Afrika Statistika

Afrika Statistika
Vol. 12 (3), 2017, pages 1367–1396.
DOI: http://dx.doi.org/10.16929/as/2017.1367.109

ISSN 2316-090X

The Kumaraswamy G Exponentiated Gumbel Type-2 Distribution

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Received July 04, 2017; August 1, 2017

Abstract. The distribution due to Okorie *et al.* (2016) is further extended to a wider family of distribution called the Kumaraswamy Generalized Exponentiated Gumbel type-2 distribution. Twenty two distributions are identified as sub-models of the new distribution. Some of its important statistical properties are explicitly derived and the parameters of the new distribution are estimated through the method of maximum likelihood estimation.

Résumé. Une distribution de probabilité proposée par Okorie *et al.* (2016) est étendue à une large famille de function de répartitions dénommée Distribution Généralisée Exponentielle Gumbel-Exponentielle de kumaraswamy de Type-2 distribution qui regroupe au moins vingt-deux en tant que sous-modèles. Une série de proptiétés statistiques importantes de cette famile sont étudiées, parmi lesquelles l'estimation de paramètres par la méthode du maximum de vraisemblance.

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Key words: Gumbel type-2; Kumaraswamy; Weibull; Fréchet; Inverse exponential and Rayleigh distribution.
AMS 2010 Mathematics Subject Classification : 60E05; 62H12; 62F10

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1. Introduction

The Gumbel type-2 distribution is a very useful model in Extreme value theory. The distribution is used for modeling extreme events in the field of Meteorology and Seismology. The distribution could also be applied in Risk management-Operational risk and life testing, for modelling lifetime data-sets with monotonic failure (or hazard) rates. On the contrary, the hazard rate function of so many complex phenomena that are regularly encountered in practice are non-monotone and cannot be modelled by the Gumbel type-2 distribution. To overcome this limitation, Okorie *et al.* (2016) proposed the Exponentiated Gumbel (EG) type-2 distribution according to Nadarajah and Kotz (2006) version of Gupta *et al.* (1998). The cumulative distribution function *(cdf)* and probability density function *(pdf)* of the EG type-2 distribution is given by

$$G(x|\alpha,\theta,\phi) = 1 - \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha} \tag{1}$$

and

$$g(x|\alpha,\theta,\phi) = \alpha\phi\theta x^{-\phi-1} e^{-\theta x^{-\phi}} \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha-1} \tag{2}$$

respectively. Where α, ϕ, $\theta > 0$ and $x > 0$, α and ϕ are the shape parameters while θ is the scale parameter
In probability and statistics, the Kumaraswamy's double bounded distribution due to Kumaraswamy (1980) is a continuous probability distributions defined on [0,1]. It shares several similarity with the Beta distribution. But, the structural simplicity and analytical tractability of the Kumaraswamy distribution makes it a favourable alternative and a competitor of the Beta distribution. The *cdf* and *pdf* of the Kumaraswamy distribution is given by

$$F'(x|a,b) = 1 - [1 - x^a]^b$$

and

$$f'(x|a,b) = abx^{a-1}(1 - x^a)^{b-1}$$

respectively. Where a(shape)> 0, b(shape)> 0 and $x \in [0,1]$.
In this paper we introduce and study a five-parameter distribution called the Kumaraswamy Generalized Exponentiated Gumbel (KwGEG) type-2 distribution as an expansion of the EG type-2 distribution. The new distribution is in principle according to the work of Cordeiro and de Castro (2011)-Kumaraswamy G (KwG) distributions. The *cdf* and *pdf* of the KwG distributions is defined as

$$F(x|a,b) = 1 - [1 - G^a(x)]^b \tag{3}$$

and

$$f(x|a,b) = abg^{a-1}(x)[1 - G^a(x)]^{b-1} \tag{4}$$

respectively. Where x is in the range of $g(x)$ and a and b are as defined above, while $G(x)$ and $g(x)$ corresponds to the *cdf* and *pdf* of the baseline distribution respectively.

The family of KwG distributions was motivated to model the failure time of a certain series system with b units where each unit has a functional subunits in parallel arrangement. The time to failure of the subunits are independent and identically distributed according to $g(x)$. This implies that the entire system fails if one out of the b components of the system fails and this can only happen if and only if all the subcomponents of a fails. i.e.;

$$\begin{aligned} F(x) &= \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X_1 > x, \ldots, X_b > x) \\ &= 1 - [1 - \mathbb{P}(X_1 \leq x)]^b \\ &= 1 - [1 - \mathbb{P}(X_{11} \leq x, \cdots, X_{1a} \leq x)]^b \\ &= 1 - [1 - G^a(x)]^b. \end{aligned}$$

For a fair account of several distributions extended according to the KwG distributions readers are referred to Nadarajah and Rocha (2015). Let *G(x)* be the *cdf* of the EG type-2 distribution in Equation (1). The *cdf* of the KwGEG type-2 distribution is defined by substituting Equations (1) and (2) into Equations (3) and (4) as

$$F(x|a,b,\alpha,\theta,\phi) = 1 - \left(1 - \left[1 - \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha}\right]^{a}\right)^{b}, \; x > 0. \tag{5}$$

Hence, the KwGEG type-2 density function with five parameters $a > 0,\ b > 0,\ \alpha > 0,\ \theta > 0,$ and $\phi > 0$ is given by

$$\begin{aligned} f(x|a,b,\alpha,\theta,\phi) &= ab\alpha\theta\phi x^{-\phi-1} e^{-\theta x^{-\phi}} \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha-1} \left(1 - \left[1 - e^{-\theta x^{-\phi}}\right]^{\alpha}\right)^{a-1} \\ &\cdot \left(1 - \left[1 - \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha}\right]^{a}\right)^{b-1}, \; x > 0. \end{aligned} \tag{6}$$

If X is a random variable with density in Equation (6), we write $X \sim$ KwGEG type-2 $(a, b, \alpha, \theta, \phi)$. The corresponding reliability and hazard functions are

$$R(x|a,b,\alpha,\theta,\phi) = \left(1 - \left[1 - \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha}\right]^{a}\right)^{b} \tag{7}$$

and

$$\begin{aligned} h(x|a,b,\alpha,\theta,\phi) &= ab\alpha\theta\phi x^{-\phi-1} e^{-\theta x^{-\phi}} \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha-1} \left(1 - \left[1 - e^{-\theta x^{-\phi}}\right]^{\alpha}\right)^{a-1} \\ &\cdot \left(1 - \left[1 - \left(1 - e^{-\theta x^{-\phi}}\right)^{\alpha}\right]^{a}\right)^{-1} \end{aligned} \tag{8}$$

respectively. The plot of the *pdf*, *cdf*, reliability and hazard rate function are shown in Figure 1, 2, 3 and 4, respectively.

The remainder of this paper contains the following sections: Section 2 gives some properties of the new distribution including a list of special cases, Section 3 presents the order statistics, Section 4 discuss some estimation issues and Section 5 is the conclusion.

2. Properties of the distribution

This section contains some of the properties of the KwGEG type-2 distribution.

2.1. Asymptotics and Shapes

When $\phi = 1$ and $a > 0$, the limiting behaviour of the density and hazard function of the Kw-EG type-2 distribution is readily established as

$$\lim_{x\to 0} f(x) = \begin{cases} 0 & \text{if } \alpha > 1 \text{ and } b \geq 1, \\ \infty & \text{if } \alpha \leq 1 \text{ and } b \geq 1, \end{cases} \qquad \lim_{x\to\infty} f(x) = 0 \; \forall \; a,\; b,\; \alpha,\; \theta \text{ and } \phi > 0$$

and

$$\lim_{x\to 0} h(x) = \lim_{x\to\infty} h(x) = \infty, \; \forall \; a,\; b,\; \alpha,\; \theta \text{ and } \phi > 0$$

respectively.

