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AN APPROXIMATION FOR THE POWER FUNCTION OF A SEMI-PARAMETRIC TEST OF FIT

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Abstract. We consider in this paper goodness of fit tests of the null hypothesis that the underlying d.f. of a sample $F(x)$, belongs to a given family of distribution functions $\mathcal{F}$. We propose a method for deriving approximate values of the power of a weighted Cramér-von Mises type test of goodness of fit. Our method relies on Karhunen-Loève [K.L] expansions on $(0,1)$ for the weighted a Brownian bridges.

1. Introduction

In this paper we investigate semi-parametric tests of fit based upon a random sample $X_1, X_2, \ldots, X_n$ with common continuous distribution function $F(x) = \mathbb{P}(X_1 \le x)$. Here $\mathcal{F} = \{G(., \theta) : \theta \in \Theta\}$ denotes a family of all distribution function which will be specified later on, and Θ is some open set in $\mathbb{R}^k$.
We seek to test the hypothesis

$$H_0 : F(.) = G(., \theta) \in \mathcal{F},$$

against an alternative which will be specified later on. We will make use of the Cramér-von Mises type statistics of the form

$$\widehat{W}^2_{n,\varphi} := n \int_{-\infty}^{\infty} \varphi\big(G(x, \widehat{\theta}_n)\big)\big[\mathbb{F}_n(x) - G(x, \widehat{\theta}_n)\big]^2 dG(x, \widehat{\theta}_n),$$

with $\mathbb{F}_n(x) = n^{-1} \sum_{i=1}^n 1\!\mathrm{I}_{\{X_i \le x\}}$ denotes the usual empirical distribution function [d.f.] and $\widehat{\theta}_n$ is a sequence of estimators of θ and φ is a positive and continuous function on $(0,1)$, fulfilling

$$(1.1) \qquad (i)\ \lim_{t \uparrow 0} t^2 \varphi(t) = \lim_{t \downarrow 1} (1-t)^2 \varphi(t) = 0 \qquad (ii) \int_0^1 t(1-t)\varphi(t) < \infty.$$

Note that, setting $Z_i = G(X_i, \widehat{\theta}_n)$ for $i = 1, \ldots, n$ and letting $\widehat{\mathbb{G}}_n(t)$ denotes the empirical d.f. based upon $Z_1, \ldots, Z_n$ then, we may write, under (H_0),

$$(1.2) \qquad \widehat{W}^2_{n,\varphi} = n \int_0^1 \varphi(t)\big(\widehat{\mathbb{G}}_n(t) - t\big)^2 dt,$$

with $Z_1, \ldots, Z_n$ being not independent and identically distributed [i.i.d.] uniform (0,1) r.v's. However, in some important cases the distribution of $Z_1, \ldots, Z_n$

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doses not depend upon θ, but only on $\mathcal{F}$. In this cases, the distribution of $\widehat{W}^2_{n,\varphi}$ is parameter free. This happens if $\mathcal{F}$ is a location scale family and $\widehat{\theta}_n$ is an equivalent estimator, a fact noted by David and Johnson [4].

2. The empirical process with estimated parameters

A general study of the weak convergence of the estimated empirical process was carried out by Durbin [6]. We present here an approach to his main results using strong approximations.
Introduce, for each $x \in \mathbb{R}$, the empirical process with estimated parameters

$$\alpha_n(x, \widehat{\theta}_n) = \sqrt{n}\big(\mathbb{F}_n(x) - G(x, \widehat{\theta}_n)\big), \tag{2.3}$$

where $\widehat{\theta}_n$ is a sequence of estimators of θ, and we assume that

$$\sqrt{n}(\widehat{\theta}_n - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(X_i, \theta) + o_{\mathbb{P}}(1), \tag{2.4}$$

where $l(X_1, \theta) = \big(l_1(X_1, \theta_1), \ldots, l_k(X_1, \theta_k)\big)$ is centered function and has finite second moments.
Suppose $F(x) = G(x, \theta) \in \mathcal{F}$ has density $f(x, \theta) = \frac{\partial G}{\partial \theta}(x, \theta)$. Take $\widehat{\theta}_n$ as the maximum Likelihood estimator: the maximizer of

$$m(\theta) = \sum_{i=1}^{n} \log f(X_i, \theta).$$

Under adequate regularity conditions $\int \frac{\partial}{\partial \theta} \log f(x, \theta) dG(x, \theta) = 0$ and

$$\int \Big(\frac{\partial}{\partial \theta} \log f(x, \theta)\Big)\Big(\frac{\partial}{\partial \theta} \log f(x, \theta)\Big)^T dG(x, \theta) = -\int \frac{\partial^2}{\partial \theta^2} \log f(x, \theta) dG(x, \theta) := I(\theta).$$

