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GENERALIZED q-GAUSSIAN VON NEUMANN ALGEBRAS WITH COEFFICIENTS I: RELATIVE STRONG SOLIDITY

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GENERALIZED q-GAUSSIAN VON NEUMANN ALGEBRAS WITH COEFFICIENTS I: RELATIVE STRONG SOLIDITY

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We define $\Gamma_q(B, S \otimes H)$, the generalized q-gaussian von Neumann algebras associated to a sequence of symmetric independent copies (π_j, B, A, D) and to a subset $1 \in S = S^* \subset A$ and, under certain assumptions, prove their strong solidity relative to B. We provide many examples of strongly solid generalized q-gaussian von Neumann algebras. We also obtain nonisomorphism and nonembedability results about some of these von Neumann algebras.

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1. Background and statement of results

1A. ***Normalizers in von Neumann algebras.*** The study of normalizers in von Neumann algebras is nowadays an intensely active area of research within the field of von Neumann algebras. For a von Neumann algebra M, we denote by $\mathcal{U}(M)$ the group of unitaries in M. Recall that for an inclusion $A \subset M$ of von Neumann algebras, *the normalizing group of* A in M is defined as $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ and *the normalizer of* A is the von Neumann algebra generated by the normalizing group, i.e., $\mathcal{N}_M(A)'' \subset M$. When A is a maximal abelian von Neumann subalgebra of a type-II_1 factor M, A is called a *Cartan subalgebra* if its normalizer is the whole of M. While some results were obtained in the early 80's, see, e.g., [Connes and Jones 1982; Jones and Popa 1982] and most notably [Connes, Feldman and Weiss 1981], the first truly significant achievement in this area is Voiculescu's ground-breaking result [1996] about the absence of Cartan subalgebras in the free-group factors $L(\mathbb{F}_n)$. After this, in their seminal work, Ozawa and Popa [2010a] established the following result:

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Theorem 1.1. *Let $\mathbb{F}_n \curvearrowright B$ be a* profinite *trace-preserving action of a free group on an amenable von Neumann algebra B. Then for every amenable von Neumann subalgebra $A \subset M = B \rtimes \mathbb{F}_n$, either $A \prec_M B$, or the normalizer of A is amenable.*

The notation $A \prec_M B$ reads "a corner of A embeds into B inside M", in the sense of [Popa 2006b, Theorem 2.1], and it roughly means that A can be conjugated into B by a partial isometry in M. When B is the scalars, this shows that the normalizer of any amenable diffuse von Neumann subalgebra of $L(\mathbb{F}_n)$ is itself amenable, not only reproving and strengthening Voiculescu's result, but also giving a surprisingly far-reaching classification of normalizer algebras in the free group factors. More than merely proving the above theorem, [Ozawa and Popa 2010a] introduced an array of innovative ideas and techniques which remain all-pervasive and highly influential in the field to this day. The results in that paper were then extended in [Ozawa and Popa 2010b] to profinite actions of weakly amenable groups having proper 1-cocycles into (a multiple of) their left regular representations. Subsequent generalizations to the case of profinite actions of groups having quasicocycles or direct products of such have been obtained in [Chifan and Sinclair 2013; Chifan, Sinclair and Udrea 2013]. Recently, Popa and Vaes obtained the definitive form of these results, by completely removing any assumption on the action of the group. Specifically, they proved the following results (see [Popa and Vaes 2014a, Theorem 1.6; 2014b, Theorem 1.4]):

Theorem 1.2. *Let Γ be a weakly amenable group having either a proper* 1*-cocycle or a proper* 1*-quasicocycle into a* (*representation which is weakly contained into*) *a multiple of its left regular representation. Let $\Gamma \curvearrowright B$ be any trace-preserving action of Γ on the finite von Neumann algebra B, and let $A \subset M = B \rtimes \Gamma$ be a von Neumann subalgebra which is amenable relative to B inside M. Then either $A \prec_M B$, or the normalizer of A is amenable relative to B inside M.*

Theorem 1.3. *Let $\Gamma \curvearrowright B$ be a p.m.p. free ergodic action, where B is abelian diffuse and Γ is weakly amenable and admits an unbounded* (*rather than proper*) 1*-cocycle into a mixing representation which is weakly contained into a multiple of the left regular representation of Γ. Then $M = B \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy.*

Popa and Vaes coined the phrase "relative strong solidity" to describe the situation in which the dichotomy in Theorem 1.2 holds. Namely, a von Neumann algebra M is *strongly solid relative to B*, for $B \subset M$ a subalgebra, if for every von Neumann subalgebra $A \subset M$ which is amenable relative to B inside M, see [Ozawa and Popa 2010a], it is either the case that $A \prec_M B$ or that the normalizer of A is amenable relative to B inside M. In the case of B abelian diffuse and of p.m.p. free ergodic actions $\Gamma \curvearrowright B$, the strong solidity of the von Neumann algebra $M = B \rtimes \Gamma$ relative to B implies its uniqueness of Cartan subalgebra, up to unitary conjugacy. Strong solidity relative to the scalars is simply termed strong solidity. Strong solidity is in turn an enhancement of Ozawa's concept of *solidity* [2004]. Ozawa called a von Neumann algebra M solid if for every diffuse von Neumann subalgebra $\mathcal{A} \subset M$ one has that $\mathcal{A}' \cap M$ is amenable. It's easy to see that a nonamenable solid factor M is automatically *prime*, i.e., cannot be written as $M = M_1 \mathbin{\overline{\otimes}} M_2$, with M_i an infinite-dimensional factor for $i = 1, 2$.

Further results pertaining to the classification of normalizers and relative strong solidity were obtained in [Sinclair 2011; Ioana 2015; Isono 2015a; 2015b; Avsec 2012; Boutonnet, Houdayer and Vaes 2018; Houdayer and Vaes 2013; Caspers 2018].

1B. ***Noncommutative probability.*** Voiculescu [1985] introduced his highly influential free probability theory in the early 80's, in order to tackle some problems related to the free group factors. Since then, free probability theory has grown into an immense industry with far-reaching ramifications. Very roughly speaking, in the realm of free probability, classical probability spaces are replaced by C^*- or W^*-algebras endowed with distinguished states (normal in the W^* case), classical random variables by operators in those algebras, classical independence by Voiculescu's free independence, and the classical distribution function by Voiculescu noncommutative distribution of a noncommutative random variable, or joint distribution in the case of a system of random variables. In particular, the normal (gaussian) distribution is replaced by Wigner's semicircular law.

1B1. *Classical gaussian random variables.* We briefly recall the construction in Section 1.1 of [Peterson and Sinclair 2012]. Let H a real Hilbert space and, for $\xi \in H$, let l_ξ be the creation operator on the symmetric Fock space of $H_{\mathbb{C}} = H \oplus iH$. Then $s_1(\xi) = \frac{1}{2}(l_\xi + l_\xi^*)$ is an unbounded self-adjoint operator in the symmetric Fock space. The operators $s_1(\xi)$ and $s_1(\eta)$ commute for all ξ and η and are independent with respect to the vacuum state whenever $\langle \xi, \eta \rangle = 0$. Define $\Gamma_1(H)$ to be the abelian von Neumann algebra generated by the spectral projections of all the $s_1(\xi)$, $\xi \in H$ (or equivalently by all the unitaries $\omega(\xi_1, \ldots, \xi_k) = \exp(i\pi s(\xi_1) \cdots s(\xi_k)))$, equipped with the trace given by the restriction of the vacuum state. For $\|\xi\| = 1$, we have the moment formula

$$\tau(s(\xi)^m) = \delta_{m \in 2\mathbb{N}} \frac{m!}{2^{\frac{m}{2}}(\frac{m}{2})!} = |P_2(m)| = \sum_{\sigma \in P_2(m)} 1^{\mathrm{cr}(\sigma)},$$

where $P_2(m)$ is the collection of pair partitions on the set $\{1, \ldots, m\}$, and for $\sigma \in P_2(m)$, $\mathrm{cr}(\sigma)$ denotes the number of crossings of σ. These are exactly the moments of a classical gaussian random variable. By commutativity, independence and multilinearity, the moment formula can be extended to

$$\tau(s_1(\xi_1) \cdots s_1(\xi_m)) = \sum_{\sigma \in P_2(m)} 1^{\mathrm{cr}(\sigma)} \prod_{\{l,r\} \in \sigma} \langle \xi_l, \xi_r \rangle.$$

One also recalls:

Theorem 1.4 (classical central limit theorem). *Let $\{X_n\}_{n \geq 1}$ be a sequence of independent, identically distributed random variables on a probability space (Ω, Σ, P), all having mean equal to zero and variance equal to* 1. *Then the averages $S_n = n^{-\frac{1}{2}} \sum_{j=1}^n X_j$ converge in distribution to a normal (gaussian) random variable with mean zero and variance* 1.

If X_n are chosen such that $\sup_{n \geq 1} \|S_n\|_\infty < \infty$, then one can restate the central limit theorem by saying that the element $S = (S_n)_n \in (L^\infty(\Omega, P)^\omega, \tau_\omega)$ has a gaussian (normal) distribution, where ω is a free ultrafilter on $\mathbb{N}$ and $(L^\infty(\Omega)^\omega, \tau_\omega)$ is the ultraproduct von Neumann algebra. In other words, one could "simulate" gaussian elements using an ultraproduct model.

1B2. *Voiculescu's free semicircular random variables.* Voiculescu [1985] constructed a functor Φ from the category of real Hilbert spaces with contractions to the category of finite von Neumann algebras with completely positive maps. For $h \in H$, the element $\Phi(h) = s_0(h)$ is concretely realized as the real part of the creation operator on the full Fock space of $H_{\mathbb{C}}$. Moreover, he proved that for an orthonormal set $\{h_1, \ldots, h_m\} \subset H$, the elements $s_0(h_1), \ldots, s_0(h_m)$ are freely independent, have semicircular distributions given by $d\mu(t) = \frac{1}{\pi}\chi_{(-1,1)}(t)\sqrt{1-t^2}\,dt$, and generate a copy of the free group factor $L(\mathbb{F}_m)$. In particular, for a finite-dimensional Hilbert space H, $\Phi(H)$ is $*$-isomorphic to the free group factor $L(\mathbb{F}_{\dim(H)})$. It is well known that the moments of a semicircular variable are given by

$$\tau(s_0(h)^m) = \delta_{m\in 2\mathbb{N}} \frac{m!}{\left(\frac{m}{2}+1\right)\left(\left(\frac{m}{2}\right)!\right)^2} = \sum_{\sigma\in P_2(m)} 0^{\mathrm{cr}(\sigma)} = \sum_{\sigma\in \mathrm{NCP}_2(m)} 0^{\mathrm{cr}(\sigma)},$$

where we denote by $\mathrm{NCP}_2(m)$ the collection of noncrossing pair partitions on the set $\{1, \ldots, m\}$ and with the convention $0^0 = 1$. By direct computation, the above formula can be extended to

$$\tau(s_0(h_1)\cdots s_0(h_m)) = \sum_{\sigma\in P_2(m)} 0^{\mathrm{cr}(\sigma)} \prod_{\{l,r\}\in\sigma} \langle h_l, h_r\rangle.$$

Let's recall Voiculescu's central limit theorem [1985]:

Theorem 1.5 (Voiculescu's central limit theorem). *Let $\{a_n\}_{n\geq 1}$ be a sequence of freely independent self-adjoint random variables in a C^*-probability space (A, φ). Assume that $\varphi(a_n) = 0$ for all n, $\sup_{n\geq 1}\|a_n\| < \infty$ and $\lim_{n\to\infty} n^{-1}\sum_{j=1}^n \varphi(a_n^2) = \frac{1}{4}$. Then the elements $S_n = n^{-\frac{1}{2}}\sum_{j=1}^n a_j$ converge in distribution to a semicircular element; i.e., their limit distribution is the semicircular law.*

In particular, this says that, beside the Fock space construction, one could create semicircular random variables by taking elements of the form

$$S = (S_n)_n = \left(n^{-\frac{1}{2}}\sum_{j=1}^n a_j\right)_n \in (M^\omega, \tau_\omega),$$

with a_n as above in a finite W^*-probability space $(A, \varphi) = (M, \tau)$ and satisfying the additional condition

$$\sup_{n\geq 1}\left\|n^{-\frac{1}{2}}\sum_{j=1}^n a_j\right\|_\infty < \infty.$$

Let us also recall an informal statement of Voiculescu's matrix limit theorem [1991, Theorem 2.2]:

Theorem 1.6 (Voiculescu's matrix limit theorem). *Any family of random matrices with size going to infinity having independent normalized gaussian entries converges in distribution to a free semicircular family.*

Again, we could interpret this as saying that one can create free semicircular families using elements of some suitable ultraproduct of matrix algebras over abelian von Neumann algebras.