For certain parameter values of a, b, α, θ and ϕ, the shapes of the density function is affected, as it could either take a unimodal or monotonic decreasing shapes, where symmetry and substantial tail variability is also evident as depicted in Figure 1. The hazard rate function could be unimodal, bathtub or upside-down bathtub shaped. This attractive shape characteristics suggests that the KwGEG type-2 distribution is suitable for modelling data-sets with non-monotonic hazard rate behaviours which are mostly encountered in practical situations.

2.2. Useful expansion of the density and cumulative density function

To motivate analytical derivation of some basic distributional properties of the KwGEG type-2 distribution we present the series representation of its *pdf* and *cdf* through the well-known generalized binomial expansion

$$(1-y)^{\beta} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma(\beta+1)}{\ell!\Gamma(\beta-\ell+1)} y^{\ell}; \; |y| < 1, \; \beta > 0,$$

where $(\beta > 0) \in \mathbb{R}$ is a non-integer. If β is an integer then the sum terminate at $\ell = \beta$. Thus, for $a, b \in \mathbb{R}$ (non-integer values) we obtain the *pdf* as

$$\begin{aligned} f(x|a,b,\alpha,\theta,\phi) &= ab\alpha\theta\phi\Gamma(a)\Gamma(b) \sum_{i,j,\ell=0}^{\infty} \frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\ &\quad \cdot x^{-\phi-1}e^{-\theta x^{-\phi}} \left[1-e^{-\theta x^{-\phi}}\right]^{\alpha(i+1)-1} \left[1-e^{-\theta x^{-\phi}}\right]^{a\ell} \end{aligned} \tag{9}$$

and the *cdf* as

$$F(x|a,b,\alpha,\theta,\phi) = 1-\Gamma(b)\sum_{i,j=0}^{\infty} \frac{\Gamma(ai)}{i!j!\Gamma(b-i+1)\Gamma(ai-j+1)} \left[1-e^{-\theta x^{-\phi}}\right]^{\alpha j} \tag{10}$$

If a, and b are integers then the sums in Equation (9) terminate at $i = a-1$, $j = b-1$ and $\ell = aj$, while the sums in Equation (10) terminate at $i = b$ and $j = ai$.

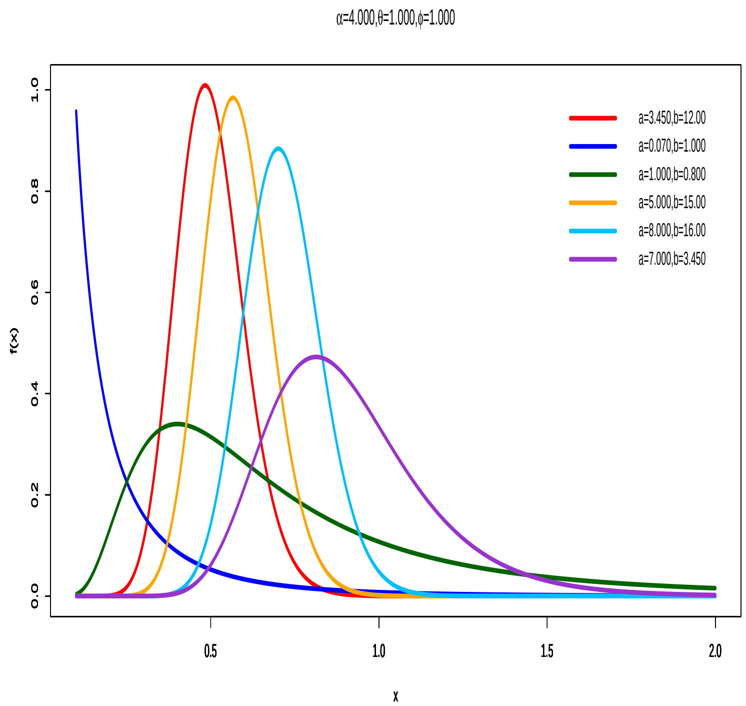

Fig. 1. *pdf* plot of the KwGEG type-2 distribution for some parameter values

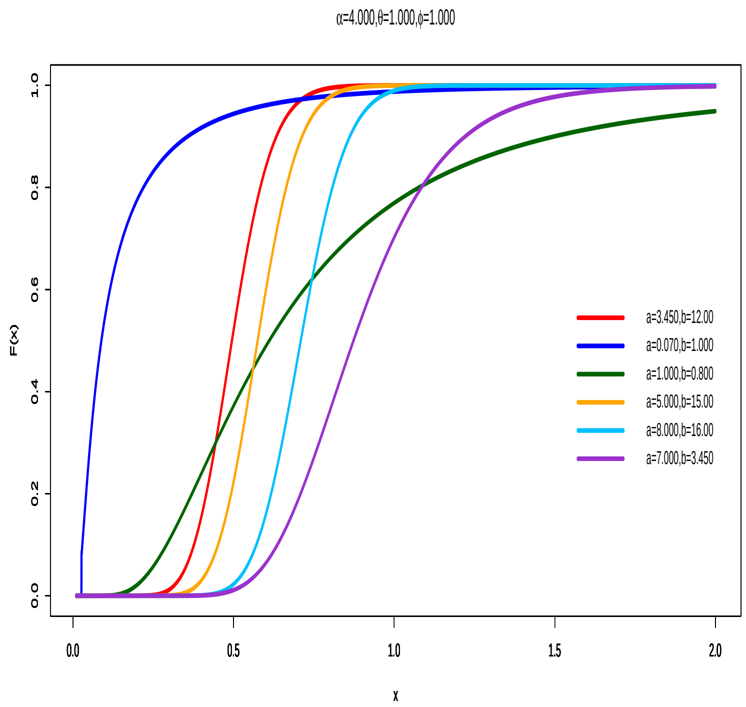

Fig. 2. *cdf* plot of the KwGEG type-2 distribution for some parameter values

2.3. Quantile function and random number generation

By inverting Equation (5) we obtain the quantile function as

$$X(Q) = F^{-1}(Q) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq Q\}$$
$$= \left[-\frac{1}{\theta}\log\left(1-\left[1-\left(1-[1-Q]^{\frac{1}{b}}\right)^{\frac{1}{a}}\right]^{\frac{1}{\alpha}}\right)\right]^{-\frac{1}{\phi}};\ Q \in (0,1]. \quad (11)$$

For selected parameter values $a,\ b,\ \alpha,\ \theta$ and ϕ, the numerical quantile values of the KwGEG type-2 distribution have been computed and listed in Table 2. When $Q = 1/2$ the quantile function reduces to the median and the numerical median values of the KwGEG type-2 distribution for some parameter values have been computed and reported in Tables 2 and 3. It is straightforward to simulate random variables from the KwGEG type-2 distribution by using $X(Q)$. For instance, if $U \sim U(0,1)$ then, it follows that $X(U)$ ∼KwGEG type-2 distribution.

