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Stationary Phase Method with Estimate of Remainder Term over a Space of Large Dimension

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Abstract.

Let $r_d(\nu)$ denote the remainder term of the stationary phase method over R^d. Then an estimate of $\nu^{d/2+1}r_d(\nu)$, as $d \to \infty$, is given under certain assumptions, which are tolerable for application to Feynman path integrals.

§1. Stationary phase method

Stationary phase method is a method to evaluate asymptotically, as $\nu \to \infty$, oscillatory integrals over R^d of the following form:

$$I(S,a,\nu) = \int_{R^d} e^{-i\nu S(x)} a(x)dx,$$

where $S(x)$ is a real valued C^∞ function called the phase function, $a(x)$ is a C^∞ function called the amplitude and ν is a large positive parameter. In the simplest case that $a(x) \in C_0^\infty(R^d)$ and that $S(x)$ has only one critical point x^*, where $\operatorname{Hess} S(x^*)$ is non-degenerate, it gives

$$I(S,a,\nu) = \left(\frac{2\pi}{i\nu}\right)^{d/2} [\det\{\operatorname{Hess} S(x^*)\}]^{-1/2}(e^{-i\nu S(x^*)}a(x^*) + r_d(\nu))$$

and an estimate of the remainder term

$$r_d(\nu) = O(\nu^{-d/2-1}).$$

If support of $a(x)$ is not compact, we have to require some additional assumption that control the behaviour of $a(x)$ at the infinity. For instance (cf. [1]), the same conclusion holds if we assume the following

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Hypothesis (H.0). (i) $\sup_x \mid \partial_x^\alpha S(x) \mid < \infty$ *for any multi-index* α *with* $\mid \alpha \mid \geq 2$. (ii) *There exists a constant* $\delta > 0$ *such that* $\mid \det \mathrm{Hess}\, S(x) \mid \geq \delta$. (iii) *For any multi-index* α, $\sup_x \mid \partial_x^\alpha a(x) \mid < \infty$.

Since the stationary phase method is closely related to the mathematical theory of Feynman path integrals (cf. [3], [4], [5] and [6]), we wish to investigate the following

Question. *Can one control* $\nu^{d/2+1} r_d(\nu)$ *as* $d \to \infty$ *?*

We give a positive answer to this question. Detailed discussions can be found in [2]. Applications are discussed in [4], [5] and [6].

§2. Statement of results

We shall treat the following oscillatory integral over $L-1$ dimensional space:

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \prod_{j=1}^{L} \Big(\frac{\nu i}{2\pi t_j}\Big)^{1/2} \int_{R^{L-1}} e^{-i\nu S(x_L, \dots, x_0)} a(x_L, \dots, x_0) \prod_{j=1}^{L-1} dx_j,$$

with large positive parameter ν and small positive parameters $\{t_j\}$. Our hypothesis for the phase function is

Hypothesis (H.1). $S(x_L, \dots, x_0)$ *is of the form*

$$S(x_L, \dots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}),$$

where

$$S_j(t_j, x_j, x_{j-1}) = \frac{\mid x_j - x_{j-1} \mid^2}{2t_j} + t_j \omega_j(t_j, x_j, x_{j-1}).$$

For any $m \geq 2$ *there exists a positive constant* κ_m *such that*

$$\sup_{x_j, x_{j-1}} \mid \partial_{x_j}^\alpha \partial_{x_{j-1}}^\beta \omega_j(t_j, x_j, x_{j-1}) \mid \leq \kappa_m$$

if $2 \leq \alpha + \beta \leq m$.

We will give two examples of phase functions satisfying hypothesis (H.1).

