Advanced Studies in Pure Mathematics 23, 1994
Spectral and Scattering Theory and Applications
pp. 17–26

Asymptotics for the Painlevé II Equation: Announcement of Result

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Submitted in honor of Professor S.T. Kuroda, from whom we have learned so much

1. Introduction

In this paper we study the asymptotics of a class of solutions of the (homogeneous) Painlevé II (PII) equation

$$u_{xx} = 2u^2 + xu \ , \qquad x \in \mathbb{R} \ , \tag{1.1}$$

as $x \to \pm\infty$. Following the work of Flaschka and Newell [FN] and Jimbo, Miwa and Ueno [JMU] the PII equation can be solved by means of a Riemann-Hilbert (RH) factorization problem as follows ([FA]; see also [IN]). Let $\Sigma^{(1)}$ denote the oriented contour consisting of six rays, $\Sigma^{(1)} = \bigcup_{k=1}^{6} \left\{ \Sigma_j^{(k)} = e^{i(k-1)\pi/3}\mathbb{R}_+ \right\}$,

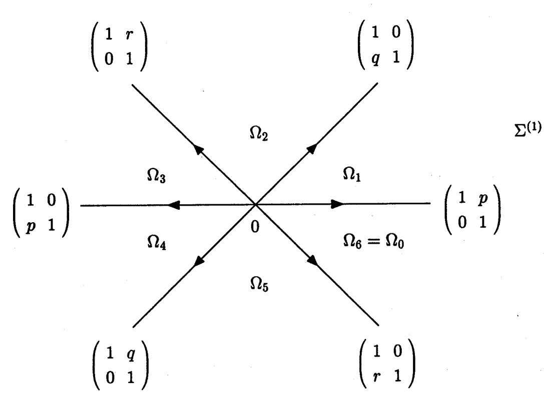

Fig. 1.2

Received February 8, 1993.

with associated jump matrix $v^{(1)}\colon \Sigma^{(1)} \to M_2(\mathbb{C})$, $v^{(1)} \restriction \Sigma_1^{(1)} = \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$, etc., where p, q and r are complex numbers satisfying the relation

$$p + q + r + pqr = 0 \ . \tag{1.3}$$

For $x \in \mathbb{R}$ and $z \in \Sigma^{(1)}$, set

$$\left.\begin{aligned} v_x^{(1)}(z) &= e^{-i(\frac{4z^3}{3}+xz)\sigma_3}\, v^{(1)}(z)\, e^{i(\frac{4}{3}z^3+xz)\sigma_3} \\ &\equiv e^{-i(\frac{4z^3}{3}+xz)ad\,\sigma_3}\, v^{(1)}(z) \end{aligned}\right\} \tag{1.4}$$

where σ_3 is the Pauli matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Now let $m^{(1)}(z) = m^{(1)}(x,z)$ be a (2×2 matrix valued) holomorphic function defined on $\mathbb{C} \setminus \Sigma^{(1)}$ solving the RH problem

$$\left.\begin{aligned} &m_+^{(1)}(z) = m_-^{(1)}(z)\, v_x^{(1)}(z) \ , \quad 0 \neq z \in \Sigma^{(1)} \ , \\ &m^{(1)}(z) \to I \quad \text{as} \quad z \to \infty \ , \end{aligned}\right\} \tag{1.5}$$

where $m_+^{(1)}(z)$ (resp. $m_-^{(1)}(z)$) denotes as usual the boundary value of $m^{(1)}(z)$ from the left (resp. right) side of the oriented contour $\Sigma^{(1)}$. (Thus for $z \in \mathbb{R}_+$ in particular, we have $m_\pm^{(1)}(z) = \lim_{\epsilon\downarrow 0} m^{(1)}(z \pm i\epsilon)$, etc.). Then

$$u(x) \equiv 2i(m_1^{(1)}(x))_{12} = -2i(m_1^{(1)}(x))_{21} \ , \tag{1.6}$$

solves PII, where

$$m^{(1)}(z) = m^{(1)}(z;x) = I + \frac{m_1^{(1)}(x)}{z} + O(\frac{1}{z^2}) \tag{1.7}$$

as $z \to \infty$.