An alternative measure of skewness and kurtosis statistics could be obtained through the quantile function for certain fractiles values, as the limitation of the classical measures are well-known. The Bowley skewness due to Bowley (1920) is given by

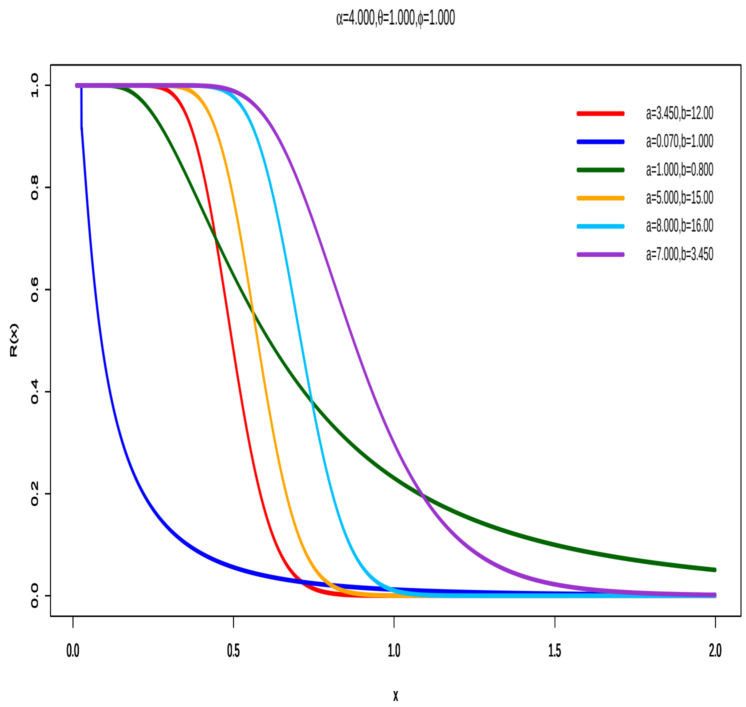

Fig. 3. *Reliability function* plot of the KwGEG type-2 distribution for some parameter values

$$B = \frac{X(3/4) + X(1/4) - 2X(2/4}{X(3/4) - X(1/4)}$$

while the Moors kurtosis due to Moors (1986) is given by

$$M = \frac{X(3/8) - X(1/8) + X(7/8) - X(5/8)}{X(6/8) - X(2/8)}$$

The advantage of the Bowley skewness and Moors kurtosis over the classical measures of skewness and kurtosis is that, they can be computed even in situation where the moments of the distribution does not exist and they are not reasonably affected by extreme values.

2.4. Sub-models

There is no question about the flexibility of the KwGEG type-2 distribution. The new distribution contains several new and existing sub-distributions. In this section we have identified and listed some of the special cases of the new distribution in Table 1.

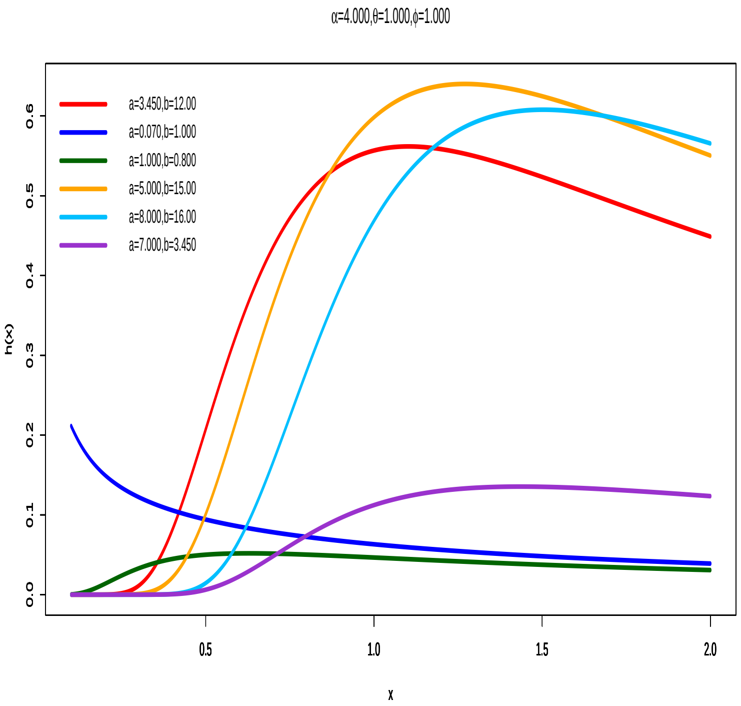

Fig. 4. *hrf* plot of the KwGEG type-2 distribution for some parameter values

2.5. Moments and generating function

In theory and practice, the moments of a random variable X, say is very useful because, many other characteristics of the distribution depends on it.

Theorem 1. *If X is distributed according to the KwGEG type-2 distribution in Equation (6) then, its kth crude moment is given by*

$$\begin{aligned} E(X^k) &= ab\alpha\theta^{\frac{k}{\phi}}\Gamma(a)\Gamma(b)\sum_{i,j,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\ &\cdot\frac{\partial^k}{\partial c^k}\left[\mathbf{B}\left(1-\frac{c}{\phi},\alpha[i+1]\right){}_2\mathsf{F}_1\left(-a\ell;1-\frac{c}{\phi};\alpha[i+1]-\frac{c}{\phi}+1;1\right)\right]\Big|_{c=0} \end{aligned}$$

Proof. We have

Table 1. Numerical quantiles of the KwGEG type-2 distribution for fixed $\theta = 2$ and $\phi = 4.5$ and selected α, a and b parameter values

	$X(Q) \rightarrow [a,\ b,\ \alpha,\ \theta,\ \phi] \downarrow$					
$Q \downarrow$	$[6.8, 1, 3]$	$[2.45, 6, 1.5]$	$[1, 3.8, 0.5]$	$[5.5, 8.34, 9]$	$[3, 0.24, 1]$	$[0.15, 0.8, 0.45]$
0.1	1.147164	0.9972949	0.9193547	0.9323342	1.477839	0.6580239
0.2	1.192949	1.0355985	0.9790485	0.9468896	1.735577	0.7227468
0.3	1.229813	1.0643626	1.0281433	0.9571645	2.014825	0.7828473
0.4	1.264369	1.0897442	1.0749034	0.9658042	2.356609	0.8474751
0.5	1.299634	1.1141875	1.1232970	0.9737739	2.811270	0.9239990
0.6	1.338211	1.1393852	1.1769598	0.9816544	3.469886	1.0233706
0.7	1.383733	1.1672584	1.2411987	0.9900036	4.536791	1.1678054
0.8	1.443560	1.2012112	1.3270609	0.9996922	6.607815	1.4170520
0.9	1.540773	1.2510065	1.4699666	1.0130373	12.555822	2.0286563

$$\begin{aligned} E(X^k) &= \int_{\text{all x}} x^k f(x) dx \\ &= ab\alpha\theta\phi\Gamma(a)\Gamma(b) \sum_{i,j,\ell=0}^{\infty} \frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\ &\cdot \int_0^{\infty} x^{k-\phi-1} e^{-\theta x^{-\phi}} \left[1 - e^{-\theta x^{-\phi}}\right]^{\alpha(i+1)-1} \left[1 - e^{-\theta x^{-\phi}}\right]^{a\ell} dx \end{aligned} \tag{12}$$

setting $y = \theta x^{-\phi}$ in Equation (12) we have

$$\begin{aligned} E(X^k) &= -ab\alpha\theta^{\frac{k}{\phi}}\Gamma(a)\Gamma(b) \sum_{i,j,\ell=0}^{\infty} \frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\ &\cdot \int_0^{\infty} y^{-\frac{k}{\phi}} e^{-y} \left[1 - e^{-y}\right]^{\alpha(i+1)-1} \left[1 - e^{-y}\right]^{a\ell} dy \end{aligned} \tag{13}$$