Example 1. Let $L(\xi, x) = \frac{1}{2}\xi^2 - V(x)$, $(\xi, x) \in R^2$, be a Lagrangian with a potential $V(x)$. Assume that the potential $V(x)$ is a real-valued C^∞-function satisfying estimates:

$$\sup_x \mid V^{(k)}(x) \mid < \infty \quad \text{for any} \quad k \geq 2.$$

Then for a small $T > 0$, there exists a unique classical orbit $\gamma^{cl}(t)$ such that $\gamma^{cl}(0) = y, \gamma^{cl}(T) = x$. Let

$$S^{cl}(T, x, y) = \int_0^T L(\dot{\gamma}^{cl}(t), \gamma^{cl}(t))dt$$

be the classical action. Then $S^{cl}(T, x, y)$ is of the form

$$S^{cl}(T, x, y) = \frac{\mid x - y \mid^2}{2T} + T\phi^{cl}(T, x, y)$$

and for any $m \geq 2$ there exists a constant C_m such that

$$\sup_x \mid \partial_x^\alpha \partial_y^\beta \phi^{cl}(T, x, y) \mid \leq C_m$$

if $2 \leq \alpha + \beta \leq m$. Therefore, $S(x_L, \ldots, x_0) = \sum_{j=1}^L S(t_j, x_j, x_{j-1})$ satisfies the hypothesis (H.1).

Example 2. Let $L(\xi, x)$ be the same lagrangian. Let $\gamma^{ln}(t)$ be the straight line connecting $(0, y)$ and (T, x) in the time-space, i.e.,

$$\gamma^{ln}(t) = \frac{t}{T}x + \frac{T-t}{T}y.$$

Let

$$S^{ln}(T, x, y) = \int_0^T L(\dot{\gamma}^{ln}(t), \gamma^{ln}(t))dt.$$

Then function $S^{ln}(T, x, y)$ is of the form

$$S^{ln}(T, x, y) = \frac{\mid x - y \mid^2}{2T} + T\phi^{ln}(T, x, y)$$

and for any $m \geq 2$ there exists a positive constant C_m such that

$$\sup_x \mid \partial_x^\alpha \partial_y^\beta \phi^{ln}(T, x, y) \mid \leq C_m$$

if $2 \leq \alpha + \beta \leq m$. Therefore, $S^{ln}(x_L, \dots, x_0) = \sum_{j=1}^{L} S^{ln}(t_j, x_j, x_{j-1})$ satisfies the hypothesis (H.1).

Under hypothesis (H.1) the critical point of the function $(x_{L-1}, \dots, x_1) \to S(x_L, x_{L-1}, \dots, x_1, x_0)$ is unique if $T_L = \sum_{j=1}^{L} t_j$ is small. We denote it by $(x_{L-1}^*, \dots, x_1^*)$. We abbreviate $S(x_L, x_{L-1}^*, \dots, x_1^*, x_0)$ as $S(\overline{x_L, x_0})$. We can write the Hessian of S at the critical poit as $H + W$, where

$$H = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2} & -\frac{1}{t_2} & 0 & 0 & \dots \\ -\frac{1}{t_2} & \frac{1}{t_2} + \frac{1}{t_3} & -\frac{1}{t_3} & 0 & \dots \\ 0 & -\frac{1}{t_3} & \frac{1}{t_3} + \frac{1}{t_4} & -\frac{1}{t_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$W = \begin{pmatrix} t_1 \partial_{x_1}^2 \omega_1 + t_2 \partial_{x_1}^2 \omega_2 & t_2 \partial_{x_1} \partial_{x_2} \omega_2 & 0 & \dots \\ t_2 \partial_{x_1} \partial_{x_2} \omega_2 & t_2 \partial_{x_2}^2 \omega_2 + t_3 \partial_{x_2}^2 \omega_3 & t_3 \partial_{x_2} \partial_{x_3} \omega_3 & \dots \\ 0 & t_3 \partial_{x_2} \partial_{x_3} \omega_3 & t_3 \partial_{x_3}^2 \omega_3 + t_4 \partial_{x_3}^2 \omega_4 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is clear that

$$\det H = \frac{T_L}{t_1 t_2 \dots t_L} \neq 0.$$

We can state our first result.

Theorem 1. *Under the hypothesis* (H.1) *there exists a positive constant* δ_1 *independent of* L *such that if* $T_L = t_1 + \dots + t_L \leq \delta_1$ *then*

$$I(\{t_j\}, S, 1, \nu)(x_L, x_0)$$
$$= \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} \left[\det(I + H^{-1}W)\right]^{-1/2} (1 + r(\nu, x_L, x_0)),$$

where the remainder term $r(\nu, x_L, x_0)$ *satisfies the estimate: For any* $K \geq 0$ *there exists positive constants* C_K *such that if* $|\alpha_0|, |\alpha_L| \leq K$

$$|\partial_{x_0}^{\alpha_0} \partial_{x_L}^{\alpha_L} r(\nu, x_L, x_0)| \leq C_K T_L^3 \nu^{-1}.$$

Remark. δ_1 and C_K are independent of L as far as T_L is bounded. Therefore, we can control $r(\nu, x_L, x_0)$ even when L tends to ∞.