Since

$$m'(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \log f(X_i, \theta) \qquad \text{and} \quad m''(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial \theta^2} \log f(X_i, \theta),$$

we obtain, from the Law of Large Number, that $\frac{1}{n} m''(\theta) \to I(\theta)$ almost surely.
Now, a Taylor expansion of $m'(\theta)$ around θ gives

$$\begin{aligned} \frac{1}{\sqrt{n}}\big(m'(\widehat{\theta}_n) - m'(\theta)\big) &= \frac{1}{n} m''(\widehat{\theta}_n)\sqrt{n}\big(\theta - \widehat{\theta}\big) + o_p(1) \\ &= -I(\theta)\sqrt{n}\big(\theta - \widehat{\theta}\big) + o_p(1), \end{aligned}$$

which, taking into account that $m'(\widehat{\theta}) = 0$, gives

$$\sqrt{n}\big(\theta - \widehat{\theta}\big) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} l(X_i, \theta) + o_p(1),$$

with $l(x,\theta) = I(\theta)^{-1}\frac{\partial}{\partial\theta}logf(x,\theta)$. Clearly $\int l(x,\theta)dG(x,\theta) = 0$, while

$$\int l(x,\theta)l(x,\theta)^T dG(x,\theta) = I(\theta)^{-1}I(\theta)I(\theta)^{-1} = I(\theta)^{-1}.\Box$$

To obtain the null asymptotic distribution of $\alpha_n(x,\widehat{\theta}_n)$, we assume that (H_0) and (2.4) and write

$$\begin{aligned}
\alpha_n(x,\widehat{\theta}_n) &= \sqrt{n}\big(\mathbb{F}_n(x) - G(x,\theta)\big) - \sqrt{n}\big(G(x,\widehat{\theta}_n) - G(x,\theta)\big) \\
&= \alpha_n\big(G(x,\theta)\big) - H(G(x,\theta),\theta)^T\int_0^1 L(t,\theta)d\alpha_n(t) + o_{\mathbb{P}}(1) \\
&= \widehat{\alpha}_n\big(G(x,\theta)\big) + o_{\mathbb{P}}(1), \qquad (2.5)
\end{aligned}$$

where $\alpha_n(.)$ denotes the uniform empirical process, $H(t,\theta) = \frac{\partial G}{\partial\theta}\Big(G^{-1}(t,\theta),\theta\Big)$, $L(t,\theta) = l\Big(G^{-1}(t,\theta),\theta\Big)$, with $G^{-1}(t,\theta) = \big\{x : G(x,\theta) \geq t\big\}$ denoting the quantile function of X_1, and

$$\widehat{\alpha}_n(t) = \alpha_n(t) - H(t,\theta)^T\int_0^1 L(s,\theta)d\alpha_n(s), \text{ for } 0 < t < 1, \qquad (2.6)$$

is the uniform estimated empirical process.

2.1. **Some notes on stochastic integration.** Equation (2.6) suggests that

$$\widehat{\alpha}_n(t) \xrightarrow{w} B(t) - H(t,\theta)^T\int_0^1 L(s,\theta)dB(s), \text{ as } n \to \infty,$$

where $\xrightarrow{w}$ denotes the weak convergence and $B(.)$ is a brownian bridge (i.e., a Gaussian process with $B(0) = B(1) = 0$, $\mathbb{E}(B(t)) = 0$, $\mathbb{E}(B(s)B(t)) = \min(s,t) - st$ for $s,t \in [0,1]$).
We cannot give $\int_0^1 L(s,\theta)dB(s)$ the meaning of a Stieltjes integral since the trajectories of $B(.)$ are not of bounded variation. It is possible, though, to make sense of expressions like $\int_0^1 f(s)dB(s)$, with $f \in L^2(0,1)$ through the following construction.
Assume first that f is simple : $(f(t) = \sum_{i=1}^n a_i\mathbb{I}_{(t_{W<},t_{i-1}]}$, with $a_i \in \mathbb{R}$ and $0 = t_0 < t_1 < \cdots < t_n = 1)$. Then

$$\int_0^1 f(s)dB(s) = \sum_{i=1}^n a_i\big(B(t_i) - B(t_{i-1})\big) := \sum_{i=1}^n a_i \triangle B_i,$$

where $\triangle B_i = B(t_i) - B(t_{i-1})$. It can be easily checked that $\mathbb{E}(\triangle B_i) = 0$ and

$$\mathbb{V}ar(\triangle B_i) = \triangle t_i(1 - \triangle t_i) \quad and \quad \mathbb{C}ov(\triangle B_i, \triangle B_j) = -\triangle t_i \triangle t_j \quad if \quad i \neq j.$$