setting $z = e^{-y}$ in Equation (13) we have

Table 2. Special cases of the KwGEG type-2 Distribution

S/No.	a	b	α	θ	ϕ	Distributions
1	1	1	1	θ	ϕ	Gumbel type-2
2	1	1[b]	α[1]	θ	ϕ	Exp Gumbel type-2 due to Okorie *et al.* (2016)
3	a	b	1	θ	ϕ	Kumaraswamy Gumbel type-2 (new)
4	a	1	α	θ	ϕ	Generalized Exp Gumbel type-2 (new)
5	1	1	1	1	ϕ	Fréchet
6	1	1[b]	α[1]	1	ϕ	Exp Fréchet due to Nadarajah and Kotz (2006)
7	a	b	1	1	ϕ	Kumaraswamy Fréchet due to **?**
8	a	1	α	1	ϕ	Generalized Exp Fréchet (new)
9	1	1	1	θ	1	Inverse Exponential
10	1	1[b]	α[1]	θ	1	Exp Inverse Exponential Flaih *et al.* (2012)
11	a	b	α	θ	1	Kumaraswamy Exp Inverse Exponential (new)
12	a	b	1	θ	1	Kumaraswamy Inverse Exponential due to Oguntunde *et al.* (2014)
13	a	1	α	θ	1	Generalized Exp Inverse Exponential due to Oguntunde *et al.* (2014)
14	1	1	1	θ	2	Inverse Rayleigh
15	1	1[b]	α[1]	θ	2	Exp Inverse Rayleigh (new)
16	a	b	1	θ	2	Kumaraswamy Inverse Rayleigh
17	a	b	α	θ	2	Kumaraswamy Exp Inverse Rayleigh due to ul Haq (2016)
18	a	1	α	θ	2	Generalized Exp Inverse Rayleigh (new)
19	1	1	1	θ^{ϕ}	ϕ	Inverse Weibull
20	1	1[b]	α[1]	θ^{ϕ}	ϕ	Exp Inverse Weibull due to Flaih *et al.* (2012)
21	a	b	1	θ^{ϕ}	ϕ	Kumaraswamy Inverse Weibull due to Shahbaz *et al.* (2012)
22	a	1	α	θ^{ϕ}	ϕ	Generalized Exp Inverse Weibull (new)

$$
\begin{aligned}
E(X^k) &= -ab\alpha\theta^{\frac{k}{\phi}}\Gamma(a)\Gamma(b)\sum_{i,j,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\
&\quad\cdot\int_1^0 \log(z)^{-\frac{k}{\phi}}\left[1-z\right]^{\alpha(i+1)-1}\left[1-z\right]^{a\ell}dz \\
&= ab\alpha\theta^{\frac{k}{\phi}}\Gamma(a)\Gamma(b)\sum_{i,j,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\
&\quad\cdot\frac{\partial^k}{\partial c^k}\int_0^1 z^{-\frac{c}{\phi}}\left[1-z\right]^{\alpha(i+1)-1}\left[1-z\right]^{a\ell}dz\Big|_{c=0} \\
&= ab\alpha\theta^{\frac{k}{\phi}}\Gamma(a)\Gamma(b)\sum_{i,j,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}\Gamma(aj+1)}{i!j!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)} \\
&\quad\cdot\frac{\partial^k}{\partial c^k}\left[\mathbf{B}\left(1-\frac{c}{\phi},\alpha[i+1]\right){}_2\mathsf{F}_1\left(-a\ell;1-\frac{c}{\phi};\alpha[i+1]-\frac{c}{\phi}+1;1\right)\right]\Big|_{c=0} \qquad (14)
\end{aligned}
$$

where

$$\int_0^1 y^{b-1}(1-y)^{c-b-1}(1-yx)^{-a}dy \;=\; \mathbf{B}(b,c-b)\,{}_2\mathsf{F}_1(a,b;c;x)$$

and

$$\mathbf{B}(b,c-b)$$

is the beta function, while

$${}_2\mathsf{F}_1(a,b;c;x) \;=\; \sum_{\ell=0}^{\infty}\frac{(a)_\ell(b)_\ell}{(c)_\ell}\frac{x^\ell}{\ell!};\; |x|<1$$

is the hypergeometric function, and

$$(p)_\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ p(p+1)\cdots(p+\ell-1) & \text{if } \ell > 0 \end{cases}$$

is the rising Pochhammer symbol. It is easy to evaluate the hypergeometric function in R by using the function hyperg_2F1(a, b, c, x) in the 'gsl' package.□

Let $E(X^k) = \mu_k'$, the variance and coefficient of variation CV are given by

$$\sigma^2 = \mu_2' - \mu_1'^2$$

and

$$CV = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\mu_2' - \mu_1'^2}}{\mu_1'} = \sqrt{\frac{\mu_2'}{\mu_1'^2} - 1}$$

respectively.

We have presented some numerical values of the variance, standard deviation and coefficient of variation of Kw-EG type-2 distribution for some selected values of the distribution parameters in Table 3.

Corollary 1. *The moment generating function of the KwGEG type-2 distribution is given by*

$$M_X(t) \;=\; ab\alpha\Gamma(a)\Gamma(b)\sum_{i,j,k,\ell,=0}^{\infty}\frac{(-1)^{i+j+\ell}\left(t\theta^{\frac{1}{\phi}}\right)^k\Gamma(aj+1)}{i!j!k!\ell!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)}$$
$$\cdot\frac{\partial^k}{\partial c^k}\left[\mathbf{B}\left(1-\frac{c}{\phi},\alpha[i+1]\right){}_2\mathsf{F}_1\left(-a\ell;1-\frac{c}{\phi};\alpha[i+1]-\frac{c}{\phi}+1;1\right)\right]\Bigg|_{c=0}$$

Table 3. Some basic statistics of 200 random sample of the KwGEG type-2 distribution for different values of parameter combinations.

$[a, b, \alpha, \theta, \phi]\downarrow$	μ'	μ'_2	σ^2	σ	CV	median
1.5,1.5,1.5,1.5,1.5	1.63857	4.04078	1.35588	1.16442	0.71063	1.34694
2.45,6.45,.2,1.44,5.1	1.90040	4.07971	0.46821	0.68426	0.36006	1.73974
1,3.8,3.3,2.7,5	0.98730	0.98058	0.00581	0.07625	0.07723	0.98426
5.5,8.34,5.4,1,1	0.56822	0.33306	0.01018	0.10091	0.17759	0.56202
3.5,24,5,3,20	1.00961	1.01937	0.00007	0.00818	0.00811	1.01008
12,8,.6,3.3,4.5	2.44959	6.14899	0.14852	0.38538	0.15733	2.40771
4.5,2.7,3.7,7,1	6.01062	40.13415	4.00656	2.00164	0.33302	5.65999
2,3,8,2.3,6.5	0.98059	0.96311	0.00155	0.03942	0.04020	0.98023
2.8,15,1.9,8,1	4.96547	26.44923	1.79335	1.33916	0.26969	4.82551
1,4,1,33,8.5	1.41380	2.00865	0.00981	0.09903	0.07004	1.40457
9.4,8,2,1.84,2.1	1.64603	2.75690	0.04747	0.21789	0.13237	1.63133
6.15,2.41,3,4.45,7.08	1.26132	1.59489	0.00397	0.06298	0.04993	1.25657
12,5.18,4,6.57,20	1.09896	1.20784	0.00013	0.01132	0.01030	1.09880
5.5,8,5,3,.5	3.31191	12.52929	1.56056	1.24922	0.37719	3.12071
4,9,6,5,7	1.12696	1.27089	0.00084	0.02905	0.02578	1.12763
22,4.8,7,5.7,14.3	1.11379	1.24065	0.00013	0.01125	0.01010	1.11363
3.4,7,8,3.85,3	1.15586	1.34083	0.00482	0.06940	0.06004	1.15620
5,3,9,1,2	0.71069	0.51011	0.00503	0.07091	0.09977	0.70685
8,.85,1,1,3	3.02219	17.33389	8.20028	2.86361	0.94753	2.39249
1,4,1.5,5,3	1.33812	1.84245	0.05188	0.22777	0.17022	1.31170

2.6. Rényi entropy

The Rényi entropy measure generalizes the Hartley, Shannon, collision and min entropy. Entropies is very useful in quantifying the diversity, uncertainty, or randomness of a system. The Rényi entropy due to Renyi (1960) is defined as

$$\mathcal{I}_R(\lambda) = \frac{1}{1-\lambda}\log\left\{\int_{\text{all x}} f^{\lambda}(x)dx\right\},\ \lambda > 0\backslash\{1\}$$