In order to state the result for general integral with amplitude $a(x)$, we require a little more preparations. Let $1 \leq k \leq l \leq L$. Then the critical point of the function $(x_{l-1},\dots,x_{k+1}) \to \sum_{j=k+1}^{l} S_j(t_j,x_j,x_{j-1})$ is unique if $t_{k+1}+\dots+t_l$ is small. Let $(x^*_{l-1},\dots,x^*_{k+1})$ denote the critical point, which is a function of x_l and x_k. We abbreviate $a(x_L,\dots,x_l,x^*_{l-1},\dots,x^*_{k+1},x_k,\dots,x_0)$ to $a(x_L,\dots,x_{l+1},\overline{x_l,x_k},x_{k-1},\dots,x_0)$.

Our hypothesis concerning the amplitude function is the following:

Hypothesis (H.2). *For any integer $K \geq 0$ there exists a positive constant A_K with the following properties:* (i) *If $\mid \alpha_j \mid \leq K$ for $j = 0,1,\dots,L$, then*

$$\mid \prod_{j=0}^{L} \partial_{x_j}^{\alpha_j} a(x_L,\dots,x_0) \mid \leq A_K.$$

(ii) *For any sequence of positive integers $\{j_1,\dots,j_s\}$ satisfying*

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < \dots < j_s - 1 < j_s < L$$

we have

$$\mid \partial_{x0}^{\alpha_0} \partial_{xL}^{\alpha_L} \prod_{k=1}^{s} \partial_{x_{j_k-1}}^{\alpha_{j_k-1}} \partial_{x_{j_k}}^{\alpha_{j_k}} a(\overline{x_L,x_{j_s}},\overline{x_{j_s-1},x_{j_{s-1}}},\dots,\overline{x_{j_1-1},x_{j_0}}) \mid \leq A_K,$$

as far as $\mid \alpha_j \mid \leq K$ for $j = 0, j_1-1, j_1,\dots,j_s-1, j_s, L$.

Before stating our second theorem, we give an example of amplitude functions satisfying hypothesis (H.2).

Example. Let $b_j(x_j,x_{j-1})$, $j = 1,\dots,L$, be functions bounded together with their derivatives of all order, i.e., for any positive integer K there exists C_K such that

$$\sup_x \mid \partial_{x_j}^{\alpha_j} \partial_{x_{j-1}}^{\alpha_{j-1}} b_j(x_j,x_{j-1}) \mid \leq C_K \qquad 0 \leq \alpha_j, \alpha_{j-1} \leq K.$$

Then $a(x_L,\dots,x_0) = e^{(\sum_{j=1}^{L} t_j b_j(x_j,x_{j-1}))}$ satisfies hypothesis (H.2) above.

Now we can state our main

Theorem 2. *Under the hypotheses* (H.1) *and* (H.2) *there exists a positive constant δ_1 such that if $0 < T_L \leq \delta_1$*

$$\begin{aligned} &I(\{t_j\},S,a,\nu)(x_L,x_0) \\ &\quad = \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} e^{-i\nu S(\overline{x_L,x_0})} \left[\det(I + H^{-1}W)\right]^{-1/2} \\ &\qquad \times \left(a(\overline{x_L,x_0}) + r(\nu,x_L,x_0)\right), \end{aligned}$$

where $r(\nu, x_L, x_0)$ *satisfies the estimate: For any* $K \geq 0$ *there exists positive constants* C_K *and* $M(K)$ *such that if* $|\alpha_0|, |\alpha_L| \leq K$ *we have*

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} r(\nu, x_L, x_0)| \leq C_K T_L \nu^{-1} A_{M(K)}.$$

Remark. δ_1, C_K and $M(K)$ are independent of L as far as T_L is bounded. Therefore, we can control $r(\nu, x_L, x_0)$ even when L tends to ∞.

