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} + \sum_{|\alpha| \leq k} y^j_{\alpha+1_i}(Z_{k+1}) \frac{\partial}{\partial y^j_\alpha}$$

and

$$\partial_i f = \frac{\partial}{\partial x^i} + \sum_{|\alpha| \leq k} \frac{\partial f}{\partial y^j_\alpha} y^j_{\alpha+1_i}$$

where $f \in F_k$, and $\partial_i$ denotes $\partial_\theta$ when $\theta = \partial/\partial x^i$.

Let $Q_k = \mathrm{Ker}(\rho^k_{k-1})_*$ be the vector bundle on $J_k$ of vertical tangent vectors with respect to $\rho^k_{k-1}$. The fiber of $Q_k$ on $Z_k$ is denoted by $Q_{Z_k}$. The dual bundle of $Q_k$ is denoted by $Q^*_k$. If $f \in F_k$, then $d(\partial_\theta f) \mid_{Q_{Z_{k+1}}}$ depends only on $df \mid_{Q_{Z_k}}$ and $\theta(z)$. So we have a map

$$d_K : T_z N \otimes Q^*_{Z_k} \to Q^*_{Z_{k+1}}$$

defined by

$$d_K(\theta_z \otimes df \mid_{Q_{Z_k}}) = d(\partial_\theta f) \mid_{Q_{Z_{k+1}}} .$$

In coordinates

$$d_K(\frac{\partial}{\partial x^i} \otimes dy^j_\alpha \mid_{Q_{Z_k}}) = dy^j_{\alpha+1_i} \mid_{Q_{Z_{k+1}}} .$$

If $Q^*_{Z_\infty} = \sum_{k\geq 0} Q^*_{Z_k}$, we define *Koszul's complex* $(\Lambda T_z \otimes Q^*_{Z_\infty}, d_K)$ by

$$d_K(v_1 \wedge \cdots \wedge v_l \otimes \mu) = \sum_{i=1}^{l} (-1)^{i+1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_l \otimes d_K(v_i \otimes \mu).$$

**Definition 2.1.** *A system of partial differential equations (SPDE) is a subsheaf of ideals $I_k$ of $F_k$ locally finitely generated. The subset $R_k$ of $J_k$,*

$$R_k = \{Z_k \in J_k : f(Z_k) = 0, \forall f \in I_k\}$$

*is the set of integral jets of $I_k$. In case $(R_k, N, \alpha_k)$ is a submersion, $I_k$ (or $R_k$) is said to be regular. The subsheaf of ideals of $F_{k+1}$ generated by*

$$(\rho_k^{k+1})^* I_k \cup \{\partial_\theta f : f \in I_k, \theta \in \Gamma(TN)\}$$

*is called the prolongation $(I_k)_{+1}$ of $I_k$.*

We shall write $I_{k+1}$ instead of $(I_k)_{+1}$. The subsheafs $I_{k+l}, l \geq 2$ are defined inductively. Suppose $(x^1, \cdots, x^n)$ is a chart on $N$, and $f_p, 1 \leq p \leq r$ a system of (local) generators of $I_k$, then a system of (local) generators of $I_{k+l}$ is given by $\{\partial_\alpha f_p : 1 \leq p \leq r, 0 \leq |\alpha| \leq l\}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\partial_\alpha f_p = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_p$. We will assume that $F_k$ is contained in $F_{k+l}$, through the inclusion $(\rho_k^{k+l})^* : F_k \to F_{k+l}$.

**Definition 2.2.** *The symbol $h_{Z_k}$ at the integral jet $Z_k$ of $I_k$ is the subset of $Q^*_{Z_k}$ defined by*

$$h_{Z_k} = \{df \mid_{Q_{Z_k}} : f \in I_k\}$$

*The family of symbols on $R_k$ is denoted by $h_k$, i.e. $(h_k)_{Z_k} = h_{Z_k}$.*

If $Z_{k+1} \in (\rho_k^{k+1})^{-1}(Z_k)$, with $Z_k \in R_k$, put $h_{Z_{k+1}} = d_K(T_z \otimes h_{Z_k})$, and $h_{Z_{k+l+1}} = d_K(T_z \otimes h_{Z_{k+l}}), l \geq 1$ for every $Z_{k+l+1} \in (\rho_k^{k+l+1})^{-1}(Z_k)$. Also, we put

$$h_{k+l} = \{h_{Z_{k+l}} : Z_{k+l} \in (\rho_k^{k+l})^{-1}(R_k)\}.$$

In case $Z_{k+l} \in R_{k+l}, h_{Z_{k+l}}$ coincides with the symbol of $I_{k+l}$ at $Z_{k+l}$, i.e. $h_{Z_{k+l}} = dI_{k+l} \mid_{Q_{Z_{k+l}}}$. Let us put $h_{Z_\infty} = \sum_{l\geq 0} h_{Z_{k+l}}$. Then $h_{Z_\infty} \subset Q^*_{Z_\infty}$, and from $d_K(T_z \otimes h_{Z_\infty}) \subset h_{Z_\infty}$ it follows that $(\Lambda T_z \otimes h_{Z_\infty}, d_K)$ is a subcomplex of Koszul's complex. The *(j,k+l+1)-th homology group* of this subcomplex is

$$H_{(j,k+l+1)}(Z_k) = \frac{\ker(d_K : \Lambda^j T_z \otimes h_{Z_{k+l+1}} \to \Lambda^{j-1} T_z \otimes h_{Z_{k+l+2}})}{d_K(\Lambda^{j+1} T_z \otimes h_{Z_{k+l}})}$$

for $l \geq 0$.We say that $h_{Z_k}$ is *r-acyclic* if $H_{j,k+l+1}(Z_k) = 0$ , for $0 \leq j \leq r$, $l \geq 0$ and $h_k$ is *r-acyclic* if $h_{Z_k}$ is r-acyclic for every $Z_k \in R_k$. Clearly , $h_k$ is 0-acyclic. If $g_{Z_k} \subset Q_{Z_k}$ is defined by $g_{Z_k}^{\perp} = h_{Z_k}$, then $g_{Z_k}$ is also called the *symbol* of $R_k$ at $Z_k$. It is proved in [9] that $h_{Z_k}$ is 1-acyclic if and only if $g_{Z_k}$ is 2-acyclic in the sense of [4].

