Weighted Multivariate Cramér-von Mises-type Statistics

by

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Abstract

In this paper, we consider weighted quadratic functionals of the multivariate uniform empirical process. By deriving the Karhunen-Loève expansion of the corresponding limiting Gaussian process, we obtain the asymptotic distribution of these statistics. Our results have direct applications to tests of goodness of fit and tests of independence by Cramér-von Mises-type statistics.

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1 Introduction and Premiminaries.

1.1 Introduction.

In this paper, we consider quadratic functionals of the form

$$\int_0^1 \cdots \int_0^1 t_1^{2\beta_1} \ldots t_d^{2\beta_d} \alpha_{n,0}^2(t_1, \ldots, t_d) dt_1 \ldots dt_d, \tag{1.1}$$

where $\alpha_{n,0}$ is an appropriate version of the uniform empirical process on the unit hypercube $[0,1]^d$ (see, e.g., (2.20) in the sequel for an explicit definition

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of this quantity). Our aim is, at first, to establish conditions on the constants $\beta_1, \ldots, \beta_d$, under which the statistic in (1.1) converges to a quadratic functional of a Gaussian process. The latter is of the form

$$\int_0^1 \ldots \int_0^1 t_1^{2\beta_1} \ldots t_d^{2\beta_d} \mathbf{B}_0^2(t_1, \ldots, t_d) dt_1 \ldots dt_d, \tag{1.2}$$

with $\mathbf{B}_0$ denoting a tied-down Brownian bridge. Second, we will establish the explicit distribution of the random variable in (1.2), by deriving the Karhunen-Loève expansion of the corresponding weighted Gaussian process.

This problem has received considerable attention in the literature from the pioneering work of Cramér [10] (see, e.g., Nikitin [26] and the references therein), up to the more recent work of Scott [32] for $d = 1$. In higher dimensions, one must refer to Blum, Kiefer and Rosenblatt [6], Cotterill and Csörgő [8, 9], Deheuvels [12], Dugué [16, 17, 18], Hoeffding [19], Kiefer [23], Martynov [27], and Smirnov [33, 35, 34], who have investigated un-weighted statistics corresponding to $\beta_1 = \ldots = \beta_d = 0$. The study of quadratic functionals of Gaussian processes has been also discussed at length, in particular, by Biane and Yor [5], Donati-Martin and Yor [15], Pitman and Yor [28, 29, 30, 31], and Yor [37, 38], among others. Recently, some important progress has been made in this framework by Deheuvels and Martynov [13], and Deheuvels, Peccati and Yor [14], whose results will be instrumental in the present paper. The theory of Bessel functions plays an essential role in the derivation of our theorems, and we refer to Bowman [7], Korenev [24] and Watson [36] for the appropriate details on these mathematical objects.

Our paper is organized as follows. In the forthcoming §1.2 and 1.3, we establish some general preliminaries which are used later on in §2, where our main results are stated. We describe the univariate case in §2.1, whereas the multivariate case, with $d \geq 2$, is discussed in §2.2. Most of the results given here turn out to follow readily from a series of dispersed references in the literature, and the proofs are obtained via the proper book-keeping arguments. In spite of the fact that the mathematical techniques we shall use are not too difficult, the resulting theorems in §2 are far from trivial, and, for this reason, very likely worth to be mentioned.

1.2 Some Preliminaries on Gaussian Process Theory.

Let $\{X(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ denote a centered Gaussian process defined on the d-dimensional hyper-cube, with $d \geq 1$. For convenience, we set $\mathbf{s} =$

$(s_1,\ldots,s_d)\in\mathbb{R}^d$ and $\mathbf{t}=(t_1,\ldots,t_d)\in\mathbb{R}^d$, and denote by

$$R(\mathbf{s},\mathbf{t})=\mathbb{E}\big(X(\mathbf{s})X(\mathbf{t})\big)\quad\text{for}\quad \mathbf{s},\mathbf{t}\in[0,1]^d, \tag{1.3}$$

the covariance function of $X(\cdot)$. Below, we will be mainly concerned with the study of the *quadratic functional*

$$\int_{[0,1]^d} X^2(\mathbf{t})d\mathbf{t}, \tag{1.4}$$

with $d\mathbf{t}$ denoting the Lebesgue measure on $\mathbb{R}^d$. To render (1.4) meaningful, we will work under the minimal assumption that

$$0<\mathbb{E}\Big(\int_{[0,1]^d} X^2(\mathbf{t})d\mathbf{t}\Big)=\int_{[0,1]^d} R(\mathbf{t},\mathbf{t})d\mathbf{t}<\infty. \tag{1.5}$$