§3. Sketch of the proof

We begin with our key lemma, which is valid under hypothesis (H.3) weaker than (H.2) and is interesting in its own sake.

Hypothesis (H.3). *For any integer* $K \geq 0$ *there exists a positive constant* A_K *such that if* $|\alpha_j| \leq K$ *for* $j = 0, 1, \ldots, L$,

$$\left|\prod_{j=0}^{L} \partial_{x_j}^{\alpha_j} a(x_L, \ldots, x_0)\right| \leq A_K.$$

We can state

Key Lemma. *Under the hypotheses* (H.1) *and* (H.3) *there exists a positive constant* δ_0 *such that if* $T_L \leq \delta_0$ *we have*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} \left[\det(I + H^{-1}W)\right]^{-1/2} b(\nu, x_L, x_0),$$

where $b(\nu, x_L, x_0)$ *satisfies the estimate: For any* $K \geq 0$ *there exists positive constants* $C_1(K)$ *and* $M(K)$ *such that if* $|\alpha_0|, |\alpha_L| \leq K$ *we have*

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} b(\nu, x_L, x_0)| \leq C_1(K)^L A_{M(K)}.$$

Remark. $C(K)$ and $M(K)$ are independent of $\{t_j\}, L, (x_L, x_0)$ and ν as long as $T_L \leq \delta_0$.

Above Lemma can be proved by modifying the proof of Theorem 6.8 in Chapt. 10 of Kumano-go [7].

Omitting the proof of lemma we proceed to the proof of Theorem 2. To make notations simpler we denote $\frac{\nu i}{2\pi}$ by E. With this notation we can write

$$\begin{aligned}
&I(\{t_j\}, S, a, \nu)(x_L, x_0) \\
&\quad = \prod_{j=1}^{L} \left(\frac{E}{t_j}\right)^{1/2} \int_{R^{L-1}} e^{-i\nu S(x_L,\dots,x_0)} a(x_L, \dots, x_0) \prod_{j=1}^{L-1} dx_j.
\end{aligned}$$

We perform integration over x_1-space. Using stationary phase method, we have

$$\begin{aligned}
&\prod_{j=1}^{2} \left(\frac{E}{t_j}\right)^{1/2} \int_R e^{-i\nu\{S_2(t_2,x_2,x_1)+S_1(t_1,x_1,x_0)\}} a(x_L, \dots, x_2, x_1, x_0) dx_1 \\
&= \left(\frac{E}{T(2,1)}\right)^{1/2} e^{-i\nu S^*_{21}(x_2,x_0)} \left(P_1 a(x_L, \dots, x_2, x_0) + R_1 a(x_L, \dots, x_2, x_0)\right).
\end{aligned}$$

Here $T(2,1) = t_2 + t_1$, $S^*_{21}(x_2, x_0)$ denotes the critical value of $S_2(t_2, x_2, x_1)$ $+S_1(t_1, x_1, x_0)$ with respect to the variable x_1, $P_1 a$ is the main part and $R_1 a$ is the remainder term of the stationary phase method.

Remark. (A) Clearly, we have

$$P_1(a)(x_L, \dots, x_2, x_0) = a(x_L, x_{L-1}, \dots, \overline{x_2, x_0}) D(S_1 + S_2; x_2, x_0)^{-1/2}$$

here

$$D(S_1+S_2; x_2, x_0) = 1 + \frac{t_1 t_2}{t_1 + t_2} \left(t_2 \partial^2_{x_1} \omega_2(t_2, x_2, x^*_1) + t_1 \partial^2_{x_1} \omega_1(t_1, x^*_1, x_0)\right).$$