## §3. The prolongation projection commutativity theorem

Let us put

$$I_{k+l}^{k+l+1} = \{f \in F_{k+l} : (\rho_{k+l}^{k+l+1})^* f \in I_{k+l+1}\}$$

for $l \geq 0$. It is clear that $I_{k+l} \subset I_{k+l}^{k+l+1}$. If $R_{k+l}^{k+l+1}$ denotes the set of integral jets of $I_{k+l}^{k+l+1}$, then $\rho_{k+l}^{k+l+1}(R_{k+l+1}) \subset R_{k+l}^{k+l+1}$. In general the equality doesn't hold. The following proposition gives a condition for this.

**Proposition 3.1.** *If $I_k$ is a regular SPDE, and $h_{k+l+1}$ is a vector bundle on $(\rho_k^{k+l+1})^{-1}(R_k)$,then*

$$R_{k+l}^{k+l+1} = \rho_{k+l}^{k+l+1}(R_{k+l+1}).$$

*Furthermore, if $f_p, 1 \leq p \leq r$ are local independent generators of $I_k$, then $I_{k+l}^{k+l+1}$ is generated by*

$$\{\partial_\alpha f_p, 1 \leq p \leq r, \mid \alpha \mid \leq l; g_t, 1 \leq t \leq s\},$$

*where $g_t = \sum_{p=1}^{r}(\sum_{|\beta|=l+1} a_t^{\beta,p} \partial_\beta f_p + b^p f_p)$ with $a_t^{\beta,p} \in F_k, b^p \in F_{k+l+1}$.*

**Proof:** Let $U_k$ be an open set in $J_k$, where $f_p, 1 \leq p \leq r$ are defined, and $U_{k+j} = (\rho_k^{k+j})^{-1}(U_k), j \geq 1$. By hypothesis, $df_1, \cdots, df_r$ are linearly independents at every $Z_k \in U_k$, then

$$U_k \cap R_k = \{Z_k \in J_k : f_p(Z_k) = 0, 1 \leq p \leq r\}.$$

Let us put $V_{k+j} = U_{k+j} \cap (\rho_k^{k+j})^{-1}(R_k), j \geq 0$. Since $h_{k+l+1}$ is a vector bundle on $V_{k+l+1}$, for every $p$,$1 \leq p \leq r$, there exist $\Lambda_p \subset \{\alpha \in \mathbf{N}^n :$

$|\alpha| = l+1\}$ such that $\{d(\partial_{\alpha_p} f_p)\mid_{Q_{Z_{k+l+1}}}, 1 \leq p \leq r, \alpha_p \in \Lambda_p\}$ is a basis of $h_{Z_{k+l+1}}, Z_{k+l+1} \in V_{k+l+1}$ (eventually shrinking $U_k$ ). Then, given $q$ and $\alpha$ , $1 \leq q \leq r, \mid \alpha \mid= l+1$, there exist functions $A^{p,\alpha_p}_{q,\alpha}$ on $V_k$, such that

$$d(\partial_\alpha f_q)\mid_{Q_{Z_{k+l+1}}} + \sum_{p=1}^{r} \sum_{\alpha_p \in \Lambda_p} A^{p,\alpha_p}_{q,\alpha}(Z_k) d(\partial_{\alpha_p} f_p)\mid_{Q_{Z_{k+l+1}}} = 0$$

for every $Z_{k+l+1} \in V_{k+l+1}$. This is so since $d(\partial_\alpha f_p)\mid_{Q_{Z_{k+l+1}}}$ depends only on $Z_k$. Let $a^{p,\alpha_p}_{q,\alpha}$ be extensions to $U_k$ of functions $A^{p,\alpha_p}_{q,\alpha}$. Then $\partial_\alpha f_p + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p$ are constant on the fibers of the submersion $\rho^{k+l+1}_{k+l}\mid_{V_{k+l+1}}: V_{k+l+1} \to V_{k+l}$. This implies that, given $q$ and $\alpha$,$1 \leq q \leq r, |\alpha| = l+1$, there are functions $g_{q,\alpha}$ on $U_{k+l}$ such that $\partial_\alpha f_q + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p - g_{q,\alpha}$ are identically zero on $V_{k+l+1}$, which is the zero set of functions $(\rho^{k+l+1}_{k+l})^* f_p$,$1 \leq p \leq r$. By implicit function theorem there are functions $b^p_{q,\alpha} \in U_{k+l+1}$ such that

$$\partial_\alpha f_q + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p - g_{q,\alpha} + \sum b^p_{q,\alpha} f_p = 0$$

in $U_{k+l+1}$.Then $I_{k+l+1}$ is generated by

$$\{\partial_\beta f_p, \mid \beta \mid\leq l; g_{p,\alpha}, \mid \alpha \mid= l+1; \partial_{\alpha_p} f_p, \alpha_p \in \Lambda_p; 1 \leq p \leq r\}$$

where $g_{p,\alpha} \in F_{k+l}$. So $I^{k+l+1}_{k+l}$ is generated by

$$\{\partial_\beta f_p, |\beta| \leq l; g_{p,\alpha}, |\alpha| = l+1; 1 \leq p \leq r\}$$

which proves the second part of Proposition. Since $\partial_{\alpha_p} f_p$ are independents, given $Z_{k+l} \in R^{k+l+1}_{k+l} \cap U_{k+l}$, there exist $Z_{k+l+1} \in U_{k+l+1}$ such that $\partial_{\alpha_p} f_p(Z_{k+l+1}) = 0$. This implies $Z_{k+l+1} \in R_{k+l+1}$, so $R^{k+l+1}_{k+l} \subset \rho^{k+l+1}_{k+l}(R_{k+l+1})$, which completes the proof.

$R_{k+l+1}$ is not necessarily a manifold, nor $R^{k+l+1}_{k+l}$. To guarantee this, we need the following Proposition, which is dual of a result in [3].