Below, we briefly discuss the meaning and implications of this assumption. The condition (1.5) entails that $X(\cdot)\in L^2\big([0,1]\big)$ a.s., so that $X(\cdot)$ belongs to the special subclass of *Hilbert space valued centered Gaussian processes* (see, e.g., Section 10 in Lifshits [25]). Moreover, making use of the Cauchy-Schwarz inequality, we see that, for each $\mathbf{s},\mathbf{t}\in[0,1]^d$,

$$R(\mathbf{s},\mathbf{t})^2=\mathbb{E}\big(X(\mathbf{s})X(\mathbf{t})\big)^2\leq\mathbb{E}\big(X(\mathbf{s})^2\big)\mathbb{E}\big(X(\mathbf{t})^2\big)=R(\mathbf{s},\mathbf{s})R(\mathbf{t},\mathbf{t}).$$

When combining this last inequality with (1.5), we obtain that

$$\|R\|_{L^2}^2:=\int_{[0,1]^d}\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})^2d\mathbf{s}d\mathbf{t}\leq\Big\{\int_{[0,1]^d} R(\mathbf{t},\mathbf{t})d\mathbf{t}\Big\}^2<\infty, \tag{1.6}$$

so that $R\in L^2\big([0,1]^d\times[0,1]^d\big)$. Routine analytical arguments show that, under (1.6) only, the Fredholm transformation $y(\cdot)\in L^2\big([0,1]^d\big)\rightarrow\widetilde{y}(\cdot)$, defined by

$$\widetilde{y}(\mathbf{t})=\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})y(\mathbf{s})d\mathbf{s}\quad\text{for}\quad\mathbf{t}\in[0,1]^d, \tag{1.7}$$

is a continuous linear mapping of $L^2\big([0,1]^d\big)$ onto itself. In particular, it is easy to check that, under (1.6), for each $y_1(\cdot),y_2(\cdot)\in L^2\big([0,1]^d\big)$,

$$\begin{aligned}\|\widetilde{y}_1-\widetilde{y}_2\|_{L^2}^2 &= \int_{[0,1]^d}\Big\{\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})\big\{y_1(\mathbf{s})-y_2(\mathbf{s})\big\}d\mathbf{s}\Big\}^2d\mathbf{t}\\ &\leq\|R\|_{L^2}^2\times\|y_1-y_2\|_{L^2}.\end{aligned}$$

The condition (1.6) also implies the existence of $\{\lambda_k,e_k(\cdot):1\leq k<K\}$ with the following properties. First, $\{\lambda_k:1\leq k<K\}$ is a sequence of

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positive constants, with $K \in \{2, \ldots, \infty\}$ denoting a possibly infinite index, such that

$$\lambda_1 \geq \ldots \geq \lambda_k \geq \ldots > 0. \tag{1.8}$$

Second, the $\{e_k(\cdot) : 1 \leq k < K\}$ form an orthonormal sequence of functions in $L^2\big([0,1]\big)$, fulfilling

$$\int_{[0,1]^d} e_k(\mathbf{t}) e_\ell(\mathbf{t}) d\mathbf{t} = \begin{cases} 1 & \text{if} \quad k = \ell, \\ 0 & \text{if} \quad k \neq \ell. \end{cases}$$

Third, both sequences are related to R through the identity

$$R(\mathbf{s}, \mathbf{t}) = \sum_{1 \leq k < K} \lambda_k e_k(\mathbf{s}) e_k(\mathbf{t}), \tag{1.9}$$

where the series in (1.9) is convergent in $L^2\big([0,1]^d\big)$. We note here that this last property entails that

$$\|R\|_{L^2} = \int_{[0,1]^d} \int_{[0,1]^d} R(\mathbf{s}, \mathbf{t})^2 d\mathbf{s} d\mathbf{t} = \sum_{1 \leq k < K} \lambda_k^2 < \infty. \tag{1.10}$$