Theorem 2. *If X is distributed according to the KwGEG type-2 distribution in Equation (6) then, its Rényi entropy is given by*

$$\mathcal{I}_R(\lambda) = \frac{1}{1-\lambda}\log\left\{(ab\alpha\theta\phi)^{\lambda}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)\phi\theta^{-\frac{2}{\phi}}}\right.$$
$$\cdot\frac{\partial}{\partial s}\left[\mathbf{B}\left(1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right],\alpha[\lambda+i+k]+1\right)\right.$$
$$\left.\left.\cdot{}_2\mathsf{F}_1\left(\lambda;1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right];\alpha[\lambda+i+k]-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]+2;1\right)\right]\Big|_{s=0}\right\}$$

Proof. For convenience let $\mathcal{W}^{\lambda} = \int_0^{\infty} f^{\lambda}(x|a,b,\alpha,\theta,\phi)dx$ then

$$\mathcal{W}^{\lambda} = (ab\alpha\theta\phi)^{\lambda} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)} \cdot \int_0^{\infty} x^{-\lambda[\theta+1]} e^{-\lambda\theta x^{-\phi}} \left[1-e^{-\lambda\theta x^{-\phi}}\right]^{\alpha[\lambda+i+k]} \left[1-e^{-\lambda\theta x^{-\phi}}\right]^{-\lambda} dx \tag{15}$$

setting $y = e^{-\theta x^{-\phi}}$ in Equation (15) we have

$$\begin{aligned}
\mathcal{W}^{\lambda} &= (ab\alpha\theta\phi)^{\lambda} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)\phi\theta^{-\frac{2}{\phi}}} \\
&\quad \cdot \int_0^1 [\log(y)]^{-\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]} (1-y)^{\alpha[\lambda+i+k]} (1-y)^{-\lambda} dy \\
&= (ab\alpha\theta\phi)^{\lambda} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)\phi\theta^{-\frac{2}{\phi}}} \\
&\quad \cdot \frac{\partial}{\partial s} \int_0^1 y^{-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]} (1-y)^{\alpha[\lambda+i+k]} (1-y)^{-\lambda} dy \bigg|_{s=0} \\
&= (ab\alpha\theta\phi)^{\lambda} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)\phi\theta^{-\frac{2}{\phi}}} \\
&\quad \cdot \frac{\partial}{\partial s} \left[\mathbf{B}\left(1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right], \alpha[\lambda+i+k]+1\right)\right. \\
&\quad \left.\left. \cdot {}_2\mathsf{F}_1\left(\lambda; 1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]; \alpha[\lambda+i+k]-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]+2; 1\right)\right]\right|_{s=0}
\end{aligned}$$

thus,

$$\begin{aligned}
\mathcal{I}_{\mathcal{R}}(\lambda) &= \frac{1}{1-\lambda} \log\left\{(ab\alpha\theta\phi)^{\lambda} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}\Gamma(\lambda[a-1]+1)\Gamma(\lambda[b-1]+1)\Gamma(aj+1)}{i!j!k!\Gamma(\lambda[a-1]+1-i)\Gamma(\lambda[b-1]+1-j)\Gamma(aj+1-k)\phi\theta^{-\frac{2}{\phi}}}\right. \\
&\quad \cdot \frac{\partial}{\partial s} \left[\mathbf{B}\left(1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right], \alpha[\lambda+i+k]+1\right)\right. \\
&\quad \left.\left.\left. \cdot {}_2\mathsf{F}_1\left(\lambda; 1-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]; \alpha[\lambda+i+k]-s\left[\frac{\lambda}{\phi}[\theta+1]+\frac{1}{\phi}+1\right]+2; 1\right)\right]\right|_{s=0}\right\}
\end{aligned}$$

3. Order statistics

The Order statistics plays a very vital role in probability and statistics. Here, we present the distribution of the *rth* order statistics for the KwGEG type-2 distribution. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables from the

KwGEG type-2 distribution while $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the corresponding ascending ordered observations; then, the distribution of the *rth* order statistics of the KwGEG type-2 distribution with *pdf* $f(x|a,b,\alpha,\theta,\phi)$ is obtained as follows

$$f_{r:n}(x) = \frac{n!f(x|a,b,\alpha,\theta,\phi)}{(r-1)!(n-r)!} F^{r-1}(x|a,b,\alpha,\theta,\phi)[1-F(x|a,b,\alpha,\theta,\phi)]^{n-r} \quad (16)$$

by substituting the *cdf* and *pdf* and using the binomial expansion we have

$$\begin{aligned} f_{r:n}(x) &= \frac{n!f(x|a,b,\alpha,\theta,\phi)}{(r-1)!(n-r)!} \sum_{q=0}^{n-r} (-1)^q \frac{\Gamma(n-r+1)}{q!\Gamma(n-r-q+1)} F^{q+r-1}(x|a,b,\alpha,\theta,\phi) \\ &= \frac{n!f(x|a,b,\alpha,\theta,\phi)}{(r-1)!(n-r)!} \sum_{q=0}^{n-r} \sum_{k=0}^{q+r-1} \sum_{t,m=0}^{\infty} (-1)^{q+k+t+m} \\ &\cdot \frac{\Gamma(n-r+1)\Gamma(q+r)\Gamma(bk+1)\Gamma(at+1)\left[1-e^{-\theta x^{-\phi}}\right]^{\alpha m}}{q!k!t!m!\Gamma(n-r-q+1)\Gamma(q+r-k)\Gamma(bk-t+1)\Gamma(at-m+1)} \\ &= \frac{n!ab\alpha\theta\phi}{(r-1)!(n-r)!} \sum_{q=0}^{n-r} \sum_{k=0}^{q+r-1} \sum_{i,j,\ell,m,t=0}^{\infty} (-1)^{i+j+k+\ell+q+t+m} \\ &\cdot \frac{\Gamma(a)\Gamma(b)\Gamma(aj+1)\Gamma(n-r+1)\Gamma(q+r)\Gamma(bk+1)\Gamma(at+1)}{i!j!k!\ell!m!q!t!\Gamma(a-i)\Gamma(b-j)\Gamma(aj-\ell+1)\Gamma(n-r-q+1)\Gamma(q+r-k)\Gamma(bk-t+1)\Gamma(at-m+1)} \\ &\cdot x^{-\phi-1} e^{-\theta x^{-\phi}} \left[1-e^{-\theta x^{-\phi}}\right]^{\alpha(i+1)-1} \left[1-e^{-\theta x^{-\phi}}\right]^{a\ell+\alpha m} \end{aligned} \quad (17)$$

It is easy to calculate the moments and other properties of interest of the order statistics of the KwGEG type-2 distribution from $f_{r:n}(x)$.

4. Estimation

Let denote the parameters of the *pdf* in Equation (6) by $\Theta = (a,b,\alpha,\theta,\phi)'$ then, the estimation of these five parameters can be obtained by the method of maximum likelihood estimation. Suppose $x_1, x_2, \cdots, x_n$ is a random sample from the KwGEG type-2 then, the Likelihood equation can be expressed as:

$$\begin{aligned} L(\Theta|x) &= \prod_{i=1}^{n} ab\alpha\theta\phi x^{-\phi-1} e^{-\theta x^{-\phi}} \left(1-e^{-\theta x^{-\phi}}\right)^{\alpha-1} \left(1-\left[1-e^{-\theta x^{-\phi}}\right]^{\alpha}\right)^{a-1} \\ &\cdot \left(1-\left[1-\left(1-e^{-\theta x^{-\phi}}\right)^{\alpha}\right]^{a}\right)^{b-1} \end{aligned} \quad (18)$$

with the Log-likelihood equation as

$$\begin{aligned}\mathcal{L}(\Theta|x) &= b\sum_{i=1}^{n}\log\left[1-\left(1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right)^{a}\right]+(a-1)\sum_{i=1}^{n}\log\left(1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right)\\ &-\sum_{i=1}^{n}\log\left[1-\left(1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right)^{a}\right]+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\theta x_i^{-\phi}}\right)-\theta\sum_{i=1}^{n}x_i^{-\phi}\\ &-\phi\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\log(x_i)+n\log(a)+n\log(b)+n\log(\alpha)\\ &+n\log(\theta)+n\log(\phi). \end{aligned} \quad (19)$$