For general p, q, r and x, $p+q+r+pqr = 0$, $x \in \mathbb{R}$, the RH problem (1.5) may fail to have a solution. However, every (local) solution of the Cauchy problem for (1.1) can be obtained from the RH problem for suitable p, q and r by the above prescription. Indeed (see e.g. [FA], [IN]) there is an inejctive map (the *Direct Transform*)

$$\begin{aligned} &(u(0), u'(0)) \mapsto \\ &\quad (p_0 = p(u(0),u'(0)),\ q_0 = q(u(0),u'(0)),\ r_0 = r(u(0),u'(0))) \\ &\qquad \in \{(p,q,r)\colon p+q+r+pqr = 0\} \end{aligned} \tag{1.8}$$

with the property that the RH problem (1.5) with $p = p_0$, $q = q_0$ and $r = r_0$ has a solution for all x in a neighborhood of zero, and if $u(x; p_0, q_0, r_0)$

is the solution of PII obtained from (1.6), then $u(x; p_0, q_0, r_0)$ is the unique (local) solution of PII with the given initial data $(u(0), u'(0))$. Moreover, allowing complex values for x in (1.4), $u(x; p_0, q_0, r_0)$ gives a meromorphic continuation of the solution to the entire complex plane.

We are interested in particular in solutions of (1.1) that exist for all x in $\mathbb{R}$. A sufficient condition (see [FZ]) for the RH problem (1.5) to have a solution for all $x \in \mathbb{R}$, is that

$$|q - \bar{p}| < 2 \quad \text{and} \quad r \in \mathbb{R} . \tag{1.8}$$

Real solutions of PII correspond to RH data with the symmetry

$$p = -\bar{q} , \qquad r \in i\mathbb{R} , \tag{1.9}$$

and pure imaginary solutions correspond to data with

$$p = \bar{q} , \qquad r \in \mathbb{R} \tag{1.10}$$

(cf. [FA], [IN]). From (1.3), (1.8), (1.9) and (1.10), we see that for any real q,

$$-1 < q < 1 , \qquad p = -q , \qquad r = 0 \tag{1.11}$$

formula (1.6) leads to a global, real solution of PII, while for any $q \in \mathbb{C}$

$$q , \quad p = \bar{q} , \quad r = -\left[(q + \bar{q})/(1 + |q|^2)\right] \in \mathbb{R} \tag{1.12}$$

formula (1.6) leads to a global, purely imaginary solution of PII. Furthermore (see below) a special argument shows that for

$$q = \pm 1 , \qquad p = \mp 1 , \qquad r = 0 \tag{1.13}$$

(1.6) also leads to a global, real solution of (1.1).

We will study the asymptotic behavior of the solutions of PII in these three cases (1.1), (1.2) and (1.3). The results are as follows.

Theorem 1.14 (global real solutions). *For*

$$-1 < q < 1 , \quad p = -q \quad \textit{and} \quad r = 0 ,$$

(1.15)

$$u(x) = \frac{\sqrt{2\nu}}{(-x)^{1/4}} \cos\left(\frac{2}{3}(-x)^{3/2} - \frac{3}{2}\nu \log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right)$$

$$\textit{as} \quad x \to -\infty ,$$

where

$$\nu = \nu(q) = \frac{-1}{2\pi} \log(1 - q^2) \tag{1.16}$$

$$\phi = -3\nu \log 2 + \log \Gamma(i\nu) + \frac{\pi}{2} \operatorname{sgn} q - \frac{\pi}{4} \tag{1.17}$$

(*here* Γ *denotes the Gamma function*) *and*

$$u(x) = \frac{q}{2\sqrt{\pi} x^{1/4}} e^{-(2/3)x^{3/2}} (1 + O(1)) \quad as \quad x \to +\infty . \tag{1.18}$$

Theorem 1.19 (global purely imaginary solutions). *For*

$$q \in \mathbb{C} , \quad p = \bar{q} \quad and \quad r = -[(q + \bar{q})/(1 + |q|^2)]$$

(1.20)

$$u(x) = \frac{i(-2\nu)^{1/2}}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} - \frac{3}{2}\nu \log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right) \quad as \quad x \to -\infty ,$$

where

$$\nu = \frac{-1}{2\pi} \log(1 + |q|^2) \tag{1.21}$$

and

$$\phi = -3\nu \log 2 + \frac{\pi}{4} + \arg \Gamma(i\nu) - \arg q . \tag{1.22}$$

For $\operatorname{Re} q \neq 0$ (*equivalently* $r \neq 0$)