1B3. *q-gaussian von Neumann algebras.* The q-gaussian von Neumann algebras $\Gamma_q(H)$, for H a real Hilbert space, were introduced by Bożejko and Speicher [1991; 1992; 1994; 1996] and further studied in [Bożejko, Kümmerer and Speicher 1997; Ricard 2005; Śniady 2001; 2004; Królak 2000; 2006; Nou 2004; 2006; Shlyakhtenko 2004; 2009; Dykema and Nica 1993; Avsec 2012; Dabrowski 2014]. For $-1 < q < 1$, Bożejko and Speicher constructed a functor Γ_q from the category of real Hilbert spaces with contractions to the category of finite von Neumann algebras with completely positive maps. $\Gamma_q(H)$ is called the q-gaussian von Neumann algebra associated to H. The generators $\Gamma_q(h) = s_q(h)$, for $h \in H$, admit a concrete representation as the real part of the creation operator by h on the q-Fock space of H; for details, see, e.g., Section 2 of [Bożejko and Speicher 1991]. When $q = 0$, the functor Γ_q coincides with Voiculescu's functor Φ, so $\Gamma_0(H) = L(\mathbb{F}_{\dim(H)})$. A direct computation using the concrete realization of the $s_q(h)$'s gives the moment formula

$$\tau(s_q(h_1)\cdots s_q(h_m)) = \sum_{\sigma\in P_2(m)} q^{\mathrm{cr}(\sigma)} \prod_{\{l,r\}\in\sigma} \langle h_l, h_r\rangle,$$

which is why, in view of the above, the $s_q(h)$'s can be called q-semicircular elements. The central limit theorem holds in the q-gaussian context as well, see, e.g., [Speicher 1992, Theorem 1; 1993, Theorems 1 and 2; Bożejko 1991; Junge and Zeng 2015, Appendix A], but its statement is very technical and we omit it. Also, the q-gaussian von Neumann algebras admit random matrix models; see [Śniady 2001, Theorem 3]. We mention that, originally, the q-gaussian von Neumann algebras were studied as concrete implementations of the canonical q-commutation relations, or as examples of nonclassical Brownian motions, see, e.g., [Bożejko and Speicher 1991; 1992; 1994], but we choose to downplay these aspects in the present work. The central limit theorem suggests that the q-gaussians can be introduced via an ultraproduct model. In fact, a concrete ultraproduct embedding which holds a great heuristic value for us is given by

$$\Gamma_q(H) \to (\Gamma_q(\ell^2\otimes H))^\omega, \quad s_q(h) \mapsto \left(n^{-\frac{1}{2}}\sum_{j=1}^{n} s_q(e_j\otimes h)\right)_n,$$

where $\{e_j\}_{j\in\mathbb{N}}$ is the standard orthonormal basis of $\ell^2 = \ell^2(\mathbb{N})$.

1C. ***Generalized q-gaussian von Neumann algebras with coefficients.*** In this article we introduce a new class of von Neumann algebras and prove some structural results about them. Specifically, we introduce the *generalized q-gaussian von Neumann algebras with coefficients* associated to a sequence of *symmetric copies* (π_j, B, A, D). A 4-tuple (π_j, B, A, D) is called a sequence of symmetric copies (of A) if B, A, D are finite tracial von Neumann algebras such that $B \subset A \cap D$ and $\pi_j : A \to D$, $j \in \mathbb{N}$, are unital trace-preserving normal $*$-homomorphisms satisfying

(1) $\pi_j|_B = \mathrm{id}_B$ for all j;

(2) $E_B(\pi_{j_1}(a_1)\cdots\pi_{j_m}(a_m)) = E_B(\pi_{\sigma(j_1)}(a_1)\cdots\pi_{\sigma(j_m)}(a_m))$ for all finite permutations σ on $\mathbb{N}$, all indices $j_1,\ldots,j_m$ in $\mathbb{N}$ and all $a_1,\ldots,a_m$ in A, where $E_B : D \to B$ is the canonical trace-preserving conditional expectation.

We mention that our copies satisfy some additional independence conditions (see Definition 3.2). Let $-1 < q < 1$ be fixed. For H an infinite-dimensional (real) Hilbert space and S a self-adjoint subset of A containing 1, the generalized q-gaussian von Neumann algebra

$$\Gamma_q(B, S \otimes H) \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$$

with coefficients in B and associated to the symmetric copies (π_j, B, A, D) is defined as the von Neumann subalgebra generated by the elements

$$s_q(a,h) = \left(n^{-\frac{1}{2}} \sum_{j=1}^{n} s_q(e_j \otimes h) \otimes \pi_j(a)\right)_n, \quad a \in BSB = \{b_1 a b_2 : b_1, b_2 \in B, a \in S\},\ h \in H.$$

Here ω is a free ultrafilter on the natural numbers and $\Gamma_q(\ell^2 \otimes H)$ is the q-gaussian von Neumann algebra. When H is finite-dimensional, one needs to further apply a "closure operation" (see Definition 3.4 and Proposition 3.14 for more details). The crucial observation here is that, since a Fock space model is not available, we are forced to introduce our generalized gaussians via an ultraproduct model. The generators $s_q(a,h)$ satisfy the moment formula

$$\tau(s_q(a_1,h_1)\cdots s_q(a_m,h_m)) = \delta_{m\in 2\mathbb{N}} \sum_{\sigma\in P_2(m)} q^{\mathrm{cr}(\sigma)} \prod_{\{l,r\}\in\sigma} \langle h_l, h_r\rangle \tau_D(\pi_{\phi_\sigma(1)}(a_1)\cdots\pi_{\phi_\sigma(m)}(a_m)),$$

as well as the B-valued moment formula

$$E_B(s_q(a_1,h_1)\cdots s_q(a_m,h_m)) = \delta_{m\in 2\mathbb{N}} \sum_{\sigma\in P_2(m)} q^{\mathrm{cr}(\sigma)} \prod_{\{l,r\}\in\sigma} \langle h_l, h_r\rangle E_B(\pi_{\phi_\sigma(1)}(a_1)\cdots\pi_{\phi_\sigma(m)}(a_m)),$$

where for every pair partition $\sigma = \{\{k_1', k_1''\}, \ldots, \{k_p', k_p''\}\} \in P_2(m)$, the function $\phi_\sigma : \{1,\ldots,m\} \to \{1,\ldots,p = \frac{m}{2}\}$ is chosen so that $\phi_\sigma(k_1') = \phi_\sigma(k_1'') = 1, \ldots, \phi_\sigma(k_p') = \phi_\sigma(k_p'') = p$. In view of all of the above, the elements $s_q(a,h)$ could thus judiciously be called "B-valued q-semicircular random variables having symmetric B-moments".

When compared to pure q-gaussians, the generalized q-gaussian von Neumann algebras with coefficients can be viewed as an analogue of the cross-product von Neumann algebras $B \rtimes \Gamma$ as opposed to pure group von Neumann algebras $L(\Gamma)$. This analogy can be given some substance along the lines of [Shlyakhtenko 1999]. However, in the present work we do not pursue this insight and use this analogy merely as a guideline for the implementation of Popa's deformation-rigidity strategy. The main result we prove about our generalized q-gaussian algebras is:

Theorem A. *Let (π_j, B, A, D) be a sequence of symmetric independent copies, H be a finite-dimensional Hilbert space and $\mathcal{A} \subset M = \Gamma_q(B, S \otimes H)$ be a diffuse von Neumann subalgebra which is amenable relative to B inside M. For every $s \geq 0$, define $D_s(S)$ to be the following right B-submodule of $L^2(D)$:*

$$\overline{\mathrm{span}}^{\|\cdot\|_2}\{E_{A_{\{1,\ldots,s\}}}(\pi_{\phi_\sigma(1)}(x_1)\cdots\pi_{\phi_\sigma(m)}(x_m)) : m \geq 1,\ \sigma \in P_{1,2}(m),\ x_i \in BSB\},$$

where $m = s + 2p$, $P_{1,2}(m)$ is the set of pair-singleton partitions of $\{1,\ldots,m\}$, σ runs over all pair-singleton partitions in $P_{1,2}(m)$ having s singletons and p pairs and for such a partition $\sigma =$

$\{\{k_1\},\dots,\{k_s\},\{k_1',k_1''\},\dots,\{k_p',k_p''\}\}$, the function $\phi_\sigma:\{1,\dots,m\}\to\{1,\dots,s+p\}$ satisfies $\phi_\sigma(k_1)=1,\dots,\phi_\sigma(k_s)=s$ and $\phi_\sigma(k_1')=\phi_\sigma(k_1'')=s+1,\dots,\phi_\sigma(k_p')=\phi_\sigma(k_p'')=s+p$. Assume that there exist constants $d,C>0$ such that $\dim_B(D_s(S))\le Cd^s$ for all $s\ge 1$. Then at least one of the following statements is true:

(1) $\mathcal{A}\prec_M B$.

(2) *The von Neumann algebra $P=\mathcal{N}_M(\mathcal{A})''$ generated by the normalizer of $\mathcal{A}$ in M is amenable relative to B inside M.*

The technical condition on the dimension of the B-modules $D_k(S)$ implies in particular that the subspace of Wick words of length k is finitely generated over B for all $k\ge 1$ (see Theorem 3.16 and Proposition 3.20). This last condition in turn is the exact analogue of the group cocycle being proper in the case of cross-product von Neumann algebras.

As a consequence of our Theorem A, we find a number of examples of generalized q-gaussians which are strongly solid (when $B=\mathbb{C}$ or B is finite-dimensional) or strongly solid relative to B for diffuse B. While the class of generalized q-gaussian von Neumann algebras with coefficients is huge (roughly speaking such a von Neumann algebra can be constructed starting from any action of the infinite symmetric group on another finite von Neumann algebra), the range of examples to which our Theorem A applies is greatly restricted by the technical assumptions we make. The examples in the corollary below are introduced in more detail in Section 4.