Because of (1.10), the sequence $\{(\lambda_k, e_k(\cdot)) : 1 \leq k < K\}$ is often called a *convergent orthonormal sequence* [c.o.n.s.]. The λ_k (resp. $e_k(\cdot)$) are the eigenvalues (resp. eigenfunctions) of the Fredholm operator (1.7), since they fulfill the relations, for each $1 \leq k < K$,

$$\widetilde{e}_k(\mathbf{t}) = \int_{[0,1]^d} R(\mathbf{s}, \mathbf{t}) e_k(\mathbf{s}) d\mathbf{s} = \lambda_k e_k(\mathbf{t}). \tag{1.11}$$

In view of (1.11), we see that the eigenvalue sequence $\{\lambda_k : 1 \leq k < K\}$ is always *uniquely defined*. On the other hand, such is not the case for the eigenvectors $\{e_k(\cdot) : 1 \leq k < K\}$. In the simple case where the λ_k are isolated and *distinct*, each $e_k(\cdot)$ is only uniquely defined up to a multiplicative factor ± 1. The situation is even more complex when an eigenvalue λ_k is multiple, in which case the choice of an orthonormal basis of the linear space spanned by the eigenvectors pertaining to λ_k is unique only up to an orthogonal transform.

In the sequel, we will make an instrumental use of the *Karhunen-Loève* [KL] representation of $X(\cdot)$, (see, e.g., Kac and Siegert [22, 21], Kac [20], Ash and Gardner [4], and Adler [2]). This representation decomposes $X(\cdot)$ into the sum of the series

$$X(\mathbf{t}) = \sum_{1 \leq k < K} Y_k \sqrt{\lambda_k}\, e_k(\mathbf{t}), \tag{1.12}$$

where $\{Y_k : 1 \leq k < K\}$ denotes a sequence of independent and identically distributed [i.i.d.] standard normal $N(0,1)$ random variables. In general, the series in (1.12) is convergent in mean square. This follows from the observation that, in terms of $\{\lambda_k : 1 \leq k < K\}$, the condition (1.5) is equivalent to

$$0 < \mathbb{E}\Big(\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} \Big) = \sum_{1 \leq k < K} \lambda_k < \infty. \tag{1.13}$$

This, in turn, readily implies that, as $k \uparrow K-1$,

$$\mathbb{E}\Big(\int_{[0,1]^d} \Big\{ X(\mathbf{t}) - \sum_{m=1}^{k} Y_m \sqrt{\lambda_m}\, e_m(\mathbf{t}) \Big\}^2 d\mathbf{t} \Big) = \sum_{m>k} \lambda_k \to 0.$$

Obviously, the condition (1.5) (or equivalently (1.13)) is strictly stronger than (1.10). Moreover, is readily checked that, under (1.5) (or equivalently (1.13)), the quadratic functional (1.4) we are interested in can be rewritten as the sum of the series

$$\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} = \sum_{1 \leq k < K} \lambda_k Y_k^2. \tag{1.14}$$

An easy argument, which we omit, shows that the series in (1.14) is a.s. convergent *if and only if* (1.5) (or equivalently (1.13)) holds. Therefore, we will assume, from now on, that this condition is satisfied.

1.3 A General Convergence Theorem.

We inherit the notation of §1.2, and let $R(\cdot,\cdot)$ be as in (1.3). We consider now a sequence $\xi_1(\cdot), \xi(2), \ldots$ of independent replicæ of a general (but not necessarily Gaussian) random process $\xi(\cdot)$, taking values in $L^2\big([0,1]^d\big)$, and fulfilling the conditions $(H.1$–2–$3)$ below.

$(H.1)$ $\quad \xi(\cdot) \in L^2\big([0,1]^d\big)$;

$(H.2)$ $\quad \mathbb{E}\big(\xi(\mathbf{t})\big) = 0$ for all $\mathbf{t} \in [0,1]^d$;

$(H.3)$ $\quad \mathbb{E}\big(\xi(\mathbf{s})\xi(\mathbf{t})\big) = R(\mathbf{s},\mathbf{t})$ for all $\mathbf{s},\mathbf{t} \in [0,1]^d$.