The random variable is centered Gaussian with variance

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$$\begin{aligned}\sum_{i=1}^{n} a_i^2 \mathbb{V}ar(\triangle B_i) + 2\sum_{1\le i<j\le n} a_i a_j \mathbb{C}ov(\triangle B_i, \triangle B_j) &= \sum_{i=1}^{n} a_i^2 \triangle t_i - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \triangle t_i \triangle t_j \\ &= \sum_{i=1}^{n} a_i^2 \triangle t_i - \big(\sum_{i=1}^{n} a_i \triangle t_i\big)^2 \\ &= \int_0^1 f^2(t)dt - \Big(\int_0^1 f(t)dt\Big)^2.\end{aligned}$$

Thus, $f \longrightarrow \int_0^1 f(s)dB(s)$ defines an isometry between the subspace of $L^2(0,1)$ consisting of centered, simple functions and its range. We can therefore extend the definition to all centered function in $L^2(0,1)$. Finally, for a general $f \in L^2(0,1)$,

$$\int_0^1 f(s)dB(s) = f \longrightarrow \int_0^1 \widehat{f}(s)dB(s),$$

where $\widehat{f}(s) = f(s) - \int_0^1 f(t)dt$. The stochastic integral $\int_0^1 f(s)dB(s)$ is centered, Gaussian random variable with variance

$$\int_0^1 f^2(t)dt - \Big(\int_0^1 L(t)dt\Big)^2.$$

In fact, if $f_1, ..., f_k \in L^2(0,1)$, then $\Big(\int_0^1 f_1(s)dB(s), \dots, \int_0^1 f_k(s)dB(s)\Big)$ has a joint centered, Gaussian law and form the isometry defining the integrals we see that

(2.7)

$$\mathbb{C}ov\Big(\int_0^1 f(s)dB(s), \int_0^1 g(s)dB(s)\Big) = \int_0^1 f(s)g(s)ds - \int_0^1 f(s)ds \int_0^1 g(s)ds.$$

We can similarly check that

$$\Big(\{B(t)\}_{t\in[0,1]}, \int_0^1 f_1(s)dB(s), \dots, \int_0^1 f_k(s)dB(s)\Big)$$

is Gaussian and

$$\mathbb{C}ov\Big(B(t), \int_0^1 f(s)dB(s)\Big) = \int_0^t f(s)ds - t\int_0^t f(s)ds$$

(take $g(s) = \mathbb{I}_{(0,1]}(s)$ in (2.7) to check it).

An integration by parts formula. Suppose $h(.)$ is simple. Then

$$\int_0^1 h(s)dB(s) = \sum_{i=1}^{n} h(t_i)\Big(B(t_i) - B(t_{i-1})\Big) = -\sum_{i=0}^{n-1} B(t_i)\Big(h(t_{i+1}) - h(t_i)\Big) = -\int_0^1 B(t)dh(t).$$

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This result can be easily extended to any $h(.)$ of bounded variation and continuous on $[0,1]$:

$$\int_0^1 h(s)dB(s) = -\int_0^1 B(t)dh(t).$$

This integration by parts formula can be used to bound the difference between stochastic integrals and the corresponding integrals with respect to the empirical process:

$$|\int_0^1 h(s)d\alpha_n(s) - \int_0^1 h(s)dB_n(s)| \le \sup_{0\le t\le 1} |\alpha_n(t) - B_n(t)| \int_0^1 d|h|(s),$$

$B_n(.)$ is a sequence of brownian bridges.

We can summarize now the above arguments in the following theorem (see, e.g., [6]).