Corollary B. *The following von Neumann algebras are strongly solid relative to B*:

(1) (*see Section 4A*) $B\mathbin{\bar\otimes}\Gamma_q(H)$ *for H a finite-dimensional Hilbert space.*

(2) (*see Section 4C2*) $\Gamma_q(B,S\otimes H)$ *associated to the symmetric independent copies (π_j,B,A,D) constructed in the following way: take a trace preserving action α of $\mathbb{Z}$ on a finite von Neumann algebra N. Let $\mathcal{H}=\langle g_j : j\ge 0\rangle$ be the Heisenberg group, take $\eta:\mathcal{H}\to\mathbb{Z}$ an onto group homomorphism and define $\beta:\mathcal{H}\curvearrowright N$ by*

$$\beta_g(x)=\alpha_{\eta(g)}(x),\quad g\in\mathcal{H},\ x\in N.$$

Let $\mathcal{H}_1=\langle g_0,g_1\rangle$ and take $B=N\rtimes\mathbb{Z}=N\mathbin{\bar\otimes}L(\mathbb{Z})$, $A=N\rtimes\mathcal{H}_1$ and $S=\{1,g_1,g_1^{-1}\}$. Define $\pi_j:A\to D$ by

$$\pi_j(xu_{g_1})=\alpha_{\eta(g_j)}(x)u_{g_j},\quad \pi_j(xu_{g_0})=xu_{g_0},\quad x\in N,\ j,k\in\mathbb{N}.$$

(3) (*see Section 4D1*) $\Gamma_q(\mathbb{C},S\otimes K)$ *associated to the symmetric copies $(\pi_j,B=\mathbb{C},A=\Gamma_{q_0}(H),D=\Gamma_q(\ell^2\otimes H))$, where $\pi_j(s_{q_0}(h))=s_q(e_j\otimes h)$ and K is a finite-dimensional Hilbert space.*

(4) (*see Section 4D2*) $\Gamma_q(B_d,S\otimes H)$ *associated to the symmetric copies (π_j,B_d,A_d,D_d), where $B_d=L(\Sigma_{[-d,0]})$, $A_d=L(\Sigma_{[-d,1]})$, $D_d=L(\Sigma_{[-d,\infty)})=\{u_\sigma:\sigma\in\Sigma_{[-d,\infty)}\}''$ and $S=\{1,u_{(01)}\}$ for a fixed $d\in\mathbb{N}\setminus\{0\}$; here $\Sigma_{\mathbb{Z}}$ is the group of finite permutations on $\mathbb{Z}$ and for a subset $F\subset\mathbb{Z}$, $\Sigma_F\subset\Sigma_{\mathbb{Z}}$ is the group of finite permutations on F naturally embedded into $\Sigma_{\mathbb{Z}}$. The copies are defined by $\pi_j(a)=u_{(1j)}au_{(1j)}$, $a\in A_d$.*

(4) (*see Section 4B*) $\Gamma_q(\mathbb{C}, S\otimes H)$ *associated to the symmetric copies* $\big(\pi_j, B=\mathbb{C}, A=L(\mathbb{Z})=\{u\}'', D=\ast_{\mathbb{N}} L(\mathbb{Z})\big)$, *where* u *is a Haar unitary, the symmetric copies* $\pi_j : A\to D$ *are defined by the relations* $\pi_j(u)=\cdots * 1 * u * 1 * \cdots$, *and* $S=\{1,u,u^*\}$.

It follows that the examples in (3), (4) *and* (5) *are strongly solid and hence solid nonamenable von Neumann algebras. In particular, they are prime von Neumann algebras. Note that when* $q=0$ *and* H *is trivial, the example in* (5) *is* $*$*-isomorphic to* $L(\mathbb{F}_\infty)$, *thus reproving the strong solidity of the free group factors.*

Using Theorem A we also deduce the following:

Corollary C. *Let* $M_i=\Gamma_{q_i}(B_i, S_i\otimes H_i)$ *be associated with two sequences of symmetric independent copies* (π_j^i, B_i, A_i, D_i) *and two subsets* $S_i\subset A_i$, *and* $-1<q_i<1$, $i=1,2$. *Assume that* $2\le\dim(H_i)<\infty$, $\dim_{B_i}(D_k(S_i))\le Cd^k$ *for fixed constants* $d, C>0$ *and* B_i *are amenable for* $i=1,2$. *If* $M_1\subset M_2$, *then* $B_1\prec_{M_2} B_2$. *Moreover, if* $M_1=M_2=M$, *it follows that* $B_1\prec_M B_2$ *and* $B_2\prec_M B_1$.

This result can be regarded as an analogue of the "uniqueness of Cartan subalgebra" results in the group measure space construction setting. Note however, that even when B is abelian, it is not a MASA in $M=\Gamma_q(B, S\otimes H)$. Indeed, B always commutes with a copy of $\Gamma_q(H)$ inside $M=\Gamma_q(B, S\otimes H)$; hence it can never be maximally abelian. Thus, even when B_1 and B_2 are both abelian diffuse, we cannot avail ourselves of Popa's results [2006a, Appendix, Theorem A.1] about unitary conjugacy of Cartan subalgebras to conclude that B_1 is unitarily conjugate to B_2, so this double intertwining result is optimal in our case. Finally, we deduce some nonisomorphism and nonembedability results for generalized q-gaussians.

Corollary D. *Under the assumptions of Corollary C, if we moreover assume that*

(1) B_1 *is finite-dimensional and* B_2 *is amenable diffuse, or*

(2) B_1 *is abelian and* B_2 *is the hyperfinite* II_1 *factor,*

then $M_2=\Gamma_{q_2}(B_2, S_2\otimes H_2)$ *cannot be realized as a von Neumann subalgebra of* $M_1=\Gamma_{q_1}(B_1, S_1\otimes H_1)$. *In particular* M_1 *and* M_2 *are not* $*$*-isomorphic.*

1D. ***Comments on the proofs and structure of the article.*** Finally, a couple of words about the main ideas behind the proof of Theorem A. We mention that actually Theorem A will be derived from the technical Theorem 7.2 much along the lines of Theorem 3.1 in [Popa and Vaes 2014a], whose statement and proof can be found in Section 7. We follow the approach of Popa and Vaes [2014a; 2014b], which in turn is a development of the original ground-breaking insight in [Ozawa and Popa 2010a; 2010b]. Let $\mathcal{A}\subset M=\Gamma_q(B, S\otimes H)$ be a diffuse von Neumann subalgebra which is amenable relative to B. The two main ingredients of the proof are, just as in [Popa and Vaes 2014a]:

(1) The fact that the embedding $\mathcal{A}\subset M$ is *weakly compact relative to* B. This is the existence of a sequence of normal states viewed as unit vectors $\xi_n\in L^2(\mathcal{N})$, where $\mathcal{N}\supset M$ is a suitable (in general nontracial) von Neumann algebra, which are asymptotically invariant to the action of the "tensor double" of the normalizer of $\mathcal{A}$ in M; the existence of these states is a consequence of the weak amenability (with Cowling–Haagerup constant 1) of the pure q-gaussian von Neumann algebras $\Gamma_q(H)$.

(2) The existence of a one-parameter group of $*$-automorphisms (α_t) of a suitable dilation $\widetilde{\mathcal{N}}$ of $\mathcal{N}$ having good properties.

The proof proceeds by applying the deformation α_t to the vectors ξ_n. Then either the deformation significantly displaces the vectors, or it does not. The first case yields the amenability of $P = \mathcal{N}_M(\mathcal{A})''$ relative to B, while the second implies that $\mathcal{A} \prec_M B$, via the fact that the maps T_t (where $t \to T_t$ is the canonical semigroup of u.c.p. maps on M) are compact over B, in the terminology of Popa and Ozawa.

While it's true that conceptually we follow closely the approach of [Popa and Vaes 2014a], it has to be strongly emphasized that the technical difficulties of our approach are vastly larger. First of all, since our objects are much more elusive and complicated than cross-product von Neumann algebras, being defined as subalgebras of an ultraproduct to begin with, the proof of Theorem 5.1 (the existence of the invariant states), which is the key ingredient in the proof of the technical theorem, is riddled with daunting difficulties. Among these, constructing the von Neumann algebras involved in the deformation-rigidity argument (e.g., $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ above) and the spaces on which they act was a particularly challenging task. Also, the complete boundedness of certain maps used in the proof turns out to be surprisingly nontrivial and requires the use of delicate operator-space techniques; in the pure Hilbert space setting, somewhat similar techniques have been used in [Avsec 2012; Nou 2004; 2006]. Second, and just as important, we cannot use the reduction to the "trivial-action case" (i.e., the tensor product case), as Popa and Vaes do. The reduction step plays a crucial role in their proof, because it is only in the tensor product setting that they are able to prove the relative weak compactness property and subsequently carry out the deformation-rigidity arguments. The reduction is essentially based on the use of the comultiplication map in the cross-product case. Since we have no good substitute for the comultiplication map, we cannot reduce to the tensor-product case, and hence everything becomes much more complicated and technically involved, including the standard forms of the von Neumann algebras involved, which are in general nontracial.

The article contains six sections beside the introduction, and is organized as follows: Section 2 contains some needed technical preliminaries. In Section 3 we introduce the generalized q-gaussian von Neumann algebras and prove their basic properties; among other things, we exhibit the canonical generators of $\Gamma_q(B, S \otimes H)$ (the Wick words), prove that they actually belong to the algebra and prove a very useful reduction result about them. Section 4 lists a rather wide range of examples of generalized q-gaussian von Neumann algebras constructed from a variety of symmetric independent copies. We devote Section 5 to the proof of the relative weak compactness of the embedding $\mathcal{A} \subset M$; the second half of this section contains some technical results about the complete boundedness of certain multipliers used in the proof. In Section 6 we prove that under the assumption of polynomial growth of the dimensions of the modules $D_k(S)$ over B, the natural deformation bimodules used in the technical theorem are weakly contained in $L^2(M) \otimes_B L^2(M)$, a fact which will be further used in combination with the technical theorem to derive Theorem A. The proof is based on a novel and "nondeterministic" approach. Indeed, the calculation of the deformation bimodules in the q-gaussian setting is a real challenge even in the case of pure $\Gamma_q(H)$ von Neumann algebras, see [Avsec 2012], and it becomes even more so when we allow q-gaussians with coefficients. Section 7 contains the proof of the main technical theorem and its

applications. Beside many examples of strongly solid generalized q-gaussian von Neumann algebras, we also obtain some nonisomorphism and nonembedability results.

2. Preliminaries

2A. ***Popa's intertwining techniques.*** We will briefly review the concept of intertwining two subalgebras inside a finite von Neumann algebra, along with the main technical tools developed by Popa [2006a; 2006b]. Let (M,τ) be a finite von Neumann algebra, let $f \in \mathcal{P}(M)$ and $Q \subset fMf$, $B \subset M$ be two von Neumann subalgebras. We say that *a corner of Q can be intertwined into B inside M* and denote it by $Q \prec_M B$ (or simply $Q \prec B$) if there exist two nonzero projections $q \in Q$, $p \in B$, a nonzero partial isometry $v \in qMp$, and a $*$-homomorphism $\psi : qQq \to pBp$ such that $v\psi(x) = xv$ for all $x \in qQq$. The partial isometry v is called an intertwiner between Q and B. Popa [2006b] proved the following intertwining criterion:

Theorem 2.1 [Popa 2006b, Corollary 2.3]. *Let M be a von Neumann algebra and let $Q \subset fMf$, $B \subset M$ be diffuse subalgebras for some projection $f \in M$. Then the following are equivalent*:

(1) $Q \prec_M B$.