Under $(H.1$–2–$3)$, it is well-known (see, e.g., Ex. 14, p. 205 in Araujo and Giné [3]) that the convergence in distribution

$$\zeta_n(\cdot) := n^{-1/2} \sum_{i=1}^{n} \xi_i(\cdot) \ \xrightarrow{d}\ X(\cdot), \tag{1.15}$$

holds *if and only if* the condition (1.5) (or equivalently (1.13)) holds, namely, when

$$\int_{[0,1]^d} \mathbb{E}\big(\xi^2(\mathbf{t})\big)d\mathbf{t} = \int_{[0,1]^d} R(\mathbf{t},\mathbf{t})d\mathbf{t} < \infty.$$

Putting together the previous arguments, we get the following more or the less straightforward theorem, which merely collects some well-known facts of the literature.

Theorem 1.1 *Under* (1.5) *and* (H.1–2–3)*, we have, as* $n \to \infty$*, the convergence in distribution*

$$\int_{[0,1]^d} \zeta_n^2(\boldsymbol{t})d\boldsymbol{t} \stackrel{d}{\to} \sum_{1\leq k<K} \lambda_k Y_k^2. \tag{1.16}$$

Proof. Under (1.5) (or equivalently (1.13)), it follows from (1.15) that

$$\int_{[0,1]^d} \zeta_n^2(\mathbf{t})d\mathbf{t} \stackrel{d}{\to} \int_{[0,1]^d} X^2(\mathbf{t})d\mathbf{t},$$

which, in turn, reduces (1.16) to a direct consequence of (1.15).□

In the next section, we provide some useful statistical applications of Theorem 1.1.

2 Weighted Empirical Processes.

2.1 The Univariate Case ($d = 1$).

Let $U_1, U_2, \dots$ be i.i.d. uniform $[0,1]$ random variables. For each $n \geq 1$, denote by

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1\!\!1_{\{U_n \leq t\}}, \tag{2.1}$$

the empirical distribution function based upon $U_1, \dots, U_n$, and let

$$\alpha_n(t) = n^{1/2}\big\{F_n(t) - t\big\} \quad \text{for} \quad t \in [0,1], \tag{2.2}$$

stand for the corresponding uniform empirical process. Recalling the notation of §1.3, fix now a constant $\beta \in \mathbb{R}$, and set, for $n \geq 1$,

$$\xi_n(t) = t^\beta\big\{1\!\!1_{\{U_n \leq t\}} - t\big\} \quad \text{for} \quad t \in [0,1]. \tag{2.3}$$

Here, we use the convention that $t^0 = 1$ for all $t \in \mathbb{R}$, when $\beta = 0$. Observe that, in agreement with (1.15), (2.2), (2.3), and the notation of §1.3,

$$\zeta_n(t) = n^{-1/2} \sum_{i=1}^{n} \xi_i(t) = t^\beta \alpha_n(t) \quad \text{for} \quad t \in [0,1]. \tag{2.4}$$

Obviously, the assumptions (H.1–2–3) in §1.3 are fulfilled with R defined by

$$R(s,t) = s^\beta t^\beta \big\{ s \wedge t - st \big\} \quad \text{for} \quad s,t \in [0,1]. \tag{2.5}$$

In view of (2.5), we see that (1.5)–(1.13) hold if and only if

$$\int_0^1 t^{2\beta} \big\{ t(1-t) \big\} dt < \infty, \tag{2.6}$$

which is equivalent to $\beta > -1$. Now, since $s \wedge t - st$ is the covariance function of a standard Brownian bridge $\{B(t) : t \in [0,1]\}$, the kernel $R(\cdot,\cdot)$ in (2.5) is nothing else but the covariance function of the weighted Brownian bridge

$$X(t) = t^\beta B(t) \quad \text{for} \quad t \in (0,1]. \tag{2.7}$$

Recently, Deheuvels and Martynov [13] have obtained the Karhunen-Loève representation of $X(\cdot)$ in (2.7). This is given as follows. Assume that $\beta \neq -1$, and set $\nu = 1/(2(1+\beta))$. Keep in mind that $\beta > -1$ is then equivalent to $\nu > 0$. For an arbitray $\nu \in \mathbb{R}$, define the Bessel function of the first kind (refer to Formula 9.1.69 in Abaramowitz and Stegun [1]) by

$$J_\nu(x) = (\tfrac{1}{2}x)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{\Gamma(\nu+k+1)\Gamma(k+1)} \ . \tag{2.8}$$