**Proposition 3.2.** *If $I_k$ is a regular SPDE such that*
*(i) $h_k$ is 1-acyclic;*
*(ii) $h_{k+1}$ is a vector bundle on $(\rho^{k+1}_k)^{-1}(R_k)$*
*then $h_{k+l+1}$ is a vector bundle on $(\rho^{k+l+1}_k)^{-1}(R_k)$ for every $l \geq 0$.*

**Proof:** By induction on $l$, suppose $h_{k+l+1}$ is a vector bundle . For every $Z_{k+l+2} \in (\rho^{k+l+2}_k)^{-1}(R^k)$ the sequence

$$\Lambda^2 T_z \otimes h_{Z_{k+l}} \overset{d_K}{\to} T_z \otimes h_{Z_{k+l+1}} \overset{d_K}{\to} h_{Z_{k+l+2}}$$

is exact by (i), then $\dim(T_z \otimes h_{Z_{k+l+1}}) = \dim h_{Z_{k+l+2}} + \dim(d_K(\Lambda^2 T_z \otimes h_{Z_{k+l}}))$. If $I_k$ is generated by $f_1, \cdots, f_r$, then $h_{k+l}$ is generated by the restrictions to $(\rho_k^{k+l})^{-1}(R_k)$ of $d(\partial_\alpha f_p) \mid_{Q_{k+l}}, 1 \leq p \leq r, |\alpha| = l$, and similarly, $d_K(\Lambda^2 T \otimes h_{k+l})$ and $h_{k+l+2}$ are generated by a finite number of $C^\infty$-sections. Since the rank of a linear system with variables coefficients is a lower semicontinuous function, $\dim h_{k+l+2}$ and $\dim(\Lambda^2 T \otimes h_{k+l})$ are lower semicontinuous functions, so by induction hypothesis and the above equality, it follows $\dim h_{k+l+2}$ and $\dim d_K(\Lambda^2 T \otimes h_{k+l})$ are constant functions, which proves $h_{k+l+2}$ is a vector bundle.

**Theorem 3.1** (Prolongation projection commutativity theorem). *If $I_k$ is a SPDE such that*

*(i) $h_k$ is 1-acyclic;*

*(ii) $h_{k+1}$ is a vector subbundle on $(\rho_k^{k+1})^{-1}(R_k)$;*

*then*

$$(I_{k+l}^{k+l+1})_{+1} = I_{k+l+1}^{k+l+2}$$

*or equivalently*

$$(R_{k+l}^{k+l+1})_{+1} = R_{k+l+1}^{k+l+2}$$

*for all $l \geq 0$.*

**Proof:** Let $f_p, 1 \leq p \leq r$ be a set of independent generators of $I_k$. It follows from Proposition3.2 that $h_{k+l+1}$ is a vector bundle for every $l \geq 0$, and applying Proposition3.1, $I_{k+l+1}^{k+l+2}$ is generated by $\partial_\beta f_p, 1 \leq p \leq r, \mid \beta \mid = l+1$, and functions

$$g_t = \sum_{p=1}^{r} \sum_{i,j=1}^{r} \sum_{|\alpha|=l} a_{i,j,t}^{\alpha,p} \partial_i \partial_j \partial_\alpha f_p + b_t^p f_p,$$

where $a_{i,j,t}^{\alpha,p} \in F_k, b_t^p \in F_{k+l+2}, a_{i,j,t}^{\alpha,p} = a_{j,i,t}^{\alpha,p}$ and $1 \leq t \leq s$. To show $I_{k+l+1}^{k+l+2} \subset (I_{k+l}^{k+l+1})_{+1}$, we must prove that $g_t \in (I_{k+l}^{k+l+1})_{+1}$, for every $1 \leq t \leq s$. If $Z_{k+l+2} \in (\rho_k^{k+l+2})^{-1}(R_k)$, then

$$0 = dg_t \mid_{Q_{Z_{k+l+2}}} = \sum a_{i,j,t}^{\alpha,p} d(\partial_i \partial_j \partial_\alpha f_p) \mid_{Q_{Z_{k+l+2}}} \tag{3.1}$$

by $f_p(Z_{k+l+2}) = 0$, $1 \leq p \leq r$. Put

$$(w_{p,\alpha})_{Z_{k+l}} = d(\partial_\alpha f_p) \mid_{Q_{Z_{k+l}}},$$

$$(w_{p,\alpha,j})_{Z_{k+l+1}} = d(\partial_j \partial_\alpha f_p) \mid_{Q_{Z_{k+l+1}}}$$

and

$$(w_{p,\alpha,j,i})_{Z_{k+l+2}} = d(\partial_i \partial_j \partial_\alpha f_p) \mid_{Q_{Z_{k+l+2}}} .$$

Then (3.1) can be written as $\sum a_{i,j,t}^{\alpha,p} w_{p,\alpha,j,i} = 0$ on $(\rho_k^{k+l+2})^{-1}(R_k)$, which is equivalent to

$$d_K(\sum a_{i,j,t}^{\alpha,p} \frac{\partial}{\partial x^i} \otimes w_{p,\alpha,j}) = 0.$$

From $(i)$ there exist functions $B_{i,j,t}^{\alpha,p}$ on $R_k$, with $B_{i,j,t}^{\alpha,p} = -B_{j,i,t}^{\alpha,p}$, such that

$$d_K(\frac{1}{2} \sum B_{i,j,t}^{\alpha,p} \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^i} \otimes w_{p,\alpha}) = \sum \frac{\partial}{\partial x^i} \otimes a_{i,j,t}^{\alpha,p} w_{p,\alpha,j}.$$

Then $\sum \frac{\partial}{\partial x^i} \otimes (B_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) w_{p,\alpha,j} = 0$, so $\sum (B_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) w_{p,\alpha,j} = 0$. Let be $b_{i,j,t}^{\alpha,p}$ extensions of $B_{i,j,t}^{\alpha,p}$ to $U_k$ so that

$$b_{i,j,t}^{\alpha,p} = -b_{j,i,t}^{\alpha,p}. \tag{3.2}$$

Then

$$\sum_{j,\alpha,p} (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) d(\partial_j \partial_\alpha f_p) \mid_{Q_{k+l+1}} = 0$$

on $(\rho_k^{k+l+1})^{-1}(R^k)$. This means $\sum_{j,\alpha,p} (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) \partial_j \partial_\alpha f_p$ is constant on the fibers of $(\rho_{k+l}^{k+l+1})^{-1}(R_k)$ over $(\rho_k^{k+l})^{-1}(R_k)$, so there exist functions $H_{i,t} \in F_{k+l}$ such that

$$\sum_{j,\alpha,p} (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) \partial_j \partial_\alpha f_p - H_{i,t} = 0$$

on $(\rho_k^{k+l+1})^{-1}(R_k)$. This set is the null set of $f_1, \cdots, f_r$, then there exist functions $c_{i,t}^p \in F_{k+l+1}$ which satisfies

$$\sum_{j,\alpha,p} (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) \partial_j \partial_\alpha f_p - H_{i,t} = \sum_p c_{i,t}^p f_p.$$

It follows that $H_{i,t} \in I_{k+l}^{k+l+1}$ and

$$\begin{aligned} \sum_i \partial_i H_{i,t} &= \sum_{i,j,\alpha,p} (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) \partial_i \partial_j \partial_\alpha f_p \\ &+ \sum_{i,j,\alpha,p} \partial_i (b_{i,j,t}^{\alpha,p} - a_{i,j,t}^{\alpha,p}) \partial_j \partial_\alpha f_p - \sum_{i,p} (c_{i,t}^p \partial_i f_p - \partial_i (c_{i,t}^p) f_p). \end{aligned}$$

From (3.2), $\sum_{i,j} b_{i,j,t}^{\alpha,p}\partial_i\partial_j\partial_\alpha f_p = 0$, so

$$\sum_i \partial_i H_{i,t} + g_t = 0, \mathrm{mod} I_{k+l+1}.F_{k+l+2}.$$

But the left side is in $F_{k+l+1}$ so $\sum_i \partial_i H_{i,t} + g_t \in I_{k+l+1}$, and consequently $g_t \in (I_{k+l}^{k+l+1})_{+1}, 1 \leq t \leq s$, which completes the proof.