(2) *There exists a finite set $\mathcal{F} \subset fMf$ and $\delta > 0$ such that for every unitary $v \in \mathcal{U}(Q)$ we have*

$$\sum_{x,y\in\mathcal{F}} \|E_B(xvy^*)\|_2^2 \geq \delta.$$

Let (M,τ) be a finite von Neumann algebra and $\Phi : M \to M$ a normal, completely positive map. We say that Φ is subtracial if $\tau \circ \Phi \leq \tau$. If Φ is subtracial, then, due to the Schwarz inequality, we automatically have

$$\|\Phi(x)\|_2^2 = \tau(\Phi(x)^*\Phi(x)) \leq \tau(\Phi(x^*x)) \leq \tau(x^*x) = \|x\|_2^2;$$

i.e., Φ is automatically $\|\cdot\|_2$-contractive, and hence extends to a bounded operator on $L^2(M)$ defined by

$$T_\Phi : L^2(M) \to L^2(M), \quad T_\Phi(\hat{x}) = \widehat{\Phi(x)}, \quad x \in M.$$

Let $B \subset (M,\tau)$ be an inclusion of finite von Neumann algebras. The basic construction (of M with B) is defined by, see, e.g., [Popa 2006a],

$$\langle M, e_B\rangle = (M \cup \{e_B\})'' = (JBJ)' \subset B(L^2(M)),$$

where $L^2(M)$ is the standard form of M and $J : L^2(M) \to L^2(M)$ is the associated conjugation. The definition of the *compact ideal space* of the basic construction (more generally of any semifinite von Neumann algebra) can be found in [Popa 2006a, 1.3.3].

Definition 2.2. Let (M,τ) be a finite von Neumann algebra, $B \subset M$ a von Neumann subalgebra and $\Phi : M \to M$ a normal, completely positive, B-bimodular, subunital, subtracial map. We say that Φ is compact over B if the canonical operator $T_\Phi : L^2(M) \to L^2(M)$ belongs to the compact ideal space of the basic construction $\langle M, e_B\rangle$.

The following result is Proposition 2.7 in [Ozawa and Popa 2010a]; see also [Popa 2006a, 1.3.3].

Proposition 2.3. *Let (M,τ) be a finite von Neumann algebra and let $B, P \subset M$ be two von Neumann subalgebras. Let $\Phi : M \to M$ be a normal, completely positive, subunital, subtracial map which is compact over B and assume that*

$$\inf_{u\in\mathcal{U}(P)} \|\Phi(u)\|_2 > 0.$$

Then $P \prec_M B$.

2B. ***Bimodules over von Neumann algebras and weak containment.*** Let M, Q be two von Neumann algebras. An M-Q Hilbert bimodule $\mathcal{K}$ is simply a Hilbert space together with a pair of normal $*$-representations $\lambda : M \to B(\mathcal{K})$, $\rho : Q^{\mathrm{op}} \to B(\mathcal{K})$ with commuting ranges. To these one can associate a $*$-representation $\pi : M \otimes_{\mathrm{bin}} Q^{\mathrm{op}} \to B(\mathcal{K})$ by

$$\pi\Big(\sum_k x_k \otimes y_k^{\mathrm{op}}\Big)\xi = \sum_k \lambda(x_k)\rho(y_k^{\mathrm{op}})\xi, \quad x_k \in M,\ y_k \in Q,\ \xi \in \mathcal{K}.$$

Definition 2.4. Let M, Q be two von Neumann algebras and $\mathcal{H}, \mathcal{K}$ be two M-Q bimodules. We say that $\mathcal{K}$ *is weakly contained in* $\mathcal{H}$ and denote it by $\mathcal{K} \prec \mathcal{H}$ if $\|\pi_{\mathcal{K}}(x)\| \le \|\pi_{\mathcal{H}}(x)\|$ for all $x \in M \otimes_{\mathrm{alg}} Q$, where $\pi_{\mathcal{H}}, \pi_{\mathcal{K}}$ are the $*$-representations canonically associated to the left and right actions on $\mathcal{H}, \mathcal{K}$ respectively.

Give an M-Q bimodule $\mathcal{K}$ and an Q-N bimodule $\mathcal{H}$ we will denote by $\mathcal{K} \otimes_Q \mathcal{H}$ their Connes tensor product, which is an M-N bimodule. For the definition and basic properties of the Connes tensor product, see Sections 2.3, 2.4 in [Popa and Vaes 2014a]. The Connes tensor product is well-behaved with respect to weak containment; see [loc. cit.].

Definition 2.5 [Popa and Vaes 2014a, Definition 2.3 and Proposition 2.4]. Let (M,τ_M) and (Q,τ_Q) be finite tracial von Neumann algebras and $P \subset M$ a von Neumann subalgebra. We say that an M-Q bimodule $\mathcal{K}$ is *left P-amenable* if one of the following equivalent conditions holds:

(1) There exists a P-central state Ω on $B(\mathcal{K}) \cap (Q^{\mathrm{op}})'$ such that $\Omega|_M = \tau_M$.

(2) $L^2(M) \prec \mathcal{K} \otimes_Q \overline{\mathcal{K}}$ as M-P bimodules.

Definition 2.6. Let (M,τ) be a tracial von Neumann algebra, and let $B, P \subset M$ be two von Neumann algebras. We say that P *is amenable relative to B inside M* if one of the following equivalent conditions holds:

(1) The M-B bimodule $L^2(M)$ is left P-amenable.

(2) $L^2(M) \prec L^2(M) \otimes_B L^2(M)$ as M-P bimodules.

Remark 2.7. Let (M,τ) be a finite von Neumann algebra and $B, P \subset M$ be two von Neumann subalgebras. Let $\mathcal{K}$ be a left P-amenable M-M bimodule such that $\mathcal{K} \prec L^2(M) \otimes_B \mathcal{H}$ for some B-M bimodule $\mathcal{H}$. Then P is amenable relative to B inside M. Indeed, we have that, as M-P bimodules,

$$L^2(M) \prec \mathcal{K} \otimes_M \overline{\mathcal{K}} \prec (L^2(M) \otimes_B \mathcal{H}) \otimes_M (\overline{\mathcal{H}} \otimes_B L^2(M)) \prec L^2(M) \otimes_B L^2(M).$$

2C. ***Standard forms of nontracial von Neumann algebras.*** In some instances we will have to consider nontracial von Neumann algebras M and their standard forms. Let us recall that a (hyper-)standard form for a von Neumann algebra is given by (M, H, J, P), where $J : H \to H$ is an antilinear unitary, $P \subset H$ is a self-dual cone such that

(i) the map $M \to M'$, $x \mapsto JxJ$, is a $*$-anti-isomorphism acting trivially on $\mathcal{Z}(M)$;

(ii) $J\xi = \xi$ for $\xi \in P$;

(iii) $xJxJ(P) \subset P$ for $x \in M$.

The standard form of M is unique up to $*$-isomorphism; see, e.g., [Haagerup 1975]. A particularly useful way of describing the standard form of M is the abstract Haagerup $L^2(M)$ space, which we briefly describe below. The reader can find more details in [Haagerup 1979; Terp 1982; Haagerup, Junge and Xu 2010]. Let (M, φ) be a von Neumann algebra endowed with a normal semifinite faithful (n.s.f.) weight. Consider $\mathcal{M} = M \rtimes_{\sigma^\varphi} \mathbb{R}$, the cross-product von Neumann algebra of M with $\mathbb{R}$ by the modular automorphism group σ_t^φ. Then $\mathcal{M}$ is semifinite and there exists an n.s.f. trace τ on $\mathcal{M}$ such that

$$(D\hat{\varphi} : D\tau)_t = \lambda(t), \quad t \in \mathbb{R},$$

where $\hat{\varphi}$ is the dual weight, $(D\hat{\varphi} : D\tau)_t$ is the Connes cocycle and $\lambda(t)$ is the group of translations on $\mathbb{R}$. Moreover, τ is the unique n.s.f. trace on $\mathcal{M}$ which satisfies

$$\tau \circ \hat{\sigma}_t^\varphi = e^{-t}\tau, \quad t \in \mathbb{R}.$$

Given another n.s.f. weight ψ on M, denote by h_ψ the Radon–Nikodym derivative of $\hat{\psi}$ with respect to τ, i.e., the unique positive self-adjoint operator affiliated to $\mathcal{M}$ such that

$$\hat{\psi}(x) = \tau(h_\psi^{\frac{1}{2}} x h_\psi^{\frac{1}{2}}), \quad x \in \mathcal{M}_+.$$

Then the following condition holds:

$$\hat{\sigma}_t^\varphi(h_\psi) = e^{-t} h_\psi, \quad t \in \mathbb{R}.$$

Moreover, the map $\psi \mapsto h_\psi$ is a bijection from the set of n.s.f. weights on M to the set of positive self-adjoint operators affiliated to $\mathcal{M}$ which satisfy the above condition. Let $L_0(\mathcal{M}, \tau)$ be the $*$-algebra consisting of all the operators on $L^2(\mathbb{R}, H)$ which are measurable with respect to $(\mathcal{M}, \tau)$. For $p > 0$, the Haagerup $L^p(M, \varphi)$ is defined by

$$L^p(M, \varphi) = \{x \in L_0(\mathcal{M}, \tau) : \hat{\sigma}_t^\varphi(x) = e^{-\frac{t}{p}} x \text{ for all } t \in \mathbb{R}\}.$$

One can define a bicontinuous linear isomorphism from M_* to $L^1(M, \varphi)$ as the linear extension of the map

$$M_*^+ \to L^1(M, \varphi), \quad \psi \mapsto h_\psi.$$

The norm $\|\cdot\|_1$ on $L^1(M, \varphi)$ is defined by requiring that the above isomorphism be isometric. One can define a norm-1 linear functional tr on $L^1(M, \varphi)$ by $\mathrm{tr}(h_\psi) = \psi(1)$, and thus $\|h\|_1 = \mathrm{tr}(|h|)$, $h \in L^1(M, \varphi)$.

This "trace" is indeed tracial; i.e.,

$$\operatorname{tr}(xy) = \operatorname{tr}(yx) \quad \text{for } x, y \in L^2(M).$$

Let $x = u|x|$ be the polar decomposition of an element $x \in L_0(\mathcal{M}, \tau)$. Then we have

$$x \in L^p(M,\varphi) \Longleftrightarrow u \in M \quad \text{and} \quad |x| \in L^p(M,\varphi) \Longleftrightarrow u \in M \quad \text{and} \quad |x|^p \in L^1(M,\varphi).$$

This allows one to introduce the $\|\cdot\|_p$-norm on $L^p(M,\varphi)$, by $\|x\|_p = \||x|^p\|_1^{\frac{1}{p}}$ for $x \in L^p(M,\varphi)$. Let's also remark that the weight φ can be recovered from the trace. Define

$$N_\varphi = \{x \in M : \varphi(x^*x) < \infty\}, \quad M_\varphi = N_\varphi^* N_\varphi = \operatorname{span}\{y^*x : x, y \in N_\varphi\}.$$

The dual weight $\hat{\varphi}$ has a Radon–Nikodym derivative with respect to τ, which will be denoted by d_φ. Then for every $x \in M_\varphi$ the operator $d_\varphi^{\frac{1}{2}} x d_\varphi^{\frac{1}{2}}$ is closable, its closure belongs to $L^1(M,\varphi)$ and we have the relation

$$\varphi(x) = \operatorname{tr}(d_\varphi^{\frac{1}{2}} x d_\varphi^{\frac{1}{2}}), \quad x \in M_\varphi.$$

If φ is a bounded functional, then $d_\varphi \in L^1(M,\varphi)$ and the above identity becomes

$$\varphi(x) = \operatorname{tr}(d_\varphi^{\frac{1}{2}} x d_\varphi^{\frac{1}{2}}) = \operatorname{tr}(x d_\varphi), \quad x \in M.$$

The Haagerup space $L^p(M,\varphi)$ does not depend on the choice of the n.s.f. weight φ up to isomorphism; hence it can simply be denoted by $L^p(M)$. It's easy to see that M is naturally represented in standard form on the Haagerup space $L^2(M)$ via the obvious left and right actions. When M is finite and τ is a faithful trace on M, the Haagerup space $L^2(M) = L^2(M,\tau)$ coincides with the usual one.

