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A Report on Isolated Singularities by Transcendental Methods

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Dedicated to Professor M. Kuranishi on his 70th birthday

1. An *isolated singularity* (of a complex analytic space) is by definition a germ of a reduced and irreducible complex analytic space at an isolated singular point. By a *model* of an isolated singularity we shall mean an irreducible complex analytic subset V of $\mathbb{C}^N$ containing the origin as the unique singular point. To any model V of an isolated singularity (V, o), one can associate three manifolds of completely different nature.

i) A nonsingular model of V: By Hironaka's desingularization theorem, there exists a complex manifold $\tilde{V}$ and a proper holomorphic map $\pi : \tilde{V} \to V$ such that $\pi|\tilde{V} \setminus \pi^{-1}(o)$ is a biholomorphism. In virtue of the existence of $\tilde{V}$, equivalence questions between the isolated singularities can be transferred to more geometric ones (cf. [G-2], [H-R]). Moreover a lot of work has been done on the classification of isolated singularities by manipulating the invariants on $\tilde{V}$ (cf. [I]).

ii) $V \cap S_\varepsilon$, where $S_\varepsilon = \{z \in \mathbb{C}^N \mid \|z\| = \varepsilon\}$ and ε is so chosen that $S_{\varepsilon'}$ and V intersect transversally for all $\varepsilon' \in (0, 2\varepsilon)$: As a differentiable manifold, $V \cap S_\varepsilon$ falls into the class of strongly pseudoconvex CR manifolds, or spc manifolds. Since the spc structure naturally yields L^2 estimates for the tangential Cauchy-Riemann operators, the method of PDE in the theory of deformation of complex manifolds is carried over to spc manifolds. As a result, several fundamental questions including the construction of the versal family of singularities have been solved by this method (cf. [A-1], [A-M] and [N-O]).

iii) $V \setminus \{o\}$: Manifolds of this type appear as the ends of locally symmetric varieties of rank one, and they were studied in a general framework by Andreotti-Grauert [A-G] as the ends of "pseudoconcave"

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spaces. Afterwards it was noticed that some questions on isolated singularities are tractable as a sort of boundary value problem on $V \setminus \{o\}$ for the $\bar{\partial}$-operator. In fact, analysis of this type turned out to be useful when one wants to understand the intersection cohomology of projective varieties (cf. [O-4,5,9]).

The purpose of this note is to report miscellaneous results on these three types of manifolds that are obtained since Professor M. Kuranishi gave a series of inspiring lectures at RIMS in 1976.

2. Given any desingularization $\pi : \tilde{V} \to V$, $\pi^{-1}(o)$ is called the *exceptional set*. Topology of the exceptional set is somewhat restrictive in the following sense.

Theorem 1. $\dim_{\mathbb{C}} H^r(\pi^{-1}(o), \mathbb{C}) \equiv 0 \pmod 2$ *if* $r \geq n$ *and* $r \equiv 1 \pmod 2$.

This result seems to have been a sort of folklore in the late seventies (cf. [L-R], [F]), but rigorous proofs appeared only gradually in [O-1,2,6] and [O-T]. Although a complete proof of Theorem 1 is only available via Artin's algebraization theorem and Deligne's mixed Hodge theory, it may be worthwhile to note that the following constitutes already a significant part of the proof, which motivated further generalization of the Hodge theory to noncompact manifolds.

Theorem 2 (cf. [O-1,2], [O-T]). *Let X be a Kähler manifold of dimension n and let $D \subset X$ be a strongly pseudoconvex domain with C^2-smooth boundary. Then there exists a complete Kähler metric ω on D such that $H^r(D, \mathbb{C})$ are canonically isomorphic to the space of L^2 harmonic forms of degree r with respect to ω for all $r > n$.*

The range $r > n$ is optimal. In fact, $\dim_{\mathbb{C}} H^n(D, \mathbb{C}) < \infty$ but the space of L^2 harmonic n-forms is infinite dimensional for any strongly pseudoconvex domain. As the above ω one may take any complete Kähler metric which is quasi-isometrically equivalent to the Levi form of a C^∞ exhaustion function with bounded gradient outside a compact subset of D. A typical example of such a metric is the Bergman metric on strongly pseudoconvex domains in $\mathbb{C}^n$ (cf. [D-F]). For the Bergman metric, the L^2 cohomology groups of type (p, q) are known to be infinite dimensional for $p + q = n$ (cf. [D-F], [O-8]). Recently the boundary values of these cohomology classes are studied for the unit ball (cf. [J-K]).