(1.23)

$$u(x) = \sigma i \sqrt{\frac{x}{2}} - \frac{\sigma i \sqrt{\nu}}{(2x)^{1/2}} \cos\left(\frac{2\sqrt{2}}{3} x^{3/2} - \frac{3}{2}\nu \log x + \phi\right) + O\left(\frac{1}{x^{(1/2)-\epsilon}}\right) \quad as \quad x \to +\infty$$

where ϵ *is any positive number and*

$$\nu = \frac{1}{\pi} \log \frac{1 + |q|^2}{2|\operatorname{Re} q|} \tag{1.24}$$

$$\phi = \frac{\pi}{4} - \frac{7}{2}\nu \log 2 + \arg \Gamma(i\nu) + \arg(1 - q^2) , \tag{1.25}$$

$$\sigma = \operatorname{sgn}(\operatorname{Re}\, q)\ , \tag{1.26}$$

and for Re $q = 0$ (*equivalently* $r = 0$)

$$u(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-(2/3)x^{3/2}} (1 + o(1)) \qquad as \qquad x \to +\infty\ . \tag{1.27}$$

Theorem 1.28 (global real solutions: singular case). *For*

$$q = \pm 1\ , \quad p = \mp 1 \quad \text{and} \quad r = 0\ ,$$

$$u(x) = \pm\Big[\Big(-\frac{x}{2}\Big)^{1/2} - \frac{1}{2^{7/2}}(-x)^{-5/2} + O\left((-x)^{-11/2}\right)\Big] \qquad as \qquad x \to -\infty \tag{1.29}$$

and

$$u(x) = \pm\frac{1}{2\sqrt{\pi}x^{1/4}} e^{-(2/3)x^{3/2}} (1 + o(1)) \qquad as \qquad x \to +\infty\ . \tag{1.30}$$

Theorem 1.14 is due to Ablowitz and Segur (see [SA1], [SA2]). A rigorous justification of the beautiful heuristic calculations in [SA1], [SA2] is given in [HM] and [CM], at least up to the phase shift (1.17), using a Gelfand-Levitan type equation derived earlier by Ablowitz and Segur in [AS] Theorem 1.28, at least to leading order in x, is due to Hastings and Mcleod and appears in [HM]. A Gelfand-Levitan approach is only possible in the special case of Theorems 1.14 and 1.28 when $r = 0$ and the contour $\Sigma^{(1)}$ for the RH problem can be reduced to a single line. Theorem 1.19 is due to Its and Kapaev [IK]. In the case $r \neq 0$, the contour does not reduce to a line and the RH problem must be solved directly as a RH problem on a nontrivial contour with self-interactions. The authors in [IK] use the so called "isomonodromy method" which they have developed, together with Novokshenov and others, in a wonderful series of papers over the last eight or nine years. An exposition of the method, together with a discussion of the many results that have been obtained, can be found in [IN]. The method is a descendent of the original method of Zakharov and Manikov [ZM] which they derived in analyzing the long-time behavior of the nonlinear Schrödinger equation. Another derivation of Theorem 1.14, using the isomonodromy method, was given by Suleimanov [S]. We note, however, that certain technical difficulties in [IK] and [S] remain, and a completely rigorous justification of the isomonodromy method poses a deep and very interesting challenge.

The above results solve the so-called "connection problem" for global solutions of PII. For example, if we observe the asymptotics of a purely imaginary solution of P II as $x \to -\infty$, then ν and ϕ in (1.20) are known and hence q can be computed from (1.21) and (1.22), and so $\nu(q)$, $\phi(q)$ and $\sigma(q)$ can be determined from (1.24), (1.25) and (1.26), which yields the asymptotics as $x \to +\infty$ through (1.23). Conversely if we observe the asymptotics of a purely imaginary solution as $x \to +\infty$, then ν, ϕ and σ in (1.23) are known. But then $r = \dfrac{-2\,\mathrm{Re}\, q}{1+|q|^2}$ is known from (1.24) and (1.26). This relation can be rewritten as $|r^{-1}+q|^2 = r^{-2}-1$. On the other hand $\arg(1+rq) = \arg((1-q^2)/1+|q|^2) = \arg(1-q^2)$, and hence $\arg(r^{-1}+q)$ can be determined from (1.25). Thus $r^{-1}+q$ and, hence q, is known. Substitution in (1.21) and (1.22) then yields the asymptotics as $x \to -\infty$ through (1.20). Thus if we know the behavior of the solution as $x \to +\infty$ (resp. $x \to -\infty$), we can "connect" the solution to its asymptotics as $x \to -\infty$ (resp. $x \to +\infty$).