2D. *W^*-Hilbert modules.* We also have to recall some facts about (right) Hilbert W^*-modules. According to [Paschke 1973; 1974], see also [Junge and Sherman 2005], a right Hilbert C^*-module X over a *von Neumann algebra* M is self-dual if and only if admits a module basis, i.e., a family $\{\xi_\alpha\} \subset X$ such that

$$X = \overline{\operatorname{span}} \sum_\alpha \xi_\alpha M \quad \text{and} \quad \langle \xi_\alpha, \xi_\beta \rangle = \delta_{\alpha\beta} e_\alpha \in \mathcal{P}(M).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the M-valued inner product. In this situation, there exists an index set I, a projection $e \in B(\ell_2(I)) \bar{\otimes} M$, and a right module isomorphism $u : X \to e(\ell_2(I)^c \bar{\otimes} M)$. Indeed, for a basis ξ_α with $\langle \xi_\alpha, \xi_\alpha \rangle = e_\alpha$, the map u is given by $u\big(\sum_\alpha \xi_\alpha m_\alpha\big) = [e_\alpha m_\alpha]$. Here $\ell_2(I)^c \bar{\otimes} M$ denotes the space of strongly convergent columns indexed by I. Then it is easy to see that the C^*-algebra $\mathcal{L}(X)$ of adjointable operators on X is indeed a von Neumann algebra, and isomorphic to $e(B(\ell_2(I)) \bar{\otimes} M)e$. Moreover, the M-compact operators $\mathcal{K}(X)$ spanned by the maps $\Phi_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle$ are weakly dense in $\mathcal{L}(X)$, because $K(\ell_2(I)) \otimes_{\min} M$ is weakly dense in $B(\ell_2(I)) \bar{\otimes} M$. With the help of a normal faithful state, we can complete X to the Hilbert space $L_2(X,\phi)$ with inner product $(\xi,\eta) = \phi(\langle \xi, \eta \rangle)$. Let $\iota_\phi : X \to L_2(X,\phi)$ be the inclusion map. Then

$$\pi : \mathcal{L}(X) \to B(L_2(X,\phi)), \quad \pi(T)(\iota_\phi(x)) = \iota_\phi(Tx),$$

defines a normal faithful $*$-homomorphism such that

$$\pi(\mathcal{L}(X)) = B(L_2(X,\phi)) \cap (M^{\mathrm{op}})'.$$

This is indeed very easy to check for $\mathcal{L}(X) = e(B(\ell_2(I)) \bar{\otimes} M)e$. See [Paschke 1973; 1974; Junge and Sherman 2005] for more details and references.

3. The generalized gaussian von Neumann algebras with coefficients: definition and basic properties

Throughout this section we will freely use the basic properties of the pure Hilbert space q-gaussian von Neumann algebras $\Gamma_q(H)$, as they can be found in Section 4 of [Junge, Longfield and Udrea 2014]; see also [Avsec 2012]. The following result is due to Voiculescu, Dykema and Nica [1992].

Proposition 3.1. *Let (M,φ) and (N,ψ) be two von Neumann algebras endowed with faithful normal tracial states. Let $(x_i)_{i=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be countable systems of generators for M and N, respectively. Assume that for every $m \geq 1$, every $i_1,\dots,i_m \in \mathbb{N}$ and every $\varepsilon_i \in \{1,*\}$ we have*

$$\varphi(x_{i_1}^{\varepsilon_1}\cdots x_{i_m}^{\varepsilon_m}) = \psi(y_{i_1}^{\varepsilon_1}\cdots y_{i_m}^{\varepsilon_m}).$$

Then there exists a $*$-isomorphism $\pi : M \to N$ such that $\psi\circ\pi = \varphi$ and $\pi(x_i) = y_i$ for all $i \geq 1$.

Definition 3.2. Let A and D be two finite tracial von Neumann algebras and B a von Neumann subalgebra of $A \cap D$. Let $\pi_j : A \to D$, $j \in \mathbb{N}$, be a countable family of unital, normal, faithful, trace-preserving $*$-homomorphisms. The 4-tuple (π_j, B, A, D) is called a sequence of *symmetric independent copies* of A if the following properties hold:

(1) $\pi_j|_B = \mathrm{id}_B$ for all j.

(2) $E_B(\pi_{j_1}(a_1)\cdots\pi_{j_m}(a_m)) = E_B(\pi_{\sigma(j_1)}(a_1)\cdots\pi_{\sigma(j_m)}(a_m))$ for all finite permutations σ on $\mathbb{N}$, all indices $j_1,\dots,j_m$ in $\mathbb{N}$ and all $a_1,\dots,a_m$ in A, where $E_B : D \to B$ is the canonical trace-preserving conditional expectation.

(3) For $i \in \mathbb{N}$ set $A_i = \pi_i(A) \subset D$ and for $I \subset \mathbb{N}$, set $A_I = \bigvee_{i\in I}\pi_i(A) = \bigvee_{i\in I} A_i \subset D$ (by convention, set $A_\varnothing = B$); then, for any finite subsets $I \subset J \subset \mathbb{N}$, $j \notin J$, $d \in A_I$ and $a, a' \in A$, we have

$$E_{A_I}(\pi_j(a)d\,\pi_j(a')) = E_{A_J}(\pi_j(a)d\,\pi_j(a')),$$

where $E_{A_I} : D \to A_I$ is the canonical conditional expectation.

(4) For any finite subsets $I, J \subset \mathbb{N}$, we have $E_{A_I}E_{A_J} = E_{A_{I\cap J}}$. Note that this automatically implies $E_{A_I}E_{A_J} = E_{A_J}E_{A_I} = E_{A_I\cap A_J}$ and in particular $A_I \cap A_J = A_{I\cap J}$.

(5) $A_{\mathbb{N}} = D$.

If the 4-tuple (π_j, B, A, D) only satisfies axioms (1) and (2), we call it a sequence of symmetric copies.

The role played by the copies $\pi_j(A)$ is analogous to that of tensor copies in a classical product probability space; in fact such an infinite product probability space over a commutative or noncommutative base

constitutes the first obvious example of symmetric independent copies. To be more precise, let (A, τ) be a tracial von Neumann algebra, let $D = \bigotimes_{i\in\mathbb{N}}(A_i, \tau_i)$, where $(A_i, \tau_i) = (A, \tau)$ for all $i \in \mathbb{N}$, let π_j be the obvious embedding of A in D as the j-th tensor copy and let $B = \mathbb{C}$. Then all the axioms (1) to (5) are satisfied. In particular, one could take $(A, \tau) = \left(L^\infty(X, \mu), \int_X d\mu\right)$ for a probability measure space (X, μ). Axiom (2), while convenient because it greatly simplifies some of our technical computations, doesn't seem to be indispensable to the development of a general theory of B-valued q-gaussian von Neumann algebras. Indeed, the generalized q-gaussian von Neumann algebras can still be introduced in the presence of a weaker "subsymmetry" assumption, but the technicalities become even more cumbersome, and it is unclear whether some of our results can still be obtained. Axioms (3) and (4) can both be viewed as describing some sort of independence of the copies over B, with (4) being the more obvious one, since for example it gives that for $I \cap J = \varnothing$, we have $E_{A_I} E_{A_J} = E_B$. In the case of an abelian D and $B = \mathbb{C}$, this amounts to classical probabilistic independence. Axiom (5), while added for completeness, can always be made redundant by shrinking the algebra D.

In what follows, the expectations E_{A_I} will be denoted by E_I.

Proposition 3.3. *Let (π_j, B, A, D) be a sequence of symmetric copies. Let $\Sigma = \mathbb{S}(\infty)$ be the group of finite permutations on $\mathbb{N}\setminus\{0\}$. Then for every $\sigma \in \Sigma$ there exists a trace-preserving automorphism α_σ of $D_0 = A_{\mathbb{N}\setminus\{0\}} \subset D$ such that*

$$\alpha_\sigma(\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)) = \pi_{\sigma(j_1)}(x_1)\cdots\pi_{\sigma(j_m)}(x_m) \quad \text{for all } x_1, \ldots, x_m \in A \text{ and } j_1, \ldots, j_m \in \mathbb{N}.$$

Moreover,

$$\Sigma \to \mathrm{Aut}(D_0, \tau), \quad \sigma \mapsto \alpha_\sigma,$$

is an action of Σ on D_0 by trace-preserving automorphisms. Additionally, if the symmetric copies satisfy axiom (4)*, then the fixed points algebra of this action is B.*

Proof. The map $V_\sigma : L^2(D_0) \to L^2(D_0)$ defined by

$$\sum \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) \mapsto \sum \pi_{\sigma(j_1)}(x_1)\cdots\pi_{\sigma(j_m)}(x_m)$$

is easily seen to be a well-defined unitary because of axiom (2). Then $\alpha_\sigma = \mathrm{Ad}(V_\sigma)|_D$ is a trace-preserving automorphism of D which satisfies the required condition. The verification of the second statement is straightforward and we leave it to the reader. □

Symmetric copies can also be introduced in the following alternative way, which is a converse to the previous proposition: assume that $\alpha : \Sigma \to \mathrm{Aut}(D, \tau)$ is a trace-preserving action by $*$-automorphism of the finite von Neumann algebra D, where Σ is now the finite permutation group on $\mathbb{N} \cup \{0\}$ instead of $\mathbb{N}$. Denote by $B = D^\Sigma$ the fixed points algebra of this action. Set

$$\Sigma_0 = \mathrm{Stab}_\Sigma(0) = \{\sigma \in \Sigma : \sigma(0) = 0\},$$
$$A = D^{\Sigma_0} = \{d \in D : \alpha_\sigma(d) = d \text{ for all } \sigma \in \Sigma_0\}.$$

Note that $\Sigma_0 \subset \Sigma$ is a subgroup isomorphic to $\mathbb{S}(\infty)$ and that $B \subset A \subset D$. For every $j \geq 1$, define $\pi_j : A \to D$ by the formula $\pi_j(a) = \alpha_{(0j)}(a)$, $a \in A$, where $(0j) \in \Sigma$ is the transposition interchanging

0 and j. Then (π_j, B, A, D) represents a sequence of symmetric copies. Indeed, for any $j \geq 1$ and $b \in B$ we have $\pi_j(b) = \alpha_{(0j)}(b) = b$ because B is the fixed points algebra of the action α, so (1) is true. Note that $\alpha_\sigma(a) = a$ for every $\sigma \in \Sigma_0$ and $a \in A$ and $E_B \circ \alpha_\sigma = E_B$ for all $\sigma \in \Sigma$, due to (1) and the facts that α is trace-preserving and the trace-preserving conditional expectation $E_B : D \to B$ is unique. Then for every $\sigma \in \Sigma_0 \cong \mathbb{S}(\infty)$ and for all $j_1, \ldots, j_m \geq 1$ and $a_1, \ldots, a_m \in A$ we have

$$\begin{aligned}
E_B(\pi_{\sigma(j_1)}(a_1) \cdots \pi_{\sigma(j_m)}(a_m)) &= E_B(\alpha_{(0\sigma(j_1))}(a_1) \cdots \alpha_{(0\sigma(j_m))}(a_m)) \\
&= E_B(\alpha_{\sigma(0j_1)\sigma^{-1}}(a_1) \cdots \alpha_{\sigma(0j_m)\sigma^{-1}}(a_m)) \\
&= E_B\big((\alpha_\sigma \circ \alpha_{(0j_1)} \circ \alpha_{\sigma^{-1}})(a_1) \cdots (\alpha_\sigma \circ \alpha_{(0j_m)} \circ \alpha_{\sigma^{-1}})(a_m)\big) \\
&= E_B(\alpha_\sigma(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m))) \\
&= E_B(\alpha_{(0j_1)}(a_1) \cdots \alpha_{(0j_m)}(a_m)) \\
&= E_B(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m)),
\end{aligned}$$

so (2) is also true. As noted before, we can also assume without loss of generality that

$$D = \bigvee_{j \geq 1} \pi_j(A) = \bigvee_{j \geq 1} A_j,$$

by simply replacing D with a von Neumann subalgebra.