**Corollary 3.1.** *Under the hypothesis of the preceding Theorem and $I_k^{k+1} = I_k$ we have $I_{k+l}^{k+l+1} = I_{k+l}$, for all $l \geq 0$.*

**Proof of Theorem1.2:** From Theorem 1.1(i) we have $I_k = \{f \in F_k : f(R_k) = 0\}$ is regular, from (ii) $h_k = g_k^\perp$ is 1-acyclic [9], from (iii) $h_{k+1}$ is a vector bundle and $I_k^{k+1} = I_k$. Applying Corollary3.1, $R_{k+l}^{k+l+1} = R_{k+l}, l \geq 0$, and from Proposition 3.1 we get $\rho_{k+l}^{k+l+1} : R_{k+l+1} \to R_{k+l}$ is onto, for every $l \geq 0$. The $g_{k+l+1}$ are vector bundles, so these projections are submersions.

## §4. Contact of hypersurfaces of $\mathbf{C}^2$

A three dimensional manifold $M$ with a codimension one distribution $\Delta \subset TM$, an operator $J$ on $\Delta$ such that $J^2 = -I$, and an one form $\theta$ such that $\theta^\perp = \Delta$ and $\theta \wedge d\theta \neq 0$ is a *Cauchy-Riemann manifold.* A real hypersurface of $\mathbf{C}^2$ has a natural structure of CR-manifold with $\Delta = TM \cap J(TM)$. From now on, $M$ and $M'$ will denote CR-manifolds. A diffeomorphism $f : M \to M'$ is a *CR-diffeomorphism* if $f_*(\Delta) = \Delta'$, and $f_*(J) = J'$. If $\Delta_{\mathbf{C}}$ is the complexification of $\Delta$, then $\Delta_{\mathbf{C}} = \Delta^{1,0} \oplus \Delta^{0,1}$, and f is a CR-diffeomorphism if and only if

$$f_*(\Delta^{1,0}) = (\Delta')^{1,0}. \tag{4.3}$$

Let $U$ be an open set of $M$, $Z_1$ a no null section of $\Delta^{1,0}\,|_U$, $Z_{\bar{1}} = \overline{Z_1}$, and

$$Z_0 = -i[Z_1, Z_{\bar{1}}]. \tag{4.4}$$

Then $Z_0, Z_1, Z_{\bar{1}}$ is a basis of $T_{\mathbf{C}}M\,|_U$ . If $h$ is a complex valued function on $U$, we will write $h_i = Z_i(h), i = 0, 1, \bar{1}$. Let $a, b, c$ be the complex valued functions defined by

$$[Z_1, Z_0] = aZ_1 + \bar{b}Z_{\bar{1}} + cZ_0, \tag{4.5}$$

which satisfy, as a consequence of Jacobi's identity

$$b_1 - a_{\bar{1}} + a\bar{c} - bc = 0$$

$$\bar{c}_1 - c_{\bar{1}} + i(a + \bar{a}) = 0.$$

Let $U'$ be an open set of $M'$, $Z'_i, i = 0, 1, \bar{1}$ as above with the corresponding functions $a', b', c'$. We denote by $D_k$ the open set of $J_k = J_k(M \times M', M, \pi_1)$ corresponding to $k$-jets of local diffeomorphisms of $M$ in $M'$, where $\pi_1$ is the canonical projection of $M \times M'$ on $M$. Put $D_k(U, U') = (\beta_k)^{-1}(U \times U') \subset D_k$. On $D_1(U, U')$ we introduce the coordinates system

$$p^i_j : D_1(U, U') \to \mathbf{C}, i, j = 0, 1, \bar{1},$$

defined by

$$f_*(Z_j(x)) = \sum_{i=0,1,\bar{1}} p^i_j(j^1_x f) Z'_i(f(x)).$$

These coordinates are not independent, and satisfy the relations

$$\overline{p^i_j} = p^{\bar{i}}_{\bar{j}}, i, j = 0, 1, \bar{1},$$

where $\bar{0} = 0, \bar{\bar{1}} = 1$ by convention. The coordinates on $D_2(U, U')$ are defined by

$$p^i_{jk}(j^2_x f) = Z_j(p^i_k(j^1 f))(x), i, j, k = 0, 1, \bar{1}.$$

Again $\overline{p^i_{jk}} = p^{\bar{i}}_{\bar{j}\bar{k}}$. If $[Z_i, Z_j] = \sum a^k_{ij} Z_k$, it follows from $f_*[Z_i, Z_j] = [f_* Z_i, f_* Z_j]$ that

$$\sum_k a^k_{ij}(x) p^m_k(j^1_x f) = p^m_{ij}(j^2_x f) - p^m_{ji}(j^2_x f) + \sum_{r,s} p^r_i(j^1_x f) p^s_j(j^1_x f) a'^m_{rs}(f(x)).$$

For instance

(4.6)
$$p^0_{1\bar{1}} - p^0_{\bar{1}1} = i(p^0_0 - p^1_1 p^{\bar{1}}_{\bar{1}} + p^1_{\bar{1}} p^{\bar{1}}_1) + c'(p^1_1 p^0_{\bar{1}} - p^0_1 p^1_{\bar{1}}) + \bar{c}'(p^{\bar{1}}_1 p^0_{\bar{1}} - p^{\bar{1}}_{\bar{1}} p^0_1).$$

Coordinates in $D_3(U, U')$ are defined by

$$p^i_{mjk}(j^3_x f) = Z_m(p^i_{jk}(j^2 f))(x), i, j, k = 0, 1, \bar{1},$$

and successively.