(B) The remainder term $R_1 a$ is a very complicated function with respect to x_2 but is simple with respect to the variable $(x_L, \dots, x_3, x_0)$. In fact, we have $\partial_{x_j}(R_1 a) = R_1 \partial_{x_j} a$ for $j = 0$ and $3 \le j \le L$. And $R_1 a$ is small in the following sense: For any integer $K \ge 0$ there exists a constant C_K such that

$$\begin{aligned}
&| \, \partial^{\alpha_0}_{x_0} \partial^{\alpha_2}_{x_2} \dots \partial^{\alpha_L}_{x_L} R_1 a(x_L, \dots, x_2, x_0) \, | \\
&\quad \le C_K \nu^{-1} \frac{t_1 t_2}{t_1 + t_2} \max \sup_{x_1} | \, \partial^{\alpha_0}_{x_0} \partial^{\beta_1}_{x_1} \partial^{\beta_2}_{x_2} \partial^{\alpha_3}_{x_3} \dots \partial^{\alpha_L}_{x_L} a(x_L, \dots, x_2, x_1, x_0) \, | \, .
\end{aligned}$$

Here max is taken with respect to β_1, β_2 for $\beta_1 \le \alpha_2 + 4, \beta_2 \le \alpha_2$.

Next we integrate the term $P_1 a$ over x_2-space and apply the stationary phase method. We obtain

$$\begin{aligned}&\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}\\ &\qquad\int_R e^{-i\nu\{S_3(t_3,x_3,x_2)+S^*_{21}(x_2,x_0)\}}P_1 a(x_L,\dots,x_2,x_0)dx_2\\ &=\left(\frac{E}{T(3,1)}\right)^{1/2} e^{-i\nu S^*_{31}(x_3,x_0)}\\ &\qquad(P_2P_1a(x_L,\dots,x_3,x_0)+R_2P_1a(x_L,\dots,x_3,x_0)).\end{aligned}$$

Here $S^*_{31}(x_3,x_0)$ denotes the critical value of the function $x_2 \to S_3(t_3,x_3,x_2)+S^*_{21}(x_2,x_0)$, P_2P_1a is the main term and R_2P_1a is the remainder. Since P_2P_1a is a simple function of x_3, we integrate it over x_3-space and apply the stationary phase method. The main term includes $P_3P_2P_1a$ and the remainder includes $R_3P_2P_1a$.

Repeating this procedure $L-1$ times, we obtain

$$A_0(x_L,x_0)=\left(\frac{E}{T(L,1)}\right)^{1/2} e^{-i\nu S^*_{L1}(x_L,x_0)}P_{L-1}\dots P_1a(x_L,x_0),$$

which is nothing but the main term of Theorem 2.

Now we must treat the remainder term. Since R_1a is a complicated function of x_2, we skip integration over x_2 space and perform integration over x_3-space. Then we obtain

$$\begin{aligned}&\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}\\ &\int_R e^{-i\nu\{S_4(t_4,x_4,x_3)+S_3(t_3,x_3,x_2)+S^*_{21}(x_2,x_0)\}}R_1a(x_L,\dots,x_4,x_3,x_2,x_0)dx_3\\ &=\left(\frac{E}{T(4,3)}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2} e^{-i\nu\{S^*_{43}(x_4,x_2)+S^*_{21}(x_2,x_0)\}}\\ &\qquad(P_3R_1a(x_L,\dots,x_4,x_2,x_0)+R_3R_1a(x_L,\dots,x_4,x_2,x_0)).\end{aligned}$$

Here $S^*_{43}(x_4,x_2)$ denotes the critical value of the function $x_3 \to S_4(t_4,x_4,x_3)+S_3(t_3,x_3,x_2)$, P_3R_1a denotes the main term and R_3R_1a is the remainder. P_3R_1a is a simple function of the variable x_4 but R_3R_1a is not. We integrate P_3R_1a over x_4-space but we skip integration of R_3R_1a over x_4-space.

Similarly, we skip integration of R_2P_1a over x_3-space and integrate it over x_4-space. We obtain

$$\left(\frac{E}{t_5}\right)^{1/2}\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2}$$
$$\int_R e^{-i\nu\{S_5(t_5,x_5,x_4)+S_4(t_4,x_4,x_3)+S^*_{31}(x_3,x_0)}R_2P_1a(x_L,\ldots,x_4,x_3,x_0)dx_4$$
$$=\left(\frac{E}{T(5,4)}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2}e^{-i\nu\{S^*_{54}(x_5,x_3)+S^*_{31}(x_3,x_0)\}}$$
$$(P_4R_2P_1a(x_L,\ldots,x_5,x_3,x_0)+R_4R_2P_1a(x_L,\ldots,x_5,x_3,x_0)).$$