Notation. Let $(j_1, \ldots, j_m)$ be an m-tuple with $1 \leq j_k \leq n,\ 1 \leq k \leq m$. We denote by $P(m)$ the set of partitions of $\{1, \ldots, m\}$ and by $\dot{0}, \dot{1}$ the finest and the coarsest partitions in $P(m)$, respectively. The notation $P_{1,2}(m)$ stands for the collection of all the partitions of $\{1, \ldots, m\}$ consisting only of singletons and pairs. For $\sigma \in P(m)$, we say that

(1) $(j_1, \ldots, j_m) \leq \sigma$ if $j_i = j_k$ whenever $i, k \in A \in \sigma$;

(2) $(j_1, \ldots, j_m) \geq \sigma$ if $j_i = j_k$ implies that there exists an $A \in \sigma$ with $i, k \in A$;

(3) $(j_1, \ldots, j_m) = \sigma$ if $j_i = j_k$ exactly when there exists an $A \in \sigma$ such that $i, k \in A$.

Given an m-tuple $(j_1, \ldots, j_m) = \dot{0}$ with $1 \leq j_k \leq n$ for $1 \leq k \leq m$, we set $\alpha_{j_1,\ldots,j_m} = \alpha_{\sigma_{j_1,\ldots,j_m}}$, where $\sigma_{j_1,\ldots,j_m}(i) = j_i,\ 1 \leq i \leq m$.

Definition 3.4. Let (π_j, B, A, D) be a sequence of symmetric independent copies, S a subset of A such that $1 \in S = S^*$, H a Hilbert space and ω a free ultrafilter on $\mathbb{N}$. Denote by $\{e_j\}$ the canonical orthonormal basis of $\ell^2 = \ell^2(\mathbb{N})$. Let $-1 < q < 1$. Define

$$\Gamma_q^0(B, S \otimes H) = (B \cup \{s_q(a, h) : a \in S,\ h \in H\})'' \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega,$$

where

$$s_q(a, h) = \left(n^{-\frac{1}{2}} \sum_{j=1}^{n} s_q(e_j \otimes h) \otimes \pi_j(a)\right)_n.$$

Finally define

$$\Gamma_q(B, S \otimes H) = (E_{\Gamma_q(\ell_n^2 \otimes H)} \otimes \mathrm{id})_n(\Gamma_q^0(B, S \otimes K)),$$

where K is an infinite-dimensional Hilbert space containing H, $\ell_n^2 = \text{span}\{e_1, \ldots, e_n\}$ and for each n

$$E_{\Gamma_q(\ell_n^2 \otimes H)} : \Gamma_q(\ell^2 \otimes K) \to \Gamma_q(\ell_n^2 \otimes H)$$

is the canonical conditional expectation.

As q will be fixed throughout this section, we will simply use the notation $s(x,h)$ instead of $s_q(x,h)$ from now on.

Remark 3.5. Due to functoriality, the definition of $\Gamma_q(B, S \otimes H)$ does not depend on the particular choice of $K \supset H$. When H is infinite-dimensional $\Gamma_q(B, S \otimes H) = \Gamma_q^0(B, S \otimes H)$.

Remark 3.6. $\Gamma_q^0(B, S \otimes H) = (\{s(a,h) : a \in B \cup S,\ h \in H\})'' \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$.

Remark 3.7. $\Gamma_q(B, S \otimes H)$ is a von Neumann algebra. Indeed, since the map

$$E = (E_{\Gamma_q(\ell_n^2 \otimes H)} \otimes \text{id})_n : (\Gamma_q(\ell^2 \otimes K) \bar{\otimes} D)^\omega \to (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$$

is a normal linear projection (i.e., idempotent map) of norm 1, it follows that $\Gamma_q(B, S \otimes H)$ is an ultraweakly closed, self-adjoint subspace of $(\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$ containing the identity. It's straightforward to see that the map E has the following bimodularity property:

$$E(x)E(y)E(z) = E(E(x)yE(z)) \quad \text{for all } x, y, z \in \Gamma_q^0(B, S \otimes K).$$

Thus, for $x, y \in \Gamma_q^0(B, S \otimes K)$ we have

$$E(x)E(y) = E(E(x)y) \in \Gamma_q(B, S \otimes H).$$

The canonical generators $s_q(a,h)$ are not easy to work with in a variety of situations. The classical q-gaussians possess a system of generators, the so-called Wick words, whose linear span is an ultraweakly dense $*$-subalgebra. Generalized q-gaussians also have such a well-behaved system of linear generators, which will be called Wick words by analogy with the classical case. In order to find these Wick words let us first define, for every $n \in \mathbb{N}$, $x \in A$ and $h \in H$,

$$u_n(x,h) = n^{-\frac{1}{2}} \left(\sum_{j=1}^{n} s(e_j \otimes h) \otimes \pi_j(x) \right) \in \Gamma_q(\ell^2 \otimes H) \bar{\otimes} D.$$

It's easy to see that $s(x,h) = (u_n(x,h))_n \in (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$ for $x \in A, h \in H$. For $x_1, \ldots, x_m \in BSB = \{b_1 a b_2 : b_1, b_2 \in B,\ a \in S\}$ and $h_1, \ldots, h_m \in H$ we will analyze the product

$$\begin{aligned} u_n(x_1,h_1)\cdots u_n(x_m,h_m) &= n^{-\frac{m}{2}} \sum_{1 \le j_1, \ldots, j_m \le n} s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes \pi_{j_1}(x_1)\pi_{j_2}(x_2)\cdots\pi_{j_m}(x_m) \\ &= \sum_{\sigma \in P(m)} \left(n^{-\frac{m}{2}} \sum_{\substack{(j_1, \ldots, j_m) = \sigma \\ 1 \le j_1, \ldots, j_m \le n}} s(e_{j_1} \otimes h_1) \cdots s(e_{j_m} \otimes h_m) \otimes \pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) \right). \end{aligned}$$

For $\sigma \in P(m)$ let's define

$$x_\sigma^n(x_1,h_1,\dots,x_m,h_m) = n^{-\frac{m}{2}} \sum_{\substack{1\le j_1,\dots,j_m\le n\\ (j_1,\dots,j_m)=\sigma}} s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\otimes \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m),$$

and

$$x_\sigma(x_1,h_1,\dots,x_m,h_m) = (x_\sigma^n(x_1,h_1,\dots,x_m,h_m))_n \in (\Gamma_q(\ell^2\otimes H)\bar{\otimes} D)^\omega.$$

To keep the notation less cumbersome, we will omit the parameters x_k, h_k whenever they are clearly understood from the context. Next we see that

$$u_n(x_1,h_1)\cdots u_n(x_m,h_m) = \sum_{\sigma\in P(m)} x_\sigma^n,$$

and also

$$s(x_1,h_1)\cdots s(x_m,h_m) = (u_n(x_1,h_1)\cdots u_n(x_m,h_m))_n = \sum_{\sigma\in P(m)} x_\sigma.$$

Lemma 3.8. *Let (π_j, B, A, D) be a sequence of symmetric copies. Then*:

(o) $\sup_n \|x_\sigma^n\|_\infty < \infty$ *for all* $m \ge 1$ *and* $\sigma \in P_{1,2}(m)$.

(i) *If* $\sigma \notin P_{1,2}(m)$ *and* $0 < p < \infty$ *then*

$$\lim_n \|x_\sigma^n\|_p = 0.$$

In particular $s(x_1,h_1)\cdots s(x_m,h_m) = \sum_{\sigma\in P_{1,2}(m)} x_\sigma$.

Proof. The proof is the same as that of Proposition 4.1 in [Junge, Longfield and Udrea 2014]. □

Proposition 3.9. *We have the following convolution formula for the multiplication of Wick words*:

$$x_\sigma(x_1,h_1,\dots,x_m,h_m)\,x_\theta(y_1,k_1,\dots,y_{m'},k_{m'}) = \sum_{\substack{\gamma\in P_{1,2}(m+m')\\ \gamma_p|_{1,\cdots,m}=\sigma_p,\ \gamma_p|_{1,\cdots,m'}=\theta_p}} x_\gamma(x_1,h_1,\dots,y_{m'},k_{m'}).$$

Moreover, item (i) *in the lemma above shows that in the summation we can restrict ourselves to pair-singleton partitions whose only additional pairings are between the singletons of σ and θ. In particular, the linear span of the Wick words is a $*$-algebra.*

Proof. We have

$$\begin{aligned}
&x_\sigma(x_1,h_1,\dots,x_m,h_m)\,x_\theta(y_1,k_1,\dots,y_{m'},k_{m'})\\
&\quad= \left(n^{-\frac{m+m'}{2}} \sum_{\substack{(j_1,\dots,j_m)=\sigma\\ (l_1,\dots,l_{m'})=\theta}} s(e_{j_1}\otimes h_1)\cdots s(e_{l_{m'}}\otimes k_{m'})\otimes\pi_{j_1}(x_1)\cdots\pi_{l_{m'}}(y_{m'})\right)\\
&\quad= \sum_{\substack{\gamma\in P_{1,2}(m+m')\\ \gamma_p|_{\{1,\dots,m\}}=\sigma_p,\ \gamma_p|_{\{1,\dots,m'\}}=\theta_p}} \left(n^{-\frac{m+m'}{2}} \sum_{(j_1,\dots,l_{m'})=\gamma} s(e_{j_1}\otimes h_1)\cdots s(e_{l_{m'}}\otimes k_{m'})\otimes\pi_{j_1}(x_1)\cdots\pi_{l_{m'}}(y_{m'})\right)\\
&\quad= \sum_{\substack{\gamma\in P_{1,2}(m+m')\\ \gamma_p|_{\{1,\dots,m\}}=\sigma_p,\ \gamma_p|_{\{1,\dots,m'\}}=\theta_p}} x_\gamma(x_1,h_1,\dots,y_{m'},k_{m'}).
\end{aligned}$$

Now if $\gamma \in P_{1,2}(m+m')$ connects a singleton in σ with a leg of a pair in θ or the leg of pair in σ with either a singleton or a leg of a pair in θ, the resulting x_γ is associated to a partition containing a 3-set or a 4-set and hence vanishes according to Lemma 3.8. So in the above sum we may only allow γ's which preserve the pair sets of both σ and θ and can only additionally pair singletons "on different sides of the marker", which ends the proof. □

Our next result provides a reduction method for the Wick words.

Lemma 3.10. *Let $\pi_j : A \to D$ be symmetric independent copies, and $1 \in S = S^* \subset A$. Let $x_1, \ldots, x_m \in BSB$, $\sigma \in P_{1,2}(m)$ having s singletons and p pairs and $\phi : \{1, \ldots, m\} \to \{1, \ldots, s+p\}$ which encodes σ, i.e., $\phi(k_t) = t$ for every singleton $\{k_t\} \in \sigma$, $1 \le t \le s$, and $\phi(k'_t) = \phi(k''_t) = t+s$ for every pair $\{k'_t, k''_t\} \in \sigma$, $1 \le t \le p$. Consider (ε_k) a sequence of Rademacher variables, i.e., Bernoulli independent random variables on a probability space (X, μ) satisfying $\varepsilon_k : X \to \{\pm 1\}$, $\mathbb{E}(\varepsilon_k = 1) = \mathbb{E}(\varepsilon_k = -1) = \frac{1}{2}$. Then*

$$\left\| \sum_{(l_1,\ldots,l_{s+p})=\dot{0}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \left(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m) - E_{l_1,\ldots,l_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) \right) \right\|_2 \le C(m, x_j) n^{\frac{m-1}{2}}.$$

In particular we have

$$\left(n^{-\frac{m}{2}} \sum_{(l_1,\ldots,l_s,l_{s+1},\ldots,l_{s+p})=\dot{0}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) \right)$$
$$= \left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes E_{l_1,\ldots,l_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) \right)$$
$$= \left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes \alpha_{l_1,\ldots,l_s}(F_\sigma(x_1, \ldots, x_m)) \right),$$

where $F_\sigma(x_1, \ldots, x_m) = E_{1,\ldots,s}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m))$ and the second equality takes place in $(L^\infty(X) \bar{\otimes} D)^\omega$.