Equation (4.3) in coordinates is

$$f_* Z_1 = p^1_1(j^1 f) Z'_1$$

or

$$p^{\bar{1}}_1(j^1 f) = p^0_1(j^1 f) = 0.$$

Let $I_1$ be the SPDE generated on $D_1(U, U')$ by

$$I_1 : \{p^{\bar{1}}_1 = p^0_1 = 0 \text{ and conjugated equations.}$$

The solutions of $I_1$ are (local) CR-diffeomorphisms from $M$ to $M'$. The prolongation $I_2$ of $I_1$ is generated by

$$(4.7) \qquad I_2 : \begin{cases} p^{\bar{1}}_1 = p^0_1 = 0 \\ p^{\bar{1}}_{11} = p^{\bar{1}}_{\bar{1}1} = p^{\bar{1}}_{01} = p^0_{11} = p^0_{\bar{1}1} = p^0_{01} = 0 \\ \text{and conjugated equations.} \end{cases}$$

It follows from (4.6) and (4.7) that

$$(4.8) \qquad p^0_0 - p^1_1 p^{\bar{1}}_{\bar{1}} = 0.$$

If $I^2_1 = \tilde{I}_1$, then $\tilde{I}_1$ is generated as

$$\tilde{I}_1 : \begin{cases} p^{\bar{1}}_1 = p^0_1 = p^0_0 - p^1_1 p^{\bar{1}}_{\bar{1}} = 0 \\ \text{and conjugated equations.} \end{cases}$$

**Proposition 4.1.** *$h_1$ is 1-acyclic.*

**Proof:** Put $\alpha = (\alpha_0, \alpha_1, \alpha_{\bar{1}}) \in \mathbf{N}^3$, and write $p^j_\alpha = p^j_{0\cdots 01\cdots 1\bar{1}\cdots\bar{1}}$, where the index $i$ appears $\alpha_i$–times. Then $h_k$ is generated by

$$h_k = \left[dp^1_\alpha, dp^{\bar{1}}_{\bar{\alpha}}, \alpha_{\bar{1}} \neq 0; dp^0_\alpha, \alpha_1 + \alpha_{\bar{1}} \neq 0; \mid \alpha \mid = k\right]$$

and

$$n_k = \dim h_k = 2\left\{\frac{(k+2)!}{k!2!} - \frac{(k+1)!}{k!1!}\right\} + \left\{\frac{(k+2)!}{k!2!} - 1\right\} = \frac{3k^2 + 5k}{2}.$$

We will show that the sequence

$$(4.9) \qquad 0 \to \Lambda^3 T \otimes h_{k-2} \overset{d_K}{\to} \Lambda^2 \otimes h_{k-1} \overset{d_K}{\to} T \otimes h_k \overset{d_K}{\to} h_{k+1} \to 0$$

is exact in $T \otimes h_k$, for $k \geq 2$. As we know $d_K(T \otimes h_k) = h_{k+1}$, it is enough to show that $\dim d_K(\Lambda^2 T \otimes h_{k-1}) = 3n_k - n_{k+1}$, for $k \geq 2$. But

$$0 \to \Lambda^3 T \otimes Q_{k-2} \overset{d_K}{\to} \Lambda^2 T \otimes Q_{k-1} \overset{d_K}{\to} T \otimes Q_k \overset{d_K}{\to} Q_{k+1} \to 0$$

is exact, so if $\omega \in \Lambda^2 T \otimes h_{k-1}$ is such that $d_K \omega = 0$, then there exists $\eta \in \Lambda^3 T \otimes Q_{k-2}$ such that $d_K \eta = \omega$. If $\eta = Z_0 \wedge Z_1 \wedge Z_{\bar{1}} \otimes \theta$, with

$\theta \in Q_{k-2}$, then $d_K\eta = Z_1 \wedge Z_{\bar{1}} \otimes \partial_0\theta - Z_0 \wedge Z_{\bar{1}} \otimes \partial_1\theta + Z_0 \wedge Z_1 \otimes \partial_{\bar{1}}\theta$,where $\partial_i\theta = d_K(Z_i \otimes \theta)$. Consequently , $\partial_i\theta \in h_{k-1}$, for $i = 0, 1, \bar{1}$, so $\theta \in h_{k-2}$. Then $\eta \in \Lambda^3 T \otimes h_{k-2}$, and this shows (4.9) is exact at $\Lambda^2 T \otimes h_{k-1}$, so

$$\begin{aligned} &\dim d_K(\Lambda^2 T \otimes h_{k-1}) \\ &\quad = \dim \Lambda^2 T \otimes h_{k-1} - \dim \Lambda^3 T \otimes h_{k-2} = 3n_{k-1} - n_{k-2}. \end{aligned}$$

The equality $3n_k - n_{k+1} = 3n_{k-1} - n_{k-2}$ is a simple verification, which shows (4.9) is exact.

**Proposition 4.2.** *For every $k \geq 1$,*

$$I_k^{k+1} = \tilde{I}_k$$

**Proof:**This follows from Theorem3.1 and Proposition4.1

The same way as above, we verify $\tilde{I}_2$ is generated by

$$\tilde{I}_2 : \left\{ \begin{array}{l} \text{equations}(4.7)(4.8) \\ p^0_{00} - p^{\bar{1}}_{\bar{1}} p^1_{01} - p^1_1 p^{\bar{1}}_{0\bar{1}} = 0 \\ \frac{p^{\bar{1}}_{\bar{1}\bar{1}}}{p^{\bar{1}}_{\bar{1}}} - 2i\frac{p^1_0}{p^1_1} - (\bar{c} - \bar{c}')p^{\bar{1}}_{\bar{1}} = 0 \\ \text{and conjugated equations.} \end{array} \right. \tag{4.10}$$

Then

$$\tilde{I}_1^2 = \tilde{I}_1$$

and if we define

$$\breve{I}_2 = \tilde{I}_2^3$$

then $\breve{I}_2$ is generated [10] by

$$\breve{I}_2 : \left\{ \begin{array}{l} \text{equations}(4.10) \\ \frac{p^{\bar{1}}_{0\bar{1}}}{p^{\bar{1}}_{\bar{1}}} - \frac{p^0_{00} + 3ip^1_0 p^{\bar{1}}_0}{2p^0_{00}} + \frac{i}{2}(d - d'p^0_0) - \frac{1}{2}(cp^1_0 - c'p^{\bar{1}}_0) = 0 \\ \text{and conjugated equations} \end{array} \right. \tag{4.11}$$