We continue this process. *The rule is that we apply the stationary phase method when we integrate over x_k-space and if R_k appears then we skip integration over x_{k+1}-space.* We finally obtain the following expression:

$$I(\{t_j\},S,a,\nu)(x_L,x_0)=A_0(x_L,x_0)+\sum^{*}A_{j_sj_{s-1}\ldots j_1}(x_L,x_0),$$

where $\sum^*$ denotes summation with respect to indices $(j_s,\ldots,j_1)$ satisfying

$$1<j_1<j_2-1<j_2<j_3-1<\ldots<j_s-1<j_s,$$

and each term is an oscillatory integral

$$A_{j_1j_2\ldots j_s}(x_L,x_0)$$
$$=\prod_{m=1}^{s}\left(\frac{E}{T(j_m,j_m-1)}\right)^{1/2}$$
$$\int_{R^s}e^{-i\nu S_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)}b_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)\prod_{m=1}^{s}dx_{j_m},$$

whose phase function is

$$S_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)$$
$$=S^*_{Lj_s}(x_L,x_{j_s})+S^*_{j_sj_{s-1}}(x_{j_s},x_{j_{s-1}})+\ldots+S^*_{j_10}(x_{j_1},x_0)$$

and the amplitude is

$$b_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)=Q_{L-1}Q_{L-2}\ldots Q_1a(x_L,x_{j_s},\ldots,x_{j_1},x_0),$$

with

$$Q_j = \begin{cases} Id, & \text{for } j = j_s, j_{s-1}, \dots, j_1, \\ R_j, & \text{for } j = j_s - 1, j_{s-1} - 1, \dots, j_1 - 1, \\ P_j, & \text{otherwise.} \end{cases}$$

Furthermore, we can prove that $b_{j_s \dots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0)$ satisfies hypothesis (H.3).

Proposition. *For any integer $K \geq 0$ there exist positive constants $C_2(K)$ and integer $m(K)$ such that*

$$\begin{aligned} &| \partial_{x_L}^{\alpha_L} \partial_{x_{j_s}}^{\alpha_{j_s}} \dots \partial_{x_{j_1}}^{\alpha_{j_1}} \partial_{x_0}^{\alpha_0} b_{j_s \dots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0) | \\ \leq & C_2(K)^s A_{m(K)} \prod_{k=1}^{s} \nu^{-1} t_{j_k}. \end{aligned}$$

Now we apply our key lemma to $A_{j_s j_{s-1} \dots j_1}(x_L, x_0)$ and use the proposition above. Then we obtain

$$A_{j_s j_{s-1} \dots j_1}(x_L, x_0) = \left(\frac{E}{T_{L,1)}} \right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} a_{j_s j_{s-1} \dots j_1}(x_L, x_0),$$

where the function $a_{j_s j_{s-1} \dots j_1}(x_L, x_0)$ satisfies the following estimates: For any integer $K \geq 0$ we have

$$| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} a_{j_s j_{s-1} \dots j_1}(x_L, x_0) | \leq C_1(K)^s C_2(M(K))^s A_{m(M(K))} \prod_{k=1}^{s} \nu^{-1} t_{j_k}.$$

This implies that the remainder term $r(\nu, x_L, x_0)$ can be written as

$$r(\nu, x_L, x_0) = \sum^{*} a_{j_s j_{s-1} \dots j_1}(x_L, x_0).$$

If $\alpha_0, \alpha_L \leq K$ we have

$$\begin{aligned} | \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} r(\nu, x_L, x_0) | \leq & \sum^{*} | \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} a_{j_s j_{s-1} \dots j_1}(x_L, x_0) | \\ \leq & \sum^{*} C_3(K)^s A_{m(M(K))} \prod_{k=1}^{s} \nu^{-1} t_{j_k} \\ \leq & A_{m(M(K))} \left(\prod_{j=1}^{L} (1 + C_3(K) \nu^{-1} t_j) - 1 \right), \end{aligned}$$

where we abbreviated $C_1(K)C_2(M(K))$ as $C_3(K)$. This proves Theorem 2.

Theorem 1 can be proved similarly.

More detailed dicussions are given by [2].

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