Applying Theorem 2, one can deduce the following Hartogs type theorem.

Theorem 3 (cf. [O-2]). *The natural restriction map*

$$\rho^{p,q} : H^{p,q}(\tilde{V}) \longrightarrow H^{p,q}(\tilde{V} \setminus \pi^{-1}(0))$$

is surjective if $p + q < n - 1$.

The range $p + q < n - 1$ is also optimal. In fact, since $\tilde{V} \setminus \pi^{-1}(o) \cong V \setminus \{o\}$ and $H^{n,0}(\tilde{V})$ is naturally identified with the set of L^2 holomorphic n-forms on $V \setminus \{o\}$, $\dim_{\mathbb{C}} \operatorname{Coker} \rho^{n,0}$ does not depend on the choice of the nonsingular model $\tilde{V}$. It is easy to see that $\dim_{\mathbb{C}} \operatorname{Coker} \rho^{n,0} = \dim_{\mathbb{C}} H^1(\tilde{V}, \mathcal{O}_{\tilde{V}})$ and to verify that $H^1(\tilde{V}, \mathcal{O}_{\tilde{V}}) \neq \{o\}$ if $\dim V = 2$ and $\tilde{V}$ contains a nonrational curve.

Van Straten [V-S] discovered a remarkable application of Theorem 3 to Zariski-Lipman conjecture by showing for the case $\dim V \geq 3$ that the germ (V, o) is nonsingular if the tangent sheaf of V is locally free.

As for the de Rham cohomology classes on $\tilde{V} \setminus \pi^{-1}(o)$, we have the same extendability result for the degrees less than $n - 1$. This range is also optimal in general, although one can prove the following by analysing a spectral sequence that abuts to $H^r(\pi^{-1}(o), \mathbb{C})$.

Theorem 4 (cf. [O-6]). *If the inclusion map* $\pi^{-1}(0) \hookrightarrow \tilde{V}$ *is a homotopy equivalence, the natural restriction map*

$$H^r(\tilde{V}, \mathbb{C}) \longrightarrow H^r(\tilde{V} \setminus \pi^{-1}(0), \mathbb{C})$$

is surjective for $r \leq n - 1$.

Corollary. *In the above situation, every cohomology class in* $H^n(\tilde{V}, \mathbb{C})$ *can be represented by a compactly supported closed form.*

Thus we are naturally led to the following

Question. *Is every closed holomorphic* $(n-1)$*-form on* $\tilde{V} \setminus \pi^{-1}(o)$ *holomorphically extendable to* $\tilde{V}$*?*

Note that this is certainly true if $n = 1$, since closed 0-forms are locally constant functions. The first nontrivial case $n = 2$ was solved affirmatively by T. Ueda [U]. Mentioning further a partial result, we have that if $\tilde{V}$ is a Zariski open subset of a nonsingular projective variety Z and the given form is extendable to $Z \setminus \pi^{-1}(o)$ then it is extendable also across $\pi^{-1}(o)$. In fact one can prove the following.

Theorem 5 (cf. [F1], [O-10]). *Let X be an irreducible projective variety with singular locus Y, and let p be a nonnegative integer satisfying $p < \operatorname{codim} Y$. Then a holomorphic p-form f is extendable holomorphically to a nonsingular model of X if and only if f is closed.*

After [O-10] was written down, S. Kosarew settled the question affirmatively by an algebraic method (personal communication).

As another question on $\tilde{V}$ we would like to mention the following which was asked by S. Nakano around 1976.

Problem. *Is any d-exact $(1,1)$-form on $\tilde{V}$ of the form $\partial\bar{\partial}\varphi$?*

In case the canonical bundle of $\tilde{V}$ is trivial, one may employ the above mentioned L^2 Hodge theory (cf. Theorem 2 and the remark) to solve it affirmatively. As a result, one has the smoothness of Kuranishi spaces for the deformation of certain isolated singularities (cf. [M]). It should be noted, however, that the answer to Nakano's question is negative in general, because it is not necessarily true that all the topologically trivial line bundles over $\tilde{V}$ arise as flat $U(1)$ bundles.