Proof. Throughout the proof we endow $L^\infty(X) \bar{\otimes} D$ with the natural trace $\mu \otimes \tau$, where τ is the faithful trace on D. The $\|\cdot\|_2$ in the first statement is the one corresponding to $\mu \otimes \tau$. The approach we take is somewhat similar to the one in [Junge and Zeng 2015].

Step 1. Let $x_1, \ldots, x_m \in BSB$ and n be fixed. Consider

$$\Omega_n = \{(C_1, \ldots, C_{s+p}) : C_1 \sqcup \cdots \sqcup C_{s+p} = \{1, \ldots, n\},\ C_i \neq \varnothing \text{ for all } i\}.$$

Make Ω_n into a probability space with the normalized counting measure. For every $(s+p)$-tuple $(l_1, \ldots, l_{s+p}) = \dot{0}$, consider the indicator function $\delta_{l_1,\ldots,l_{s+p}} : \Omega_n \to \{0, 1\}$ which is 1 if $l_i \in C_i$ for all $1 \le i \le s+p$ and 0 otherwise. According to the proof of Lemma 3.6 in [Junge and Zeng 2015], $\mathbb{E}(\delta_{l_1,\ldots,l_{s+p}}) = (s+p)^{-s-p} = C$. Put

$$F(l_1, \ldots, l_{s+p}) = \varepsilon_{l_1} \cdots \varepsilon_{l_s} \otimes (\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)) - E_{l_1,\ldots,l_s}(\pi_{l_{\phi(1)}}(x_1) \cdots \pi_{l_{\phi(m)}}(x_m)).$$

Then we have

$$\begin{aligned}
&\Big\| \sum_{(l_1,\dots,l_{s+p})=\dot{0}} F(l_1,\dots,l_{s+p}) \Big\|_2 \\
&\qquad = C^{-1} \Big\| C \sum_{(l_1,\dots,l_{s+p})=\dot{0}} F(l_1,\dots,l_{s+p}) \Big\|_2 \\
&\qquad = C^{-1} \Big\| \sum_{(l_1,\dots,l_{s+p})=\dot{0}} CF(l_1,\dots,l_{s+p}) \Big\|_2 \\
&\qquad = C^{-1} \Big\| \sum_{(l_1,\dots,l_{s+p})=\dot{0}} \mathbb{E}(\delta_{l_1,\dots,l_{s+p}}) F(l_1,\dots,l_{s+p}) \Big\|_2 \\
&\qquad = C^{-1} \Big\| \frac{1}{|\Omega_n|} \sum_{(l_1,\dots,l_{s+p})=\dot{0}} \sum_{(C_1,\dots,C_{s+p})\in\Omega_n} \delta_{l_1,\dots,l_{s+p}}((C_1,\dots,C_{s+p})) F(l_1,\dots,l_{s+p}) \Big\|_2 \\
&\qquad = C^{-1} \Big\| \frac{1}{|\Omega_n|} \sum_{(C_1,\dots,C_{s+p})\in\Omega_n} \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} F(l_1,\dots,l_{s+p}) \Big\|_2 = C^{-1} \|\mathbb{E}(G)\|_2 \\
&\qquad \le C \sup_{(C_1,\dots,C_{s+p})\in\Omega_n} \|G((C_1,\dots,C_{s+p}))\|_2 \\
&\qquad = C \sup_{(C_1,\dots,C_{s+p})\in\Omega_n} \Big\| \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} F(l_1,\dots,l_{s+p}) \Big\|_2,
\end{aligned}$$

where we define $G : \Omega_n \to L^\infty(X) \mathbin{\bar\otimes} D$ by

$$G((C_1,\dots,C_{s+p})) = \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} F(l_1,\dots,l_{s+p}).$$

Step 2. It suffices thus to estimate $\big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} F(l_1,\dots,l_{s+p})\big\|_2$ for a fixed nondegenerate partition $C_1,\dots,C_{s+p}$ of $\{1,\dots,n\}$. Fix such an arbitrary partition. We define the sets

$$I_l = C_1 \cup \cdots \cup C_{s+p-1} \cup (\{1,\dots,l\} \cap C_{s+p})$$

and for $l \in C_{s+p}$

$$\begin{aligned}
d_l = \sum_{\substack{l_1\in C_1,\dots,l_s\in C_s \\ l_{s+1}\in C_{s+1},\dots,l_{p-1}\in C_{s+p-1}}} &\varepsilon_{l_1}\cdots\varepsilon_{l_s} \otimes \big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m) \\
&- E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m))\big).
\end{aligned}$$

Note that $D_l = L^\infty(X) \mathbin{\bar\otimes} A_{I_l}$, $l \in C_{s+p}$, form an increasing finite sequence of von Neumann subalgebras of $L^\infty(X) \mathbin{\bar\otimes} D$. Now $d_l \in D_l$ and $E_{D_{l-1}}(d_l) = 0$ for all $l \in C_{s+p}$. The orthogonality together with the Cauchy–Schwarz inequality yields

$$\Big\| \sum_{l\in C_{s+p}} d_l \Big\|_2 \le n^{\frac{1}{2}} \sup_{l\in C_{s+p}} \|d_l\|_2.$$

On the other hand, since the products $\varepsilon_{l_1}\cdots\varepsilon_{l_s}$ are mutually orthogonal for different s-tuples $(l_1,\dots,l_s)$, we see that

$$\begin{aligned}\|d_l\|_2 &= \Bigg\| \sum_{\substack{l_1\in C_1,\dots,l_s\in C_s\\ l_{s+1}\in C_{s+1},\dots,l_{s+p-1}\in C_{s+p-1}}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m)\\ &\qquad -E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg\|_2\\ &= \Bigg\| \sum_{l_1\in C_1,\dots,l_s\in C_s} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\\ &\qquad \otimes\Bigg(\sum_{l_{s+1}\in C_{s+1},\dots,l_{s+p-1}\in C_{s+p-1}} \big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m)\\ &\qquad -E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg)\Bigg\|_2\\ &\le n^{\frac{s}{2}}\Bigg\| \sum_{l_{s+1}\in C_{s+1},\dots,l_{s+p-1}\in C_{s+p-1}} \big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)-E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg\|_2\\ &\le n^{\frac{s}{2}}n^{p-1}\|\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\|_\infty \le n^{\frac{m-2}{2}}\|x_1\|_\infty\cdots\|x_m\|_\infty.\end{aligned}$$

According to axiom (3) we have

$$\begin{aligned}&E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m))\\ &\qquad = E_{C_1,\dots,C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_l(x_{k'_p})\cdots\pi_l(x_{k''_p})\cdots\pi_{l_{\phi(m)}}(x_m));\end{aligned}$$

hence

$$\begin{aligned}&\Bigg\| \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)-E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg\|_2\\ &\quad = \Bigg\| \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)-E_{I_{l-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg\|_2\\ &\quad = \Bigg\| \sum_{l\in C_{s+p}} d_l\Bigg\|_2\\ &\quad \le \|x_1\|_\infty\cdots\|x_m\|_\infty n^{\frac{m-1}{2}} = C'(x_1,\dots,x_m)n^{\frac{m-1}{2}}.\end{aligned}$$

Steps 1 and 2 so far imply that

$$\Bigg\| \sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m) - E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Bigg\|_2 \le C'n^{\frac{m-1}{2}}.$$

Step 3. Now we may proceed inductively. Set $y = \pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)$. Then, using axiom (4) and because the conditional expectations commute, we see that

$$\begin{aligned}
&y - E_{C_1\cup\cdots\cup C_{s+p-2}}(y)\\
&\quad = y - E_{C_1\cup\cdots\cup C_{s+p-1}}(y) + E_{C_1\cup\cdots\cup C_{s+p-1}}(y) - E_{C_1\cup\cdots\cup C_{s+p-2}}(y)\\
&\quad = y - E_{C_1\cup\cdots\cup C_{s+p-1}}(y) + E_{C_1\cup\cdots\cup C_{s+p-1}}(y) - E_{C_1\cup\cdots\cup_{C_{s+p-2}}\cup C_{s+p}}(E_{C_1\cup\cdots\cup C_{s+p-1}}(y))\\
&\quad = y - E_{C_1\cup\cdots\cup C_{s+p-1}}(y) + E_{C_1\cup\cdots\cup C_{s+p-1}}(y - E_{C_1\cup\cdots\cup C_{s+p-2}\cup C_{s+p}}(y)).
\end{aligned}$$

Using the previous steps and the fact that the conditional expectations are $\|\cdot\|_2$-contractive, we obtain

$$\begin{aligned}
&\Big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m) - E_{C_1\cup\cdots\cup C_{s+p-2}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big\|_2\\
&\le \Big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\\
&\qquad\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m) - E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big\|_2\\
&\quad + \Big\|(\mathrm{id}\otimes E_{C_1\cup\cdots\cup C_{s+p-1}})\\
&\qquad\Big(\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\\
&\qquad\qquad - E_{C_1\cup\cdots\cup C_{s+p-2}\cup C_{s+p}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big)\Big\|_2\\
&\le \Big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\\
&\qquad\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m) - E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big\|_2\\
&\quad + \Big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\\
&\qquad\qquad - E_{C_1\cup\cdots\cup C_{s+p-2}\cup C_{s+p}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big\|_2\\
&\le 2C' n^{\frac{m-1}{2}}.
\end{aligned}$$

After using the triangle inequality p times, we get

$$\Big\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m) - E_{C_1\cup\cdots\cup C_s}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\big)\Big\|_2 \le pC' n^{\frac{m-1}{2}} = C'' n^{\frac{m-1}{2}}.$$

Now we claim that

$$E_{C_1\cup\cdots\cup C_s}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)) = E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)).$$

This can be established using axioms (3) and (4). Indeed, since

$$l_{s+p}\notin C_1\cup\cdots\cup C_{s+p-1}\supset\{l_1,\dots,l_{s+p-1}\},$$

by applying axiom (3) we see that

$$\begin{aligned}
&E_{\{l_1,\dots,l_{s+p-1}\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\\
&\qquad=\pi_{l_{\phi(1)}}(x_1)\cdots E_{\{l_1,\dots,l_{s+p-1}\}}(\pi_{l_{s+p}}(x_{k_p'})\cdots\pi_{l_{s+p}}(x_{k_p''}))\cdots\pi_{l_{\phi(m)}}(x_m)\\
&\qquad=\pi_{l_{\phi(1)}}(x_1)\cdots E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{s+p}}(x_{k_p'})\cdots\pi_{l_{s+p}}(x_{k_p''}))\cdots\pi_{l_{\phi(m)}}(x_m)\\
&\qquad=E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)),
\end{aligned}$$

and then

$$\begin{aligned}
E_{C_1\cup\cdots\cup C_s}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))&=E_{C_1\cup\cdots\cup C_s}(E_{C_1\cup\cdots\cup C_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)))\\
&=E_{C_1\cup\cdots\cup C_s}(E_{l_1,\dots,l_{s+p-1}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)))\\
&=E_{(C_1\cup\cdots\cup C_s)\cap\{l_1,\dots,l_{s+p-1}\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\\
&=E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)),
\end{aligned}$$

which proves the claim. Now the claim, together with the last inequality, gives

$$\left\|\sum_{l_1\in C_1,\dots,l_{s+p}\in C_{s+p}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\bigl(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)-E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\bigr)\right\|_2 \le C''n^{\frac{m-1}{2}}.$$