It is well-known (refer to Watson [36]) that, whenever $\nu > -1$, the positive zeros of J_ν (that is, the values of $z > 0$ for which $J_\nu(z) = 0$) are isolated and form an infinite increasing sequence $\{z_{\nu,k} : k \geq 1\}$, such that

$$0 < z_{\nu,1} < z_{\nu,2} < \dots, \tag{2.9}$$

and, as $k \to \infty$,

$$z_{\nu,k} = \big\{ k + \tfrac{1}{2}(\nu - \tfrac{1}{2}) \big\} + o(1). \tag{2.10}$$

In view of this notation and basic facts, Theorem 1.4 in [13] asserts that, whnever $\beta > -1$, the Karhunen-Loève representation of $X(t) = t^\beta B(t)$ is given by

$$X(\mathbf{t}) = t^\beta B(t) = \sum_{k=1}^{\infty} Y_k \sqrt{\lambda_k}\, e_k(\mathbf{t}), \tag{2.11}$$

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where, as usual, $\{Y_k : k \geq 1\}$ is a sequence of independent and identically distributed [i.i.d.] standard normal $N(0,1)$ random variables, and, for $k = 1, 2, \ldots$,

$$\lambda_k = \Big\{ \frac{2\nu}{z_{\nu,k}} \Big\}^2, \tag{2.12}$$

and

$$e_k(t) = t^{\frac{1}{2\nu} - \frac{1}{2}} \Big\{ \frac{J_\nu(z_{\nu,k} t^{\frac{1}{2\nu}})}{\sqrt{\nu}\, J_{\nu-1}(z_{\nu,k})} \Big\} \quad \text{for} \quad 0 < t \leq 1. \tag{2.13}$$

We refer to Deheuvels and Martynov [13] for further details concerning this theorem and the related properties of the Bessel functions used in (2.13). Putting everything together, we get the following theorem.

Theorem 2.1 *For any $\beta > -1$, setting $\nu = 1/(2(1+\beta))$, we have, as $n \to \infty$, the convergence in distribution*

$$\int_0^1 t^{2\beta} \alpha_n^2(t) dt \xrightarrow{d} \int_0^1 t^{2\beta} B^2(t) dt = \sum_{k=1}^\infty \Big\{ \frac{2\nu}{z_{\nu,k}} \Big\}^2 Y_k^2, \tag{2.14}$$

where $\{Y_k : k \geq 1\}$ is an i.i.d. sequence of normal $N(0,1)$ random variables.

Proof. In view of (2.12)–(2.13), it is a direct consequence of Theorem 1.1, when combined with the arguments above.□

2.2 The Multivariate Case ($d \geq 2$).

We consider now an arbitrary $d \geq 2$. The following notation will be useful in this multivariate framework. When $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$ are two vectors of $\mathbb{R}^d$, we denote by $\mathbf{s} \leq \mathbf{t}$ the fact that $s_j \leq t_j$ for $j = 1, \ldots, k$, and set, accordingly,

$$\mathbf{s} \wedge \mathbf{t} = \big(s_1 \wedge t_1, \ldots, s_d \wedge t_d\big).$$

Letting $\mathbf{U} = (U(1), \ldots, U(d)) \in [0,1]^d$ stand for a random vector with a uniform distribution on $[0,1]^d$, we denote by $\mathbf{U}_n = (U_n(1), \ldots, U_n(d)) \in [0,1]^d$, $n = 1, 2, \ldots$ a sequence of i.i.d. replicæ of $\mathbf{U}$. For each $n \geq 1$, the empirical distribution function based upon $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is denoted by

$$F_n(\mathbf{t}) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{\{\mathbf{U}_i \leq \mathbf{t}\}}, \tag{2.15}$$