3. Applying Theorem 4, one can describe some topological properties of $V \cap S_\varepsilon$.

Theorem 6. *Let $\alpha_i \in H^{r_i}(V \cap S_\varepsilon, \mathbb{C})$, $i = 1, 2, \ldots, m$. Then the cup product $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m$ is zero whenever $\sum_{i=1}^m r_i \geq n$ and $r_i \leq n-1$ for all i.*

Proof. Let $\tilde{V}_\varepsilon := \{w \in \tilde{V} \mid \|\pi(w)\| < 2\varepsilon\}$. Then, by Theorem 4 there exist $\tilde{\alpha}_i \in H^{r_i}(\tilde{V}_\varepsilon, \mathbb{C})$ such that $\tilde{\alpha}_i | V \cap S_\varepsilon = \alpha_i$. Since $\alpha_1 \cup \cdots \cup \alpha_m = (\tilde{\alpha}_1 \cup \cdots \cup \tilde{\alpha}_m) | V \cap S_\varepsilon$, we obtain the conclusion from the corollary of Theorem 4.

Corollary. *If $n \geq 2$, there does not exist an isolated singularity (V, o) for which $V \cap S_\varepsilon$ is homotopically equivalent to $\underbrace{S^1 \times \cdots \times S^1}_{2n-1}$.*

Such a phenomenon was first noticed by D. Sullivan for hypersurface singularities of dimension 2 (cf. [Ka]).

As we have mentioned earlier, there is a natural abstract notion of spc manifolds. Recall that a $(2n-1)$-dimensional differentiable manifold M of class C^∞ is called a CR manifold if there are subbundles T, T', F of the complexified tangent bundle $T_M \otimes \mathbb{C}$ such that T is involutive, $T = \overline{T'}$, $\operatorname{rank}_{\mathbb{C}} F = 1$ and $T_M \otimes \mathbb{C} = T \oplus T' \oplus F$. For any local frame

$\{\omega_1, \dots, \omega_{n-1}\}$ for T and a local frame θ for F with $\bar{\theta} = -\theta$, one has a matrix valued function (c_{ij}) defined by $[\omega_i, \bar{\omega}_j] = c_{ij}\theta \pmod{T \oplus T'}$. We say M is a strongly pseudoconvex CR manifold, or shortly an spc manifold, if one can choose the frames $\{\omega_i\}$ and θ around each point of M so that (c_{ij}) is positive definite. $V \cap S_\varepsilon$ is then an spc manifold if one puts $T = (T_{V\cap S_\varepsilon} \otimes \mathbb{C}) \cap T^{1,0}_{\mathbb{C}^N}$, $T' = \bar{T}$ and $F = (T \oplus T')^\perp$. In the same manner, the boundary of any strongly pseudoconvex domain is, if it is of class C^∞, regarded as an spc manifold. The following result ensures that passage from $V \cap S_\varepsilon$ to the class of spc manifolds is a good abstraction.

Theorem 7(cf. [B], [O-3]). *Every connected compact spc manifold of dimension ≥ 5 is the boundary of a strongly pseudoconvex domain in a complex manifold.*

Therefore, combining Theorem 7 with the remark preceding Theorem 4, one has the following by the same argument as in the proof of Theorem 6.

Theorem 8. *Let M be a connected compact spc manifold of dimension $2n-1$ with $n \geq 3$. Then the cup product $\alpha_1 \cup \cdots \cup \alpha_m$ of $\alpha_i \in H^{r_i}(M, \mathbb{C})$ is zero whenever $\sum_{i=1}^m r_i \geq n+1$ and $r_i \leq n-2$ for all i.*

Question. *Is there any direct proof of Theorem 8 that does not use Theorem 7 and the Hodge theory on strongly pseudoconvex domains?*

A recent work of T. Akahori [A-2] may lead to an answer to it.

For three dimensional spc manifolds, it is well known that they are even locally not embeddable as a real hypersurface of a complex manifold (cf. [Ni]). As for the recent embeddability and non-embeddability results, the reader is referred to articles of C. Epstein [E-1,2].

4. Although the compactness of $V \cap S_\varepsilon$ is a great advantage for using analytic tools, one might also be inclined to study analytic objects on the manifold $V \setminus \{o\}$ because it carries a complete Kähler metric by a theorem of Grauert (cf. [G1]). Since Grauert's Kähler metric on $V \setminus \{o\}$ is of the form $\partial\bar{\partial}\varphi$, where φ is bounded near o, one can immediately deduce from Bochner-Nakano's formula that the $\bar{\partial}$-equation $\bar{\partial} f = g$ has an L^2 solution near o for any $\bar{\partial}$-closed L^2 (p,q)-form g on V, provided that $p + q > n$. In order to proceed further, we need the following observation due to Donnelly and Fefferman [D-F] (See [O-T] for a simplified proof).