Theorem 2.1. *Provided $H(t,\theta)$ is continuous on $[0,1]$ and $L(s,\theta)$ is continous and bounded variation on $[0,1]$ we can define, on a sufficiently rich probability space, $\alpha_n(.)$ and $B_n(.)$ such that*

$$\sup_{0\le t\le 1} |\widehat{\alpha}_n(t) - \widehat{B}_n(t)| = O(\frac{\log n}{\sqrt{n}}) \text{ almost surly [a.s.]},$$

where $\widehat{B}_n(t) = B_n(t) - H(t,\theta)^T \int_0^1 L(s,\theta)dB_n(s)$ is a centered Gaussian process with function covariance

$$\begin{aligned} \widehat{K}_\theta(s,t) &= \min(s,t) - st - H(t,\theta)^T \int_0^s L(x,\theta)dx - H(s,\theta)^T \int_0^t L(x,\theta)dx \\ &+ H(s,\theta)^T \Big[\int_0^1 L(x,\theta)L(x,\theta)^T dx\Big] H(t,\theta). \end{aligned} \tag{2.8}$$

Note that this covariance function can be expressed as $s\wedge t - \sum_{j=1}^k \phi_j(s)\phi_j(t)$ for some real functions $\phi_j(.)$. A very complete study of the Karhunen-Loève expansion of Gaussian processus with this type of covariance function was carried out in [11].

Exemple 1. We consider $\mathcal{F} = \{G_0(\frac{\cdot-\mu}{\sigma}) : \theta = (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}_+^*\}$ is a location scale family $\Big(G_0(.)$ is a standard distribution function with density $g_0\Big)$. Then

$$H(t,\theta) = -\frac{1}{\sigma} g_0\big(G_0^{-1}(t)\big) \begin{bmatrix} 1 \\ G_0^{-1}(t) \end{bmatrix}$$

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and

$$I(\theta) = \frac{1}{\sigma^2}\begin{bmatrix} \int \frac{g_0^2(x)}{g_0(x)}dx & \int x\frac{g_0^2(x)}{g_0(x)}dx \\ \int x\frac{g_0^2(x)}{g_0(x)}dx & \int x^2\frac{g_0^2(x)}{g_0(x)}dx - 1 \end{bmatrix}.$$

We can now write

$$I(\theta)^{-1} = \sigma^2 \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

with σ_{ij} *depending only on* G_0, *but not on* μ *or* σ *and*

$$\widehat{K}(s,t) = \min(s,t) - st - \phi_1(s)\phi_1(t) - \phi_2(s)\phi_2(t).$$

Here

$$\phi_1(t) = -\sqrt{\big(\sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}\big) g_0\big(G_0^{-1}(t)\big)}$$

and

$$\phi_2(t) = -\frac{\sigma_{12}}{\sqrt{\sigma_{22}}} g_0\big(G_0^{-1}(t)\big) - \sqrt{\sigma_{22}}\, g_0\big(G_0^{-1}(t)\big)G_0^{-1}(t).$$

If $\mathcal{F}$ *is the Gaussian family* $G_0(x) = \Phi(x)$, $g_0(x) = \phi(x)$, ${g'}_0(x) = -x\phi(x)$ *and*

$$I(\theta) = \frac{1}{\sigma^2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, $\sigma_{11} = 1$, $\sigma_{22} = \frac{1}{2}$, $\sigma_{12} = \sigma_{21} = 0$ *and*

$$\widehat{K}(s,t) = \min(s,t) - st - \phi\big(\Phi^{-1}(s)\big)\phi\big(\Phi^{-1}(t)\big) - \frac{1}{2}\phi\big(\Phi^{-1}(s)\big)\Phi^{-1}(s)\phi\big(\Phi^{-1}(t)\big)\Phi^{-1}(t).$$

In this Gaussian case L *is not of bounded variation on* $[0,1]$, *but the above argument can be modified and still prove that*

$$\{\widehat{\alpha}_n(t)\}_t \xrightarrow{w}$$

$$\Big\{B(t) + \phi\big(\Phi^{-1}(s)\big)\int_0^1 \big(\Phi^{-1}(s)\big)dB(s) + \frac{1}{2}\phi\big(\Phi^{-1}(t)\big)\Phi^{-1}(t)\int_0^1 \big(\Phi^{-1}(s)^2 - 1\big)dB(s)\Big\}_t$$

as random variable in $D[0,1]$ *or* $L^2[0,1]$.

Theorem 2.1 provided, as an easy corollary, the asymptotic distribution of a variety of $\widehat{W}^2_{n,\varphi}$ statistics under the null hypothesis. In fact, Durbin's results also give a valuable tool for studying its asymptotic power because they include too the asymptotic distribution of the estimated empirical process under contiguous alternatives. A survey of results connected to Theorem 2.1 as well as a simple derivation of it based on Skorohod embedding can be founnd in [10].