The first-order partial derivatives of Equation (19) with respect to the five parameters are:

$$\begin{aligned}\frac{\partial\mathcal{L}(\Theta|x)}{\partial a} &= -(b-1)\sum_{i=1}^{n}\frac{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\ &+\sum_{i=1}^{n}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]+\frac{n}{a} \end{aligned} \quad (20)$$

$$\frac{\partial\mathcal{L}(\Theta|x)}{\partial b} = \sum_{i=1}^{n}\log\left[1-\left(1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right)^{a}\right]+\frac{n}{b}, \quad (21)$$

$$\begin{aligned}\frac{\partial\mathcal{L}(\Theta|x)}{\partial \alpha} &= b\sum_{i=1}^{n}\frac{a\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\ &-(a-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{-\phi}}\right]}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\\ &b\sum_{i=1}^{n}\frac{a\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\ &+\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i^{-\phi}}\right]+\frac{n}{\alpha}, \end{aligned} \quad (22)$$

$$\begin{aligned}\frac{\partial \mathcal{L}(\Theta|x)}{\partial \theta} &= b\sum_{i=1}^{n}\frac{a\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}} \\ &-(a-1)\sum_{i=1}^{n}\frac{\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} \\ &-\sum_{i=1}^{n}\frac{a\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}} \\ &+(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{-\phi}e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}-\sum_{i=1}^{n}x_i^{-\phi}+\frac{n}{\theta}, \qquad (23)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathcal{L}(\Theta|x)}{\partial \phi} &= -b\sum_{i=1}^{n}\frac{a\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}} \\ &+(a-1)\sum_{i=1}^{n}\frac{\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} \\ &+\sum_{i=1}^{n}\frac{a\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}} \\ &-(\alpha-1)\sum_{i=1}^{n}\frac{\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}+\theta\sum_{i=1}^{n}x_i^{-\phi}\log(x_i) \\ &-\sum_{i=1}^{n}\log(x_i)+\frac{n}{\phi}. \qquad (24)\end{aligned}$$

The maximum likelihood estimates (MLEs) $\hat{a}$, $\hat{b}$, $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\phi}$ could be obtained by solving the system of non-linear equations $\left(\frac{\partial \mathcal{L}(\Theta|x)}{\partial a}=0, \frac{\partial \mathcal{L}(\Theta|x)}{\partial b}=0, \frac{\partial \mathcal{L}(\Theta|x)}{\partial \alpha}=0, \frac{\partial \mathcal{L}(\Theta|x)}{\partial \theta}=0, \frac{\partial \mathcal{L}(\Theta|x)}{\partial \phi}=0\right)'$ for the parameters. But, due to the non-linear form of the equations in the system above, it is impossible to obtain the solutions analytically except by using numerical computations e.g; the Newton-Raphson iteration method.

4.1. Fisher Information

Here we calculate the Fisher information matrix for the KwGEG type-2 distribution. In statistical inference (estimation and hypothesis testing), the Fisher information matrix plays a very vital role. The Fisher information matrix for any observed random sample say $x_1, x_2, \cdots, x_n$ from the KwGEG type-2 distribution is defined as

$$H(\Theta)_{5\times5} = -E\begin{pmatrix} \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a^2} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a\partial b} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a\partial\alpha} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a\partial\theta} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a\partial\phi} \\ \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial b\partial a} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial b^2} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial b\partial\alpha} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial b\partial\theta} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial b\partial\phi} \\ \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\alpha\partial a} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\alpha\partial b} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\alpha^2} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\alpha\partial\theta} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial a\partial\phi} \\ \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta\partial a} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta\partial b} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta\partial\alpha} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta^2} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta\partial\phi} \\ \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\phi\partial a} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\phi\partial b} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\phi\partial\alpha} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\phi\partial\theta} & \frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\phi^2} \end{pmatrix}_{5\times5},$$

where the elements of the matrix are listed in the Appendix.

The inverse matrix $H^{-1}(\Theta)_{5\times5}$ is called the Variance-Covariance matrix from which statistical inference is mainly based.

4.2. Simulation study

We assess the performance of the maximum likelihood estimates through a simulation study in R-statistical software with the nlm function in the stats package. The nlm function carries out a minimization of the log-likelihood function using a Newton-type algorithm. 5000 (N) Monte Carlo simulations was replicated. The following sample sizes $n = 25, 50, 75, \cdots, 600$ were considered and the evaluation of the estimates is based on the empirical bias and mean squared errors of the parameter estimates ($\hat{\Theta}$) which are calculated by

$$Bias_\Theta(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta)$$

and

$$\text{MSE}_\Theta(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta)^2$$

where $\Theta = a, b, \alpha, \theta, \phi$. The results are given in Table 4 and Figure 5. The values in Table 4 indicates that the MLEs of the parameters stabilizes to the actual value 0.5 as n increases, while Figure 5 shows that the Bias and MSE tends to 0 as n becomes large as we expect.

5. Conclusion

In recent times, statisticians have become more interested in developing more flexible distributions to improve the modeling of survival (or reliability) data and significant success have been recorded in this direction. However, this paper is a contribution to the growing literature

Table 4. The mean parameter estimates and standard errors on 10000 simulations of the KwGEG type-2 distribution with $a = 0.5$, $b = 0.5$, $\alpha = 0.5$, $\theta = 0.5$ and $\phi = 0.5$, with $n = 25, 50, 75, \cdots, 600$.

	MLEs									
n	$\bar{\hat{a}}$	$\bar{\hat{b}}$	$\bar{\hat{\alpha}}$	$\bar{\hat{\theta}}$	$\bar{\hat{\phi}}$	$SE(\bar{\hat{a}})$	$SE(\bar{\hat{b}})$	$SE(\bar{\hat{\alpha}})$	$SE(\bar{\hat{\theta}})$	$SE(\bar{\hat{\phi}})$
25	0.75755	0.67917	0.78424	0.73896	0.73263	1.77614	1.00772	1.42492	1.12700	0.35595
50	0.57031	0.64986	0.62195	0.75920	0.63471	0.58749	0.78192	0.43030	0.94227	0.24491
75	0.56865	0.62307	0.59381	0.74134	0.59440	0.43907	0.60510	0.38235	0.91565	0.19996
100	0.57857	0.58169	0.58559	0.68337	0.57529	0.43400	0.49383	0.34286	0.76556	0.18036
125	0.55066	0.60224	0.57342	0.67554	0.55960	0.36452	0.52091	0.34735	0.66633	0.15697
150	0.53949	0.55770	0.60241	0.63718	0.56606	0.35185	0.51418	0.32676	0.63942	0.14878
175	0.55508	0.57321	0.56980	0.63399	0.55152	0.33146	0.45917	0.30175	0.62279	0.13372
200	0.53473	0.55395	0.59616	0.65093	0.53964	0.32346	0.40639	0.31635	0.57826	0.12960
225	0.54324	0.54298	0.59053	0.62339	0.54440	0.31042	0.39672	0.29317	0.56845	0.12435
250	0.56271	0.54535	0.57281	0.60727	0.54294	0.32083	0.38736	0.27499	0.54368	0.12363
275	0.55386	0.55462	0.56871	0.58380	0.53726	0.29470	0.34937	0.29473	0.49580	0.11432
300	0.54888	0.53941	0.57474	0.58416	0.53465	0.27535	0.34507	0.27640	0.48414	0.10997
325	0.53610	0.55279	0.55940	0.62714	0.52365	0.26574	0.34100	0.26025	0.48769	0.10674
350	0.53550	0.53491	0.56161	0.60458	0.52939	0.26089	0.34379	0.24481	0.48705	0.10127
375	0.54560	0.53853	0.54916	0.58535	0.52807	0.26320	0.31181	0.23810	0.45537	0.09702
400	0.53540	0.51021	0.56254	0.57722	0.52960	0.25484	0.25920	0.23678	0.42848	0.09325
425	0.54385	0.52631	0.56255	0.56187	0.53115	0.24591	0.29264	0.24710	0.43333	0.09754
450	0.54066	0.53690	0.55357	0.58187	0.52052	0.23081	0.28507	0.23356	0.43612	0.09213
475	0.54148	0.52726	0.54387	0.56040	0.52568	0.25415	0.28061	0.21708	0.39203	0.08815
500	0.52175	0.51708	0.55833	0.56785	0.52202	0.22333	0.25388	0.22669	0.37259	0.08305
525	0.52971	0.52392	0.55258	0.58965	0.51928	0.22586	0.26404	0.21537	0.43520	0.08484
550	0.53940	0.51929	0.54996	0.56871	0.51890	0.23879	0.24869	0.21416	0.39266	0.08200
575	0.52751	0.50196	0.56139	0.56846	0.52536	0.23188	0.22753	0.21895	0.40565	0.08452
600	0.51447	0.51407	0.54617	0.56842	0.52106	0.20263	0.23183	0.20779	0.37231	0.07986