We will set, for convenience

$$F(\mathbf{t}) = \mathbb{P}\big(\mathbf{U} \leq \mathbf{t}\big) = \prod_{j=1}^{d} t_j, \tag{2.16}$$

for the (*exact*) distribution function of $\mathbf{U}$, and set

$$\alpha_n(\mathbf{t}) = n^{1/2}\big(F_n(\mathbf{t}) - F(\mathbf{t})\big) \quad \text{for} \quad \mathbf{t} \in [0,1]^d, \tag{2.17}$$

for the corresponding uniform empirical process. Making use of the arguments in §1.3, it is easily checked that the following convergence in distribution holds (for processes in $L^2\big([0,1]^d\big)$, which, in the present framework, is sufficient for our needs). As $n \to \infty$, we have

$$\alpha_n(\cdot) \ \stackrel{d}{\to}\ \mathbf{B}(\cdot), \tag{2.18}$$

where $\{\mathbf{B}(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ is a standard multivariate Brownian bridge. Namely, $\mathbf{B}(\cdot)$ is a centered Gaussian process, with covariance function

$$\begin{aligned} &\mathbb{E}\big(\mathbf{B}(\mathbf{s})\mathbf{B}(\mathbf{t})\big) = \mathbb{E}\big(\alpha_n(\mathbf{s})\alpha_n(\mathbf{t})\big) \\ &= \mathbb{E}\big(\mathbb{1}_{\{\mathbf{U} \leq \mathbf{s} \wedge \mathbf{t}\}}\big) - \mathbb{E}\big(\mathbb{1}_{\{\mathbf{U} \leq \mathbf{s}\}}\big)\mathbb{E}\big(\mathbb{1}_{\{\mathbf{U} \leq \mathbf{t}\}}\big) \\ &= \prod_{j=1}^{d}\big\{s_j \wedge t_j\big\} - \prod_{j=1}^{d}\big\{s_j t_j\big\}. \end{aligned} \tag{2.19}$$

Unfortunately, the Karhunen-Loève decomposition of $\mathbf{B}(\cdot)$, with the covariance function given in (2.19), is not known explicitly for $d \geq 2$. We may, however, define a more tractable *tied-down* empirical process $\alpha_{n,0}(\cdot)$ as follows. Set

$$\begin{aligned} \alpha_{n,0}(\mathbf{t}) &= \alpha_n(\mathbf{t}) - \sum_{1 \leq j \leq d} t_j\ \alpha_n(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_d) \\ &+ \sum_{1 \leq j < \ell \leq d}^{d} t_j t_\ell\ \alpha_n(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_{\ell-1}, 1, t_{\ell+1}, \ldots, t_d) \\ &+ \ldots + (1)^d t_1 \ldots t_d\ \alpha_n(1, \ldots, 1). \end{aligned} \tag{2.20}$$

We note that $\alpha_n(1, \ldots, 1) = 0$, but this term is nevertheless stated here to render the construction more explicit. Now, making use again of the

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arguments in §1.3, it is easily checked that the following convergence in distribution holds. As $n \to \infty$, we have

$$\alpha_{n,0}(\cdot) \ \stackrel{d}{\to} \ \mathbf{B}_0(\cdot), \tag{2.21}$$

where $\{\mathbf{B}_0(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ is a tied-down multivariate Brownian bridge. Namely, $\mathbf{B}_0(\cdot)$ is a centered Gaussian process, with covariance function

$$\mathbb{E}\big(\mathbf{B}_0(\mathbf{s})\mathbf{B}_0(\mathbf{t})\big) = \prod_{j=1}^{d} \big\{s_j \wedge t_j - s_j t_j\big\}. \tag{2.22}$$

We have the following easy consequence of the results of Deheuvels and Martynov [13] (see also Deheuvels, Peccati and Yor [14]).

Theorem 2.2 *Let $\beta_1, \ldots, \beta_d$ be constants such that $\beta_j > -1$ for $j = 1, \ldots, d$. Set $\nu_j = 1/(2(1+\beta_j)) > 0$ for $j = 1, \ldots, d$. Then, the Karhunen-Loève decomposition of the centered Gaussian process*

$$X(\boldsymbol{t}) = t_1^{\beta_1} \ldots t_d^{\beta_d} \boldsymbol{B}_0(\boldsymbol{t}) \quad for \quad \boldsymbol{t} \in (0,1]^d, \tag{2.23}$$

is given by

$$X(\boldsymbol{t}) = \sum_{k_1=1}^{\infty} \ldots \sum_{k_d=1}^{\infty} \sqrt{\lambda_{k_1,\ldots,k_d}}\, Y_{k_1,\ldots,k_d}\, e_{k_1,\ldots,k_d}(\boldsymbol{t}), \tag{2.24}$$