Step 1 now implies

$$\left\|\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\bigl(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)-E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\bigr)\right\|_2 \le CC''n^{\frac{m-1}{2}},$$

which proves the first statement in the lemma. For the second statement, we begin by noting that $E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))$ only depends on $l_1,\dots,l_s$, and not on $l_{s+1},\dots,l_{s+p}$. Indeed, let $(l_1,\dots,l_s,l'_{s+1},\dots,l'_{s+p})=\dot{0}$ be another $(s+p)$-tuple with the same first s entries. Take a finite permutation σ such that $\sigma(l_i)=l_i,\ i\le s$, and $\sigma(l_{s+i})=l'_{s+i},\ i\le p$. Then α_σ is the identity on $A_{l_1,\dots,l_s}$; hence

$$\begin{aligned}
&E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{s+i}}(x_{k_i'})\cdots\pi_{l_{s+i}}(x_{k_i''})\cdots\pi_{l_{\phi(m)}}(x_m))\\
&\qquad=(E_{\{l_1,\dots,l_s\}}\circ\alpha_\sigma)(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{s+i}}(x_{k_i'})\cdots\pi_{l_{s+i}}(x_{k_i''})\cdots\pi_{l_{\phi(m)}}(x_m))\\
&\qquad=E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l'_{s+i}}(x_{k_i'})\cdots\pi_{l'_{s+i}}(x_{k_i''})\cdots\pi_{l_{\phi(m)}}(x_m)),
\end{aligned}$$

which proves the claim. Now the first statement of the lemma together with an easy counting argument shows that

$$\begin{aligned}
&\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\right)\\
&\qquad=\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\right)\\
&\qquad=\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes E_{\{l_1,\dots,l_s\}}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\right).
\end{aligned}$$

Finally, let's note that

$$E_{l_1,\dots,l_s}\circ\alpha_{l_1,\dots,l_{s+p}}=\alpha_{l_1,\dots,l_s}\circ E_{1,\dots,s},$$

which implies

$$\begin{aligned}E_{l_1,\dots,l_s}(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))&=(E_{l_1,\dots,l_s}\circ\alpha_{l_1,\dots,l_{s+p}})(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m))\\&=(\alpha_{l_1,\dots,l_s}\circ E_{1,\dots,s})(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m))\\&=\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m)).\end{aligned}$$

□

Theorem 3.11. *Let (π_j, B, A, D) be a sequence of symmetric independent copies, $x_1,\dots,x_m\in A$, $\sigma\in P_{1,2}(m)$ having s singletons and p pairs and $\phi:\{1,\dots,m\}\to\{1,\dots,s+p\}$ which encodes σ. Then*

$$\begin{aligned}&x_\sigma(x_1,h_1,\dots,x_m,h_m)\\&\quad=\left(n^{-\frac{m}{2}}\sum_{(j_1,\dots,j_m)=\sigma}s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\otimes\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)\right)\\&\quad=f_\sigma(h_1,\dots,h_m)\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\otimes\alpha_{\{l_1,\dots,l_s\}}(F_\sigma(x_1,\dots,x_m))\right)\\&\quad=f_\sigma(h_1,\dots,h_m)W_\sigma(x_1,h_1,\dots,x_m,h_m),\end{aligned}$$

where

$$\begin{aligned}F_\sigma(x_1,\dots,x_m)&=E_{\{1,\dots,s\}}(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m)),\\f_\sigma(h_1,\dots,h_m)&=q^{\mathrm{cr}(\sigma)}\prod_{\{k,l\}\in\sigma}\langle h_k,h_l\rangle\end{aligned}$$

and $\{k_1,\dots,k_s\}$ are the singletons of σ. The elements

$$W_\sigma(x_1,h_1,\dots,x_m,h_m)=\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\otimes\alpha_{\{l_1,\dots,l_s\}}(F_\sigma(x_1,\dots,x_m))\right)$$

will be called reduced Wick words.

Proof. We will use the previous lemma. Let

$$\widehat{B}=B,\quad \hat{A}=\Gamma_q(H)\,\bar{\otimes}\,A,\quad \widehat{D}=\Gamma_q(\ell^2\otimes H)\,\bar{\otimes}\,D$$

and $\hat{\pi}_j:\hat{A}\to\Gamma_q(\ell^2\otimes H)\,\bar{\otimes}\,D$ be the $*$-homomorphisms given by

$$\hat{\pi}_j(s(h)\otimes x)=s(e_j\otimes h)\otimes\pi_j(x).$$

Then $(\hat{\pi}_j, B, \hat{A}, \widehat{D})$ represents a sequence of independent symmetric copies. Moreover, it is easy to see that $\hat{A}_I=\Gamma_q(\ell^2(I)\otimes H)\,\bar{\otimes}\,A_I$. Now according to the previous lemma we have

$$\begin{aligned}
&\Bigg(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} \varepsilon_{j_{k_1}} \cdots \varepsilon_{j_{k_s}} \otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\otimes \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)\Bigg)\\
&\quad= \Bigg(n^{-\frac{m}{2}} \sum_{(l_1,\dots,l_{s+p})=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m)\otimes \pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\Bigg)\\
&\quad= \Bigg(n^{-\frac{m}{2}} \sum_{(l_1,\dots,l_{s+p})=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes \hat{\pi}_{l_{\phi(1)}}(s(h_1)\otimes x_1)\cdots\hat{\pi}_{l_{\phi(m)}}(s(h_m)\otimes x_m)\Bigg)\\
&\quad= \Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes \hat{\alpha}_{l_1,\dots,l_s}(\widehat{F}_\sigma(s(h_1)\otimes x_1,\dots,s(h_m)\otimes x_m))\Bigg)\\
&\quad= \Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes \hat{\alpha}_{l_1,\dots,l_s}\big(E_{\hat{A}_{1,\dots,s}}(\hat{\pi}_{\phi(1)}(s(h_1)\otimes x_1)\cdots\hat{\pi}_{\phi(m)}(s(h_m)\otimes x_m))\big)\Bigg)\\
&\quad= \Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes \hat{\alpha}_{l_1,\dots,l_s}\big(E_{\Gamma_q(\ell^2_s\otimes H)\bar{\otimes}A_{1,\dots,s}}\big(s(e_{\phi(1)}\otimes h_1)\cdots s(e_{\phi(m)}\otimes h_m)\\
&\qquad\qquad\otimes\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m)\big)\big)\Bigg)\\
&\quad= f_\sigma(h_1,\dots,h_m)\Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes \hat{\alpha}_{l_1,\dots,l_s}\big(s(e_1\otimes h_{k_1})\cdots s(e_s\otimes h_{k_s})\\
&\qquad\qquad\otimes E_{1,\dots,s}(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m))\big)\Bigg)\\
&\quad= f_\sigma(h_1,\dots,h_m)\Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes\big(s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\\
&\qquad\qquad\otimes\alpha_{l_1,\dots,l_s}(E_{1,\dots,s}(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m)))\big)\Bigg)\\
&\quad= f_\sigma(h_1,\dots,h_m)\Bigg(n^{-\frac{s}{2}} \sum_{(l_1,\dots,l_s)=\dot{0}} \varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\otimes\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\Bigg).
\end{aligned}$$

To see why the fifth equality is true, note that

$$\begin{aligned}
&s(e_{\phi(1)}\otimes h_1)\cdots s(e_{\phi(m)}\otimes h_m)\\
&\qquad= \sum_{\theta\in P_{1,2}(m)} f_\theta(e_{\phi(1)}\otimes h_1\otimes\cdots\otimes e_{\phi(m)}\otimes h_m)W((e_{\phi(1)}\otimes h_1\otimes\cdots\otimes e_{\phi(m)}\otimes h_m)_\theta),
\end{aligned}$$

where the notation $(\cdot)_\theta$ means that the pair positions of θ have been removed. After the application of $E_{\Gamma_q(\ell^2_s\otimes H)}$, we see that the only surviving partition is $\theta=\sigma$ and

$$\begin{aligned}
E_{\Gamma_q(\ell^2_s\otimes H)}(s(e_{\phi(1)}\otimes h_1)\cdots s(e_{\phi(m)}\otimes h_m)) &= f_\sigma(h_1,\dots,h_m)W(e_1\otimes h_{k_1}\cdots e_s\otimes h_{k_s})\\
&= f_\sigma(h_1,\dots,h_m)s(e_1\otimes h_{k_1})\cdots s(e_s\otimes h_{k_s}).
\end{aligned}$$

Now, let's define

$$\tilde{s}(x,h)=\Bigg(n^{-\frac{1}{2}}\sum_{j=1}^{n}\varepsilon_j\otimes s(e_j\otimes h)\otimes\pi_j(x)\Bigg)\in(L^\infty(X)\,\bar{\otimes}\,\Gamma_q(\ell^2\otimes H)\,\bar{\otimes}\,D)^\omega.$$

We claim that the new Wick words $\tilde{x}_\sigma$ associated to the variables $\tilde{s}(x,h)$ have the same moments as x_σ and hence they generate an isomorphic von Neumann algebra. Indeed, fix $\sigma\in P_{1,2}(m)$. Note that for

$(l_1,\dots,l_s)=\dot{0}$, we have $\mu(\varepsilon_{l_1}\cdots\varepsilon_{l_s})=\mu(\varepsilon_{l_1})\cdots\mu(\varepsilon_{l_s})=\delta_{s=0}$, due to the fact that ε_j are mean-zero, independent random variables. Then

$$\begin{aligned}
&\tau_\omega(\tilde{x}_\sigma(x_1,h_1,\dots,x_m,h_m))\\
&=\tau_\omega\left(\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m)\otimes\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\right)\right)\\
&=\lim_n\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\mu(\varepsilon_{l_1}\cdots\varepsilon_{l_s})\,\tau(s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m))\,\tau_D(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\right)\\
&=\delta_{s=0}\lim_n\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\tau(s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m))\,\tau_D(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\right)\\
&=\delta_{\sigma\in P_2(m)}\lim_n\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\tau(s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m))\,\tau_D(\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m))\right)\\
&=\tau_\omega(x_\sigma(x_1,h_1,\dots,x_m,h_m)).
\end{aligned}$$

Define $\mathcal{M}\subset(\Gamma_q(\ell^2\otimes H)\bar{\otimes}D)^\omega$ to be the von Neumann algebra generated by all the Wick words x_σ. Also define $\widetilde{\mathcal{M}}\subset(L^\infty(X)\bar{\otimes}\Gamma_q(\ell^2\otimes H)\bar{\otimes}D)^\omega$ to be the von Neumann algebra generated by the elements $\tilde{x}_\sigma$. Using the claim, the convolution formula and Proposition 3.1 we see that the map

$$\mathcal{M}\to\widetilde{\mathcal{M}},\quad \sum x_\sigma\mapsto\sum\tilde{x}_\sigma,$$

is a $*$-isomorphism. Applying the inverse of this isomorphism to the equality

$$\begin{aligned}
&\left(n^{-\frac{m}{2}}\sum_{(l_1,\dots,l_{s+p})=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes s(e_{l_{\phi(1)}}\otimes h_1)\cdots s(e_{l_{\phi(m)}}\otimes h_m)\otimes\pi_{l_{\phi(1)}}(x_1)\cdots\pi_{l_{\phi(m)}}(x_m)\right)\\
&=f_\sigma(h_1,\