of generalized family of probability distributions. We introduce a new five-parameter Kumaraswamy Generalized Exponentiated Gumbel type-2 (KwGEG type-2) distribution as an extension of the Exponentiated Gumbel type-2 distribution due to Okorie *et al.* (2016). The distribution could either have a decreasing or unimodal shape while its hazard rate function could be increasing and decreasing, bathtub and upside-down bathtub shaped. The KwGEG type-2 distribution is a more flexible model and contain over twenty sub-distributions. Some of the basic statistical properties of the distribution are derived and expressed in closed forms and their corresponding numerical values are also presented where possible. Moreover, method of maximum likelihood estimation was used to estimate the parameters of the distribution and a simulation study was carried out to investigate the performance of the maximum likelihood method.

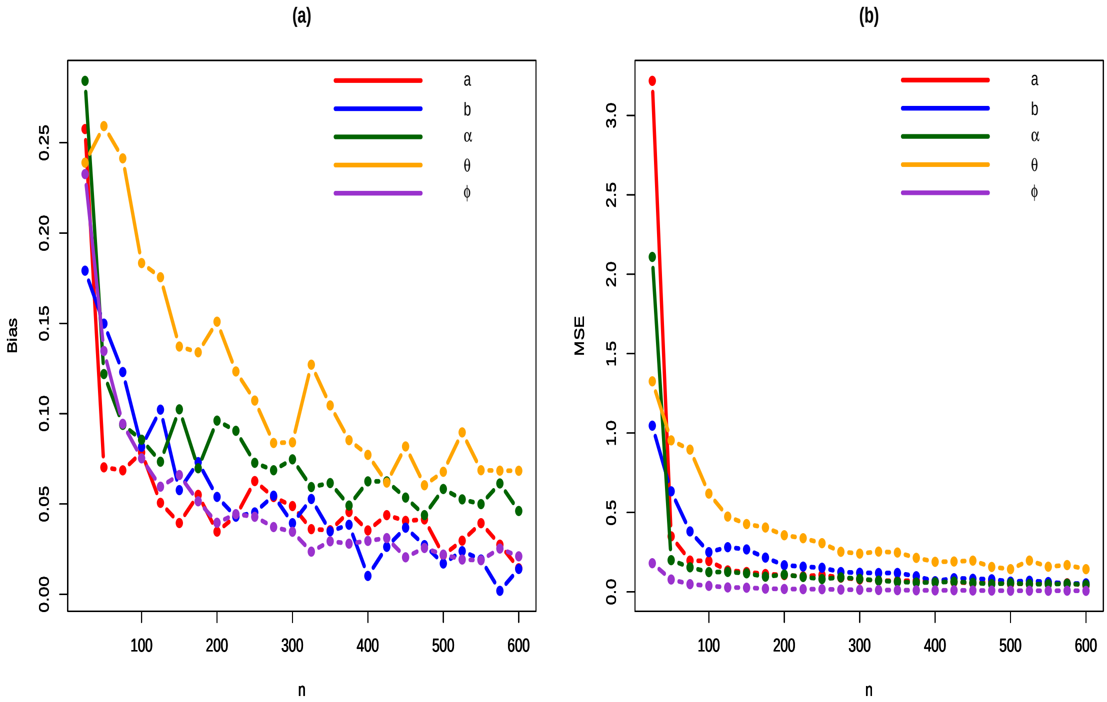

Fig. 5. Plots of the Bias and MSE the KwGEG type-2 parameters from the simulation result

Future work.

A sequel article will feature a detailed application of the Kw-G EG type-2 distribution applied to several real life data-sets ranging from climatic, biological to reliability settings and its fitting performance will be compared with thirty related distributions including the special cases with the following methods of parameter estimation: maximum likelihood estimation, moments, L-moments, trimmed L-moments (TL-moments), probability weighted moments (PWM), and the generalized probability weighted moments (GPWM). The paper will further examine the performance of those methods of parameter estimation in estimating the parameters of the new Kw-G EG type-2 distribution through a simulation study.

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APPENDIX

$$\begin{aligned}\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial a^2} &= -(b-1)\sum_{i=1}^{n}\left[\frac{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\log^2\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\right.\\ &\left.+\frac{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2a}\log^2\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^2}\right]-\frac{n}{a^2}.\end{aligned}$$

$$\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial a\partial b} = -\sum_{i=1}^{n}\frac{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}.$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial a\partial \alpha} &= (b-1)\sum_{i=1}^{n}\left[\frac{\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\log\left(1-e^{-\theta x_i^{-\phi}}\right)\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\right.\\ &+\frac{a\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\log\left(1-e^{-\theta x_i^{-\phi}}\right)\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\ &\left.+\frac{a\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\log\left(1-e^{-\theta x_i^{-\phi}}\right)\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^2}\right]\\ &-\sum_{i=1}^{n}\frac{\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\log\left(1-e^{-\theta x_i^{-\phi}}\right)}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}}.\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial a \partial \theta} &= (b-1)\sum_{i=1}^{n}\left[\frac{\alpha x_i^{-\phi} e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\right.\\
&+ \frac{a\alpha x_i^{-\phi} e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\
&+ \left.\frac{a\alpha x_i^{-\phi} e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^{2}}\right]\\
&- \sum_{i=1}^{n}\frac{\alpha x_i^{-\phi} e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial a \partial \phi} &= -(b-1)\sum_{i=1}^{n}\left[\frac{\alpha\theta x_i^{-\phi}\log(x_i) e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\right.\\
&- \frac{a\alpha\theta\log(x_i) e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\
&- \left.\frac{a\alpha\theta\log(x_i) e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2a-1}\log\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^{2}}\right]\\
&+ \sum_{i=1}^{n}\frac{\alpha\theta\log(x_i) e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}}.
\end{aligned}$$

$$\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial b^2} = -\frac{n}{b^2}.$$

$$\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial b \partial \alpha} = \sum_{i=1}^{n} \frac{a\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha} \log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}$$

$$\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial b \partial \theta} = \sum_{i=1}^{n} \frac{a\alpha x_i^{-\phi} e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}$$