where

$$\lambda_{k_1,\ldots,k_d} \quad = \quad \prod_{j=1}^{d} \Big\{\frac{2\nu_j}{z_{\nu_j,k_j}}\Big\}^2 =: \prod_{j=1}^{d} \mathcal{L}(\nu_j, k_j), \tag{2.25}$$

and

$$\begin{aligned} e_{k_1,\ldots,k_d}(\boldsymbol{t}) \quad &= \quad \prod_{j=1}^{d} \Bigg[t_j^{\frac{1}{2\nu_j}-\frac{1}{2}} \Bigg\{ \frac{J_{\nu_j}(z_{\nu_j,k}\, t_j^{\frac{1}{2\nu_j}})}{\sqrt{\nu_j}\ J_{\nu_j-1}(z_{\nu_j,k})} \Bigg\} \Bigg] \\ &=: \quad \prod_{j=1}^{d} \mathcal{E}(\nu_j, t_j). \end{aligned} \tag{2.26}$$

Proof. By (2.22) the covariance function of $X(\mathbf{t})$ in (2.23) is given by

$$R(\mathbf{s},\mathbf{t}) = \prod_{j=1}^{d} s_j^{\beta_j} t_j^{\beta_j} \big\{s_j \wedge t_j - s_j t_j\big\} =: \prod_{j=1}^{d} \mathcal{R}(s_j, t_j). \tag{2.27}$$

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Therefore, because of (2.12)–(2.13), it is straightforward that $\lambda_{k_1,\ldots,k_d}$ is an eigenvalue of the Fredholm operator (1.7) pertaining to the eigenfunction $e_{k_1,\ldots,k_d}(\cdot)$. To conclude, it is enough to show that *all* eigenvalues of the kind are obtained by this construction. For this, we combine (1.10) with (2.27), to write that

$$\int_{[0,1]^d}\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})^2 d\mathbf{s}d\mathbf{t} = \prod_{j_1=1}^{\infty}\cdots\prod_{j_d=1}^{\infty}\int_0^1\int_0^1 \mathcal{R}(s_j,t_j)^2 ds_j dt_j$$
$$= \prod_{j_1=1}^{\infty}\cdots\prod_{j_d=1}^{\infty}\Big\{\sum_{k_j=1}^{\infty}\mathcal{L}(\nu_j,k_j)^2\Big\} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\lambda^2_{k_1,\ldots,k_d}.$$

This shows obviously that there is no other remaining eigenvalue of (1.7), which suffices for our needs.□

We now give the main theorem of the present paper, which turns out to be an easy consequence of the preceding results.

Theorem 2.3 *Let $\beta_1,\ldots,\beta_d$ be constants such that $\beta_j > -1$ for $j = 1,\ldots,d$. Set $\nu_j = 1/(2(1+\beta_j)) > 0$ for $j = 1,\ldots,d$. Then, we have, as $n \to \infty$, the convergence in distribution*

$$\int_{[0,1]^d} t_1^{2\beta_1}\ldots t_d^{2\beta_d}\alpha^2_{n,0}(\boldsymbol{t})d\boldsymbol{t} \stackrel{d}{\to} \int_{[0,1]^d} t_1^{2\beta_1}\ldots t_d^{2\beta_d}B^2_0(\boldsymbol{t})d\boldsymbol{t}$$
$$= \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\Big\{\prod_{j=1}^{d}\Big\{\frac{2\nu_j}{z_{\nu_j,k_j}}\Big\}^2\Big\}Y^2_{k_1,\ldots,k_d}, \tag{2.28}$$

where $\{Y_{k_1,\ldots,k_d} : k_1 \geq 1,\ldots,k_d \geq 1\}$ is an i.i.d. array of normal $N(0,1)$ random variables.

It turns out that the limiting distribution given in Theorem 2.3 is identical to the limiting distribution of the Blum-Kiefer-Rosenblatt statistic (see, e.g., [6]), when $d = 2$ and $\beta_1 = \ldots = \beta_d = 0$.

Conclusion. Unlike the univariate case, the eigenvalues $\lambda_{k_1,\ldots,k_d}$ in the Karhunen-Loève decomposition (2.25)–(2.26) are multiple. This renders the numerical computation of the distribution quantiles of the test statistic in (2.28) for $d \geq 2$ slightly than in the univariate case. The derivation of the properties of the tests based upon this theory will be investigated elsewhere.

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