where

$$d = \frac{1}{2}(c_{\bar{1}} + i(a - 2\bar{a}). \tag{4.12}$$

It follows from (4.11) that

$$\breve{I}_1^2 = \tilde{I}_1.$$

**Proposition 4.3.** *$\tilde{h}_2$ is 1-acyclic.*

**Proof:** It is easy to see that $\tilde{h}_k$ is generated by

$$\begin{aligned}\tilde{h}_k &= \Big[dp^1_\alpha, dp^{\bar{1}}_{\bar{\alpha}}, \alpha \neq (k-1,1,0), (k,0,0); dp^0_\alpha, \alpha \neq (k,0,0);\\ &\qquad dp^0_{(k,0,0)} - p^{\bar{1}}_{\bar{1}} dp^1_{(k-1,1,0)} - dp^{\bar{1}}_{(1,k-1,0)}\Big]\end{aligned}$$

and $\tilde{n}_k = \dim \tilde{h}_k = 3\frac{(k+2)!}{k!2!} - 4$. As in the proof of Proposition4.1, $3\tilde{n}_k - \tilde{n}_{k+1} = 3\tilde{n}_{k-1} - \tilde{n}_{k-2}$ for $k \geq 3$. Observe that equality doesn't hold for $k = 2$, so $\tilde{h}_1$ is not 2-acyclic.

**Proposition 4.4.** *For every $k \geq 2$,*

$$\tilde{I}^{k+1}_k = \breve{I}_k.$$

**Proof:** The same as Proposition4.2.
Let be now

$$\hat{I}_2 = \breve{I}^3_2.$$

Then (cf. [10]) $\hat{I}_2$ is generated by

$$(4.13) \qquad \hat{I}_2 : \begin{cases} \text{equations (4.11)} \\ \dfrac{p^{\bar{1}}_{00}}{p^{\bar{1}}_{\bar{1}}} \quad -\dfrac{p^{\bar{1}}_0 p^0_{00}}{p^{\bar{1}}_{\bar{1}} p^0_0} + (\kappa - \kappa' p^1_1 p^0_0) - \dfrac{i p^1_0 (p^{\bar{1}}_0)^2}{p^{\bar{1}}_{\bar{1}} p^0_0} \\ \qquad -c' \dfrac{p^1_0 p^{\bar{1}}_0}{p^{\bar{1}}_{\bar{1}}} - b' p^1_1 p^1_0 + (id' - \overline{a}') p^1_1 p^{\bar{1}}_0 = 0 \\ \text{and conjugated equations} \end{cases}$$

with

$$(4.14) \qquad \kappa = -\frac{i}{3}(c_0 - id_1 + icd + ac - \bar{b}\bar{c}).$$

**Proposition 4.5.** *$\breve{h}_2$ is 1-acyclic.*

**Proof:** The fiber bundle $\breve{h}_k, k \geq 2$ is generated by

$$\begin{aligned}\breve{h}_k &= \Big[dp^1_\alpha, dp^{\bar{1}}_{\bar{\alpha}}, \alpha \neq (k-1,1,0), (k,0,0); dp^0_\alpha, \alpha \neq (k,0,0);\\ &\qquad 2p^{\bar{1}}_{\bar{1}} dp^1_{(k-1,1,0)} - dp^0_{(k,0,0)}; 2p^1_1 dp^{\bar{1}}_{(k-1,0,1)} - dp^0_{(k,0,0)}\Big].\end{aligned}$$

Define $\breve{h}_1$, doing $k = 1$ above. If $\breve{n}_k = \dim \breve{h}_k = \frac{3(k+2)!}{k!2!} - 3$, then $3\breve{n}_k - \breve{n}_{k+1} = 3\breve{n}_{k-1} - \breve{n}_{k-2}$ for $k \geq 3$, and the proof are in the same lines of Proposition4.1.

**Proposition 4.6.** *For every $k \geq 2$*

$$\breve{I}_k^{k+1} = \hat{I}_k.$$

**Proof:** As in Proposition4.2.

**Proposition 4.7.** *$\hat{h}_2$ is 1-acyclic.*

**Proof:** We have

$$\begin{aligned}\hat{h}_2 \quad = \quad & \Big[dp_{ij}^0, (i,j) \neq (0,0); dp_{ij}^1, (i,j) \neq (0,0), (1,0); dp_{00}^1 - \frac{p_0^1}{p_0^0} dp_{00}^0; \\ & dp_{01}^1 - \frac{1}{2p_{\bar{1}}^{\bar{1}}} dp_{00}^0; \text{ and conjugated elements}\Big]\end{aligned}$$

and $\hat{h}_k = Q_k^*$, for $k \geq 3$. It is enough to show $d_K(\Lambda^2 T \otimes \hat{h}_2) = d_K(\Lambda^2 T \otimes Q_2^*)$, or, $d_K(\Lambda^2 T \otimes [dp_{00}^0]) \subset d_K(\Lambda^2 T \otimes \hat{h}_2)$, and this is consequence of

$$d_K(e_1 \wedge e_{\bar{1}} \otimes dp_{00}^0) = d_K(e_0 \wedge e_{\bar{1}} \otimes dp_{01}^0 - e_0 \wedge e_1 \otimes dp_{0\bar{1}}^0)$$

and

$$\begin{aligned}d_K(e_0 \wedge e_1 \otimes dp_{00}^0) \quad = \quad & \frac{2}{p_{\bar{1}}^1} d_K \Bigg[e_0 \wedge e_{\bar{1}} \otimes \left(dp_{01}^{\bar{1}} - \frac{p_0^{\bar{1}}}{p_0^0} dp_{01}^0\right) \\ & + e_1 \wedge e_0 \left(dp_{0\bar{1}}^{\bar{1}} - \frac{1}{2} p_{\bar{1}}^1 dp_{00}^0 - \frac{p_0^{\bar{1}}}{p_0^0} dp_{0\bar{1}}^0\right) \\ & - e_1 \wedge e_{\bar{1}} \left(dp_{00}^{\bar{1}} - \frac{p_0^{\bar{1}}}{p_0^0} dp_{00}^0\right)\Bigg].\end{aligned}$$

The SPDE $\hat{I}_2^3$ is generated by (cf[10])

$$(4.15) \qquad \hat{I}_2^3 : \begin{cases} \text{equations (4.13)} \\ p_{\bar{1}}^{\bar{1}} r - p_{\bar{1}}^1 (p_0^0)^2 r' = 0 \\ \text{and conjugated equations} \end{cases}$$

where

$$(4.16) \qquad r = \kappa_1 - \bar{b}_0 - 2c\kappa - \bar{b}(a + \bar{a} - id);$$

If we define

$$R \mid_U = r Z_{\bar{1}}^* \wedge Z_0^* \otimes Z_0^* \otimes Z_{\bar{1}} + \bar{r} Z_{\bar{1}}^* \wedge Z_0^* \otimes Z_0^* \otimes Z_1$$

then $R$ is a tensor on $M$,i.e., $R \in \Gamma(\Lambda^2 T^* \otimes T^* \otimes T)$.