$$\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial b \partial \phi} = -\sum_{i=1}^{n} \frac{a\alpha\theta \log(x_i) e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}.$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial \alpha^2} = & -(b-1)\left[\sum_{i=1}^{n}\left[\frac{a^2\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha}\log^2\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^2}\right.\right. \\
& + \frac{a(a-1)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha}\log^2\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-2}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}} \\
& \left.\left. - \frac{a\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\log^2\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\right]\right] \\
& - (a-1)\left[\sum_{i=1}^{n}\left[\frac{\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\log^2\left[1-e^{-\theta x_i^{-\phi}}\right]}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}}\right.\right. \\
& \left.\left. + \frac{\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha}\log^2\left[1-e^{-\theta x_i^{-\phi}}\right]}{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^2}\right]\right] - \frac{n}{\alpha^2}.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\alpha\partial\theta} &= (b-1)\sum_{i=1}^{n}\left[-\frac{\alpha a^2 x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{2\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^2}\right.\\
&+ \frac{a x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha-1}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{2(a-1)}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\
&- \frac{\alpha a(a-1) x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{2\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-2}}{1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}}\\
&+ \left.\frac{a\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a-1}}{\left[1-\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^{a}\right]^2}\right]\\
&+ (a-1)\sum_{i=1}^{n}\left[-\frac{x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha-1}}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}} - \frac{\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha-1}\log\left(1-e^{-\theta x_i^{-\phi}}\right)}{1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}}\right.\\
&- \left.\frac{\alpha x_i^{-\phi}e^{-\theta x_i^{-\phi}}\left(1-e^{-\theta x_i^{-\phi}}\right)^{2\alpha-1}\log\left(1-e^{-\theta x_i^{-\phi}}\right)}{\left[1-\left(1-e^{-\theta x_i^{-\phi}}\right)^{\alpha}\right]^2}\right]\\
&+ \sum_{i=1}^{n}\frac{x_i^{-\phi}e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}.
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial\alpha\partial\phi} &= (b-1)\sum_{i=1}^{n}\left[\frac{a^2\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^{2}}\right.\\
&+\frac{a(a-1)\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\\
&-\frac{a\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\\
&\left.-\frac{a\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right]\\
&+(a-1)\sum_{i=1}^{n}\left[\frac{\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\right.\\
&+\frac{\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\\
&\left.+\frac{\alpha\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2\alpha-1}\log\left[1-e^{-\theta x_i^{-\phi}}\right]}{\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2}}\right]\\
&-\sum_{i=1}^{n}\frac{\theta x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial \theta^2} &= -\frac{n}{\theta^2} - (\alpha-1)\sum_{i=1}^{n}\left[\frac{x_i^{-2\phi}e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}} + \frac{x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}}{\left[1-e^{-\theta x_i^{-\phi}}\right]^2}\right] \\
&- (a-1)\sum_{i=1}^{n}\left[\frac{\alpha(\alpha-1)x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-2\phi}}\right]^{\alpha-2}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} - \frac{\alpha x_i^{-2\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\right. \\
&+ \left.\frac{\alpha^2 x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}}{\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^2}\right] \\
&-b\sum_{i=1}^{n}\left[\frac{a(a-1)\alpha^2 x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right. \\
&- \frac{a\alpha(\alpha-1)x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \frac{a\alpha x_i^{-2\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \left.\frac{a^2\alpha^2 x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^2}\right] \\
&- \sum_{i=1}^{n}\left[\frac{a(a-1)\alpha^2 x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right. \\
&- \frac{a\alpha(\alpha-1)x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \frac{a\alpha x_i^{-2\phi}e^{-\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \left.\frac{a^2\alpha^2 x_i^{-2\phi}e^{-2\theta x_i^{-\phi}}\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^2}\right].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\mathcal{L}(\Theta|x)}{\partial\theta\partial\phi} &= (\alpha-1)\sum_{i=1}^{n}\left[\frac{\theta x_i^{-2\phi}\log(x_i)e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}+\frac{\theta x_i^{-2\phi}\log(x_i)e^{-2\theta x_i^{-\phi}}}{\left[1-e^{-\theta x_i^{-\phi}}\right]^2}-\frac{x_i^{-\phi}\log(x_i)e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}\right] \\
&+ (a-1)\sum_{i=1}^{n}\left[\frac{\theta\alpha(\alpha-1)e^{-2\theta x_i^{-2\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\right. \\
&+ \frac{\alpha e^{-2\theta x_i^{-\phi}}x_i^{-\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} \\
&- \left.\frac{\theta\alpha e^{-\theta}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}+\frac{\theta\alpha^2e^{-2\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}}{\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^2}\right] \\
&+ b\sum_{i=1}^{n}\left[\frac{\theta\alpha^2a(a-1)e^{-2\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right. \\
&+ \frac{a\theta\alpha e^{-\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&- \frac{a\theta\alpha(\alpha-1)e^{-2\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&- \frac{a\alpha e^{-\theta x_i^{-\phi}}x_i^{-\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \left.\frac{a^2\alpha^2\theta e^{-2\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^2}\right] \\
&- \sum_{i=1}^{n}\left[\frac{\theta\alpha^2a(a-1)e^{-2\theta x_i^{-\phi}}x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right.
\end{aligned}
$$

to be continued on next page ...

$$
\begin{aligned}
&+\frac{a\theta\alpha e^{-\theta x_i^{-\phi}} x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\\
&-\frac{a\theta\alpha(\alpha-1) e^{-2\theta x_i^{-\phi}} x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\\
&-\frac{a\alpha e^{-\theta x_i^{-\phi}} x_i^{-\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\\
&+\left.\frac{a^2\alpha^2\theta e^{-2\theta x_i^{-\phi}} x_i^{-2\phi}\log(x_i)\left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^{2}}\right]\\
&+\sum_{i=1}^{n} x_i^{-\phi}\log(x_i)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}(\Theta|x)}{\partial \phi^2} &= -\frac{n}{\phi^2} \\
&- (\alpha-1)\sum_{i=1}^{n}\left[\frac{\theta^2 \log^2(x_i) x_i^{-2\phi} e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}} + \frac{\theta^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}}}{\left[1-e^{-\theta x_i^{-\phi}}\right]^2} - \frac{\theta \log^2(x_i) x_i^{-\phi} e^{-\theta x_i^{-\phi}}}{1-e^{-\theta x_i^{-\phi}}}\right] \\
&- (a-1)\sum_{i=1}^{n}\left[\frac{\alpha(\alpha-1)\theta^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}}\right. \\
&- \frac{\alpha\theta^2 \log^2(x_i) x_i^{-2\phi} e^{-\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} + \frac{\alpha\theta \log^2(x_i) x_i^{-\phi} e^{-\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1}}{1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}} \\
&+ \left.\frac{\alpha^2\theta^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)}}{\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^2}\right] \\
&- b\sum_{i=1}^{n}\left[\frac{a(a-1)\theta^2\alpha^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)} \left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}\right. \\
&- \frac{a(a-1)\theta^2\alpha \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-2} \left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&- \frac{a\theta\alpha \log^2(x_i) x_i^{-\phi} e^{-\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1} \left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \\
&+ \frac{a\theta^2\alpha \log^2(x_i) x_i^{-2\phi} e^{-\theta x_i^{-\phi}} \left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha-1} \left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1-\left[1-\left[1-e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}
\end{aligned}
$$

$$+ \left. \frac{a^2\theta^2\alpha^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^2} \right]$$

$$+ \sum_{i=1}^{n} \left[\frac{a(a-1)\theta^2\alpha^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-2}}{1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}} \right.$$

$$- \frac{a(a-1)\theta^2\alpha \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha-2} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}$$

$$- \frac{a\theta\alpha \log^2(x_i) x_i^{-\phi} e^{-\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha-1} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}$$

$$+ \frac{a\theta^2\alpha \log^2(x_i) x_i^{-2\phi} e^{-\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha-1} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a-1}}{1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}}$$

$$+ \left. \frac{a^2\theta^2\alpha^2 \log^2(x_i) x_i^{-2\phi} e^{-2\theta x_i^{-\phi}} \left[1 - e^{-\theta x_i^{-\phi}}\right]^{2(\alpha-1)} \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{2(a-1)}}{\left[1 - \left[1 - \left[1 - e^{-\theta x_i^{-\phi}}\right]^{\alpha}\right]^{a}\right]^2} \right]$$

$$- \theta \sum_{i=1}^{n} \log^2(x_i) x_i^{-\phi}$$