Theorem 9. *Let (M,ω) be a connected complete Kähler manifold of dimension n such that there exists a C^∞ strictly plurisubharmonic function φ with bounded gradient on M such that $\partial\bar\partial\varphi$ is quasi-isometrically equivalent to ω. Then the $L^2\bar\partial$-cohomology group of (M,ω) of type (p,q) vanishes if $p+q \neq n$.*

Metrics satisfying the hypothesis of Theorem 9 arise very naturally. Instances are the metric $\partial\bar\partial(-\log(-\log\|z\|))$ on the punctured unit ball $\mathbb{B}_* := \{z \in \mathbb{C}^N \mid 0 < \|z\| < 1\}$ and its restriction to $V \cap \mathbb{B}_*$. As another immediate instance, one can mention the Bergman metric on a strongly pseudoconvex domain in $\mathbb{C}^n$. A remarkable fact attached to the L^2 cohomology vanishing in Theorem 9 is that the L^2 estimates

$$\|u\| \leq C(\|\bar\partial u\| + \|\bar\partial^* u\|)$$

hold for $C = 3\sup|\partial\varphi|_\omega$. This allows us to study the L^2 cohomology of $V \cap \mathbb{B}_*$ with respect to non-complete metrics that are the limits of $\partial\bar\partial\varphi_t$ satisfying the uniformity condition $\partial\bar\partial\varphi_t \geq \partial\varphi_t \cdot \bar\partial\varphi_t$. Among such metrics is the restriction of the Euclidean metric $\partial\bar\partial\|z\|^2$ to $V \cap \mathbb{B}_*$. To state results of this kind, let us denote by $H^{p,q}_{(2)}(U)$, for any neighbourhood U of o in V, the L^2 $\bar\partial$-cohomology group of $U \setminus \{o\}$ of type (p,q). For the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N \mid \|z\| < 1\}$, it has long been known that $H^{0,q}_{(2)}(\mathbb{B}) = \{0\}$ for $q \geq 1$ (cf. [Hö]). By the above mentioned argument one can show that $H^{p,q}_{(2)}(\mathbb{B}\cap V) = \{0\}$ if $p+q > n$ (cf. [O-4]). If one denotes by $H^{p,q}_{(2),0}(\mathbb{B}\cap V)$ the L^2 $\bar\partial$-cohomology of $\mathbb{B}_* \cap V$ with respect to $\partial\bar\partial\|z\|^2$ with supports contained in compact subsets of V, one has also the dual vanishing $H^{p,q}_{(2),0}(\mathbb{B} \cap V) = \{0\}$, $p+q < n$ (cf. [O-7]). Similarly one has also the vanishing of the L^2 de Rham cohomology groups for the corresponding degrees. Moreover, with an additional technical effort one can manage to prove

Theorem 10 (cf. [O-7,9]).

$$H^r_{(2)}(V \cap \mathbb{B}) = \{0\} \quad \textit{for} \quad r \geq n$$

and

$$H^r_{(2),0}(V \cap \mathbb{B}) = \{0\} \quad \textit{for} \quad r < n.$$

Here $H^r_{(2)}(V\cap\mathbb{B})$ (resp. $H^r_{(2),0}(V\cap\mathbb{B})$) denotes the r-th L^2 de Rham cohomology group of $V\cap\mathbb{B}$ with respect to $\partial\bar\partial\|z\|^2$ (resp. that with relatively compact supports in V).

Corollary. *For any projective variety $X \subset \mathbb{P}^N$ whose singular points are isolated, the L^2 de Rham cohomology group of X is canonically isomorphic to the intersection cohomology group of X in the sense of Goresky-MacPherson.*

As a concluding remark we would like to indicate a next interesting topic in the analysis of isolated singularities. This will be a question of estimating $\dim H^{p,q}_{(2)}(V \cap \mathbb{B}_*)$ or $\dim H^r_{(2)}(V \cap \mathbb{B}_*)$ with respect to complete Kähler metrics on $V \setminus \{o\}$ that does not satisfy the condition of Theorem 9. Such metrics arise naturally by adding Kähler metrics on $\tilde{V}$. Therefore it seems that something like the following must have an answer.

Question. *Let ω_1 and ω_2 be complete Kähler metrics on $V \setminus \{o\}$. Is it true that $\omega_1 \geq \omega_2$ implies $\dim H^r_{(2)}(V \cap \mathbb{B}_*)_{\omega_1} \geq \dim H^r_{(2)}(V \cap \mathbb{B}_*)_{\omega_2}$?*

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