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3. Results (Asymptotic power of the $\widehat{W}^2_{n,\varphi}$ test of fit)

Assume that (1.1) and (2.4), under the null hypothesis (H_0), the limiting distribution of $\widehat{W}^2_{n,\varphi}$ in (1.2) coincides with the distribution of the random variable

$$\widehat{W}^2_{\varphi} := \int_0^1 \varphi(t) B^2(t,\theta) dt,$$

where $B(t,\theta)$ is a Gaussian random process with zero mean and covariance function

$$K_\varphi(s,t) = \sqrt{\varphi(s)\varphi(t)}\widehat{K}_\theta(s,t), \tag{3.9}$$

where $\widehat{K}_\theta(s,t)$ has been described above in (2.8).
We chose the sequence of local alternatives which depend on the parameters $\theta = (\theta_1, \ldots, \theta_k)$ given by

$$H_a : F(.) = F^{(n)}(.,\theta),$$

where $F^{(n)}(.,\theta)$ is chosen as a proper distribution function such that $F^{(n)}(.,\theta) \to G(.,\theta)$, as $n \to \infty$, and with $R_n(.) := \sqrt{n}\Big(F^{(n)}(.,\theta) - G(.,\theta)\Big) \to R(.,\theta)$ in the mean square, as $n \to \infty$, and $R(.,\theta)$ is known and satisfies the condition $\int_{-\infty}^{+\infty} R(x,\theta)dx < \infty$.
These kinds of alternatives were proposed and discussed, in particular, by Chibisov [2]. Setting $t = G(x,\theta)$, $\delta(t,\theta) = R(G^{-1}(t,\theta),\theta)$ and assuming that

$$\int_0^1 \varphi(t)\delta^2(t,\theta)dt < \infty. \tag{3.10}$$

Under (H_a), with $\delta(.,\theta)$ satisfies the condition (3.10), the limiting distribution (as $n \to \infty$) of statistic $\widehat{W}^2_{n,\varphi}$ coincides (see, e.g., [2]) with the distribution of r.v:

$$\begin{aligned} \widehat{W}^2_{(\delta,\varphi)} &= \int_0^1 \varphi(t)\big[B(t,\theta) + \delta(t,\theta)\big]^2 dt \\ &= \int_0^1 \varphi(t)B^2(t,\theta) + 2\int_0^1 \delta(t,\theta)\varphi(t)B(t,\theta)dt + \int_0^1 \delta(t,\theta)\varphi^2(t). \end{aligned} \tag{3.11}$$

For a fixed parameter θ and a level of significance $\alpha \in (0,1)$, there is a threshold of confidence $t_\alpha := t_\alpha(\theta)$ satisfying the identity

$$\mathbb{P}(\int_0^1 \varphi(t)B^2(t,\theta)dt \geq t_\alpha) = \alpha. \tag{3.12}$$

(see, e.g., [5] for a tabulation of numerical values of t_α for the particular cases $\varphi(t) = t^{2\beta}$, $\beta > -1$, and, $\alpha =$0.1, 0.05, 0.01, 0.005, 0.001).

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In the case above, the asymptotic power of the test of fit based upon $\widehat{W}^2_{n,\varphi}$, under the sequence of local alternatives specified by (H_a), is specified by

$$\mathbb{P}\Big(\widehat{W}^2_{(\delta,\varphi)} \geq t_\alpha\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\widehat{W}^2_{n,\varphi} \geq t_\alpha | H_a\Big). \tag{3.13}$$

Recalling the definitions (1.1) of φ, (3.9) of $K_\varphi(.,.)$ and, (3.12) of t_α, we set

(3.14)

$$g(t,\theta) := \sqrt{\varphi(t)}\delta(t,\theta), \qquad x := \frac{t_\alpha - \int_0^1 K_\varphi(t,t)dt - \int_0^1 \varphi(t)\delta^2(t,\theta)dt}{2},$$

$$A := \int_0^1 K^2_\varphi(s,s)ds, \qquad B := \int_0^1 \Big[\int_0^1 g(s,\theta)K_\varphi(s,t)ds\Big]^2 dt,$$

$$C := \int_0^1\int_0^1 \Big[\int_0^1 g(u,\theta)K_\varphi(s,u)du \int_0^1 g(v,\theta)K_\varphi(s,v)dv\Big]^2 K_\varphi(s,t)dsdt,$$

$$D^2 : = \int_0^1\int_0^1 g(s,\theta)K_\varphi(s,t)g(t,\theta)dsdt. \tag{3.15}$$

Let ϕ (resp. Φ) be the probability density (resp. distribution) function of the standard normal $\mathcal{N}(0,1)$ distribution. Namely,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x f(u)du.$$

Then, for calculating the power function defined in (3.13), we have the following theorem. Recall the definitions (3.14)-(3.15) of x, A, B, C and, D.