The six Painlevé equations PI-PVI were introduced by Painlevé and Gambier at the beginning of this century on purely mathematical grounds, but recently they have appeared in a wide range of physical applications, including self-similar solutions of the Korteweg deVries equation, correlation functions for the transverse Ising chain in the infinite temperature limit, nonperturbative 2D quantum gravity, amongst many others. A comprehensive survey of recent results and applications of Painlevé equations can be found in [FI]. It is increasingly recognized that the Painlevé equations play a role in modern mathematical physics analogous to the role played by the classical special functions of the last century. Many of the applications of classical special function theory rest on the fact that the asymptotics and the associated connection problem for the special functions can be solved explicitly. Theorems 1.14, 1.19 and 1.28 should be viewed as providing the analogous information for P II.

We now make some additional remarks about the asymptotic formulae in the above theorems. Note that if $q = \pm 1$, $p = \pm 1$ and $r = \mp 1$ in (1.24)–(1.26) then $\nu = 0$ and the lower order oscillatory term in (1.23) is absent. This case plays a special role in our analysis of PII and provides a model problem by means of which the general case of Theorem 1.19 can be analyzed. Moreover, in this case, the solution has full fractional expansion as $x \to +\infty$ identical in form to (1.29).

It is interesting to consider Theorems 1.14 and 1.28 from the following point of view. Observe that for large positive x, PII reduces to the Airy equation. *Question* (see [HM]): for real q does there exist a real global solution of P II that is asymptotic to $qA_i(x)$ as $x \to +\infty$? (Here

$A_i(x)$ is the standard Airy function.) Theorems 1.14 and 1.28 show that this is so for $|q| \leq 1$ (the asymptotics for $A_i(x)$ can be found, for example, in [AbSt]). However, if $|q| > 1$ and the solution $u(x) \sim q\, A_i(x)$ as $x \to +\infty$, then $u(x)$ must blow up for some x. This result is due to Hastings and McLeod (see [HM]). In fact, as shown by Kapaev and Novokshenov, $u(x)$ blows up at an infinite number of points $x_n \to -\infty$ (see [KN], [IN]).

An analysis of the asymptotics of solutions $u(x)$ of P II as $x \to \infty$ along a ray in the complex plane has been given by Boutroux [B]. Further interesting developments can be found in Novokshenov [N] and Kapaev [K], who use the isomonodromy method.

Recently the authors have introduced a new and general nonlinear steepest descent-type method for analyzing the asymptotics of oscillatory RH problems [DZ1]. The method has been used to derive rigorously the long-time asymptotics for the modified Korteweg de Vries (MKdV) equation [DZ1], for the nonlinear Schrödinger (NLS) equation [DIZ], and for the doubly infinite, compactly perturbed Toda lattice [Kam]. The method has also been used to announce the derivation of the collisionless shock region of Ablowitz-Segur for the Korteweg de Vries (KdV) equation [DZ3], and to obtain the long-time asymptotics for the autocorrelation function of the transverse Ising chain at the critical magnetic field [DZ2].

As indicated above, many of the results in Theorems 1.14, 1.19 and 1.28 have not yet been justified rigorously. Moreover, the methods of the authors, and in particular the isomonodromy method, require an a priori ansatz for the form of the solution. The purpose of this paper is to derive Theorems 1.14, 1.19 and 1.28 rigorously and directly with error bounds using the steepest descent method of [DZ1]. Our approach is algorithmic and requires no ansatz for the asymptotic form of the solution. The method proceeds by deforming contours, and in the simplest cases, we are left with the localized RH problem near the points of stationary phase. These localized RH problems can then be solved explicitly in terms of classical special functions. This is the case for MKdV in the similarity region and also for the asymptotics on (1.15) and (1.20). This is not the case, however, for the asymptotics in (1.23) and (1.29): here the RH problem localizes on a line segment rather than at the stationary phase points. A similar situation arises in the analysis of the collisionless shock region in KdV (see [DZ3]). This is a new and essentially nonlinear feature of the steepest descent method, and its resolution occupies the main part of the work.

Acknowledgments. The authors would like to thank A.R. Its for many useful and informative discussions. The work of the authors was supported in part by NSF Grants DMS-9203771 and DMS-9204804, respecively. The second author was also supported in part by a Yale Junior Faculty Fellowship.

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