**Definition 4.1.** *The tensor $R$ is the curvature tensor of the CR-manifold $M$. We say $M$ is umbilic at $x \in M$ if $R(x) = 0$, otherwise $M$ is said non-umbilic at $x \in M$; $M$ is said umbilic(non-umbilic) if $M$ is umbilic (non-umbilic) at every $x \in M$*

**Example:** The *quadric* $Q$ is defined by $Q = \{(z, w) \in \mathbf{C}^2 : w - \bar{w} = 2iz\bar{z}\}$. If $Z_1 = \frac{i}{2}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial w}$ then $Z_0 = -\frac{1}{2}(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}})$ . Then $a = b = c = 0$ and $R = 0$, so $Q$ is umbilic.

**Proposition 4.8.** *The diagram*

$$
\begin{array}{ccccccccc}
I_6 & \to & \tilde{I}_5 & \to & \breve{I}_4 & \to & \hat{I}_3 & \to & \hat{I}_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_5 & \to & \tilde{I}_4 & \to & \breve{I}_3 & \to & \hat{I}_2 & \to & \tilde{I}_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I_4 & \to & \tilde{I}_3 & \to & \breve{I}_2 & \to & \tilde{I}_1 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
I_3 & \to & \tilde{I}_2 & \to & \tilde{I}_1 & & & & \\
\downarrow & & \downarrow & & & & & & \\
I_2 & \to & \tilde{I}_1 & & & & & & \\
\downarrow & & & & & & & & \\
I_1 & & & & & & & &
\end{array}
\tag{4.17}
$$

*is commutative, with horizontal arrows surjective and the arrows representing the projection of projectable functions.*

**Proof:** It is a consequence of the above propositions.

**Theorem 4.1.** *Given CR-manifolds $M$ and $M'$ and points $x \in M$ and $x' \in M'$ there exist a fifth order jet of CR-diffeomorphism doing a fifth order contact between $M$ and $M'$ at points $x$ and $x'$.*

**Proof:** Proposition4.8 says that $\beta_5 : I_5 \to M \times M'$ is surjective, then there exists $X \in I_5$ such that $\beta_5(X) = (x, x')$.

**Theorem 4.2.** *If $M'$ is umbilic, then it is locally CR-diffeomorphic to the hyperquadric $Q$.*

**Proof:**Let be $M = Q$; then $r$ and $r'$ are 0, and from (4.15) we get $\hat{I}_3$ is onto $\hat{I}_2$. As $\hat{h}_2$ is 1-acyclic, Corollary3.1 says $\hat{I}_2$ is formally integrable. But $\hat{h}_3 = Q_3^*$ , then $\hat{I}_2$ is completely integrable (cf[5]), so there exists a neighborhood $U$ of $x \in Q$ and a CR-diffeomorphism $f : U \to f(U) \subset M'$ solution of $\hat{I}_2$ .

**Corollary 4.1.** *If $M'=Q$, then $\hat{I}_2 \cap \beta_2^{-1}(M,0)$ is a G-structure associated to M, where the group G is the group of CR-automorphisms of Q.*

Suppose now that $M$ and $M'$ are non-umbilic. Then $\hat{I}_2^3$ is a regular SEDP , and in (4.15) we can replace the new equation by

$$p_1^1 = \epsilon \frac{\lambda}{\lambda'}, \epsilon = \pm 1$$

where

$$\lambda = \frac{\sqrt{r}}{\sqrt[8]{r\bar{r}}} \tag{4.18}$$

($\sqrt[8]{r\bar{r}}$ taken as positive root) and $\lambda'$ defined similarly. Then

$$\hat{I}_2^3 : \begin{cases} \text{equations (4.13)} \\ p_1^1 = \epsilon\lambda/\lambda', \epsilon = \pm 1 \\ \text{and conjugated equations} \end{cases} \tag{4.19}$$

Defining

$$\bar{I}_2 = \hat{I}_2^3$$

we can verify

$$\bar{I}_2^3 : \begin{cases} \text{equations (4.19)} \\ \alpha = \alpha' \\ \beta = \beta' \\ \text{and conjugated equations} \end{cases} \tag{4.20}$$

where

$$\alpha = \frac{\bar{\lambda}_1}{\bar{\lambda}} + \frac{\lambda_1}{2\lambda} - \frac{c}{2} \tag{4.21}$$

and

$$\beta = 2\left(\alpha - \frac{\bar{\lambda}_1}{\bar{\lambda}}\right)\left(\bar{\alpha} - \frac{\lambda_{\bar{1}}}{\lambda}\right) + \frac{\lambda_{\bar{1}}\bar{\lambda}_1}{\lambda\bar{\lambda}} - i\left(\frac{\lambda_0}{\lambda} - \frac{\bar{\lambda}_0}{\bar{\lambda}}\right) - d. \tag{4.22}$$

As $\bar{h}_2 = \hat{h}_2$, we obtain in non-umbilic case an extension of (4.17):

$$
\begin{array}{ccccccccccc}
I_7 & \rightarrow & \tilde{I}_6 & \rightarrow & \breve{I}_5 & \rightarrow & \hat{I}_4 & \rightarrow & \bar{I}_3 & \rightarrow & \bar{I}_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I_6 & \rightarrow & \tilde{I}_5 & \rightarrow & \breve{I}_4 & \rightarrow & \hat{I}_3 & \rightarrow & \bar{I}_2 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
I_5 & \rightarrow & \tilde{I}_4 & \rightarrow & \breve{I}_3 & \rightarrow & \hat{I}_2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
I_4 & \rightarrow & \tilde{I}_3 & \rightarrow & \breve{I}_2 & \rightarrow & \tilde{I}_1 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
I_3 & \rightarrow & \tilde{I}_2 & \rightarrow & \tilde{I}_1 & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
I_2 & \rightarrow & \tilde{I}_1 & & & & & & & & \\
\downarrow & & & & & & & & & & \\
I_1 & & & & & & & & & &
\end{array}
\tag{4.23}
$$

where all horizontal arrows are onto.