Theorem 3.1. Under the assumptions above, we have

$$1 - \mathbb{P}\big(\widehat{W}^2_{(\delta,\varphi)} \geq t_\alpha\big)$$

$$= \Phi(\frac{x}{D}) + \Big\{\frac{A}{2D^2}H_1(\frac{x}{D}) + \frac{B}{2D^{\frac{3}{2}}}H_2(\frac{x}{D}) + \frac{C}{4D^4}H_3(\frac{x}{D}) + \frac{B^2}{8D^6}H_5(\frac{x}{D})\Big\}\phi(\frac{x}{D}) + \varepsilon(x).$$

Here $H_j(.)$ are Hermite polynomial and, $\varepsilon_k(.)$ is a remainder term fulfilling

$$\sup_y |\varepsilon(y)| \leq \frac{C_1}{\Big(D^2 - \frac{B}{\lambda_1}\Big)^{\frac{3}{2}}}, \tag{3.16}$$

where C_1 is a constant and, λ_1 is the first eigenvalue of the Fredholm transformation $h \to \int_0^1 K_\varphi(s,.)h(s)ds$.

Remark 1.

The following particular cases are of interest. If, we replace $g(.,\theta)$ by $\gamma g(.,\theta)$ in the alternatives of (3.10) (for some real parameter $\gamma > 0$), we obtain that

$$\sup_y |\varepsilon(y)| = o\big(\gamma^{-\frac{3}{2}}\big) \quad \text{as} \ \ \gamma \to \infty. \tag{3.17}$$

Proof. The proof of this theorem resembles that which was published (in the case non-parametric) in another article (see, e.g.,[1]). □

4. Numerical example

As an illustration, we will consider approximate calculation of the power of $\widehat{W}^2_{n,\varphi}$ test for verifying the hypothesis of normal distribution.

Here, we consider $\mathcal{F} = \{\Phi(\frac{\cdot-\mu}{\sigma}) : (\mu,\sigma) \in \mathbb{R} \times \mathbb{R}^*_+\}$, $\theta = (\mu,\sigma)$, $\widehat{\theta} = (\overline{X}, S^2)$ and,

$$H_0 : F(y) = G(y,\theta) := \Phi(\frac{y-\mu}{\sigma}).$$

We chose as an alternative,

$$(H_a) : F(y) = F^{(n)}(y,\theta) := \Phi(\frac{y-\mu}{\sigma}) + \gamma\frac{R(\frac{y-\mu}{\sigma})}{\sqrt{n}} + O(\frac{1}{n}),$$

where $R(x) = \frac{1}{4\sqrt{2\pi}}(3x - x^3)e^{-\frac{x^2}{2}}$ and, γ is a real parameter positive.
Setting $t = \Phi(\frac{y-\mu}{\sigma})$ and, $\delta(t) = R(\Phi^{-1}(t))$, we obtain

$$K_\varphi(s,t) = \sqrt{\varphi(s)\varphi(t)}\widehat{K}_\theta(s,t)$$
$$= \sqrt{\varphi(s)\varphi(t)}\Big\{\min(s,t) - st - \Big(1 + \frac{1}{2}\Phi^{-1}(s)\Phi^{-1}(t)\Big)\phi\big(\Phi^{-1}(s)\big)\phi\big(\Phi^{-1}(t)\big)\Big\}.$$

According to the preceding theorem, the asymptotic power of the test of fit based upon $W^2_{n,\varphi}$, under the sequence of local alternatives specified by (H_a) in the case above, is calculated for various γ and α. The accompanying table gives values of the power $\beta_\gamma = \mathbb{P}\big(\widehat{W}^2_{(\delta,\varphi)} > t_\alpha\big)$ for $\varphi \equiv 1$.

$\alpha = 0.01$	γ	β_γ	$\alpha = 0.001$	γ	β_γ
	3	0.2		3	0.085
	4	0.53		4	0.21
	5	0.851		5	0.532
	6	0.98		6	0.847

Table. *Approximate power for the test goodness of fit*

The second column gives various values of the parameter γ. The third as well as last the columns give power values for β_γ. They are compared with the exact values obtained by Martynov [8].

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