**Proposition 4.9.** *There exists a sixth order contact between two CR-manifolds at two non-umbilic points.*

**Proof:** It follows from $\rho_2^6(I_6) = \bar{I}_2$
The following theorem is in [2]:

**Theorem 4.3.** *There exists a seventh order contact between a real hypersurface of $\mathbf{C}^2$ at a non-umbilic point and the hypersurface defined by*

$$v = z\bar{z} + 2Re\left\{ z^4\bar{z}^2[1 + \frac{16}{5}\alpha(0)z + i(\frac{275}{128}\alpha(0)\bar{\alpha}(0) - \beta(0))u] \right\}$$

*where $\alpha, \beta$ are the functions defined in (4.21),(4.22)*

**Proof:**Let be $M = \{(z, w) \in \mathbf{C}^2 : v = F(z, \bar{z}, u), \text{with}\, w = u + iv\}$. Choosing coordinates $(z, u)$ on $M$, take

$$Z_1 = \frac{\partial}{\partial z} - A\frac{\partial}{\partial u}; \tag{4.24}$$

then from (4.4)

$$Z_0 = \frac{2B}{(1 + f_u^2)}\frac{\partial}{\partial u} \tag{4.25}$$

where $A = f_z/(f_u + i)$ and $B = -f_{z\bar{z}} + \bar{A}f_{uz} + Af_{u\bar{z}} - A\bar{A}f_{uu}$ . It follows from (4.5)

$$a = b = 0, c = A_u - 2\frac{f_u f_{uz} - Af_u f_{uu}}{1 + f_u^2} + \frac{B_z - AB_u}{B} \tag{4.26}$$

$$f(z, \bar{z}, u) = \sum a_{jkl} z^j \bar{z}^k u^l \tag{4.27}$$

with $a_{jkl} = 0, j + k + l \leq 5, (j, k, l) \neq (1, 1, 0); a_{1,1,0} = 1; a_{kjl} = \overline{a_{jkl}}$ . From (4.12),(4.14),(4.16),(4.24),(4.26)

$$r = -\frac{1}{6}c_{11\bar{1}} - \frac{i}{3}c_{10} + c\left(\frac{2}{3}ic_0 + \frac{i}{2}c_{1\bar{1}} - \frac{1}{3}cc_{\bar{1}}\right) + \frac{1}{6}c_1 c_{\bar{1}}. \tag{4.28}$$

From (4.24), (4.25), (4.27), (4.28)

$$Z_1(0) = \frac{\partial}{\partial z}; Z_0(0) = -2\frac{\partial}{\partial u}$$

$$c(0) = c_1(0) = c_{\bar{1}}(0) = c_0(0) = c_{1\bar{1}}(0) = c_{10}(0) = 0$$

$$c_{11\bar{1}}(0) = 4!2!a_{420}$$

$$r(0) = a_{420}.$$

As $r(0) \neq 0$, by Proposition4.9, we can choose $a_{420} = a_{240} = 1$, and all others coefficients of sixtieth-order nulls, so $r(0) = 1$, and

$$v = z\bar{z} + 2Re(z^4\bar{z}^2) + o(7) \tag{4.29}$$

Again from (4.24),(4.25),(4.28),(4.29)

$$c_{111\bar{1}}(0) = 5!2!a_{520}; c_{\bar{1}11\bar{1}}(0) = 4!3!a_{430}; c_{011\bar{1}}(0) = -4!2!2a_{421} \tag{4.30}$$

and from (4.28);(4.30)

$$r_1(0) = 5a_{520}; r_{\bar{1}}(0) = 3a_{430}; r_0(0) = -2a_{421}. \tag{4.31}$$

From (4.17)

$$\lambda_j = \frac{\lambda}{8}\left(3\frac{r_j}{r} - \frac{\bar{r}_j}{\bar{r}}\right), j = 0, 1, \bar{1} \tag{4.32}$$

and from (4.18),(4.31)

$$\lambda(0) = 1; \lambda_1(0) = \frac{1}{8}(15a_{520} - 3a_{430});$$

$$\lambda_{\bar{1}}(0) = \frac{1}{8}(9a_{430} - 5a_{520}); \lambda_0(0) = \frac{1}{4}(a_{241} - a_{421}).$$

From (4.20), (4.21), (4.32)

$$\alpha(0) = \frac{5}{16}(3a_{340} + a_{520})$$

$$\begin{aligned}\beta(0) &= \frac{9}{128}(5\alpha(0) - 16a_{340})(5\bar{\alpha}(0) - 16a_{430}) \\ &+\frac{1}{64}(5\alpha(0) - 24a_{340})(5\bar{\alpha}(0) - 24a_{430} + \mathrm{Im}(a_{241}).\end{aligned}$$

Therefore we can choose

$$a_{520} = \frac{16}{5}\alpha(0); a_{421} = -i(\beta(0) - \frac{275}{128}\alpha(0)\bar{\alpha}(0))$$

$$a_{250} = \overline{a_{520}}; a_{241} = \overline{a_{421}}$$

all the others coefficients nulls, and the theorem follows.

## References


[ 1 ] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes*, Oeuvres Complètes, partieII, Éditions du CNRS, 1984, 1231-1304

[ 2 ] S.S.Chern;J.Moser, *Real hypersurfaces in complex manifolds*, Acta Math.-**133**(1974)219-271

[ 3 ] H.Goldschmidt, *Existence theorems for analytic linear partial differential equations*, Ann. of Math.**86**(1967)246-270

[ 4 ] H.Goldschmidt, *Integrability criteria for systems of non-linear partial differential equations*, J.Differential Geometry**1**(1967)269-307

[ 5 ] M.Kuranishi, *Lectures on Involutive Systems of Partial Differential Equations*, Publ. Soc. Matemática São Paulo,1967

[ 6 ] D.G.Quillen, *Formal properties of over-determined systems of linear partial differential equations* ,Ph.D.Thesis,Harvard University,1964

[ 7 ] C.Ruiz, *Prolongement formel des systèmes différentiels extérieur d'ordre supérieur*, C.R.Acad.Sc.Paris**285**(1977)1077-1080

[ 8 ] C.Ruiz,*Complexe de Koszul du symbole d'un systéme différentiel extérieur*,C.R.Acad.Sc.Paris**286**(1978)55-58

[ 9 ] C.Ruiz,*Propriétètes de dualité du prolongement formel des systémes différentiels extérieurs***286** (1978)99-101

[10] J.M.Veloso;J.Verderesi,*Three- dimensional Cauchy-Riemann manifolds*, Dinamical Systems and Partial Differential Equations: Proceedings VII ELAM,Equinoccio,1986

*Departamento de Matemática*
*Universidade Federal do Pará*
*CEP 66075-110 Belém-PA*
*Brasil*
*veloso@marajo.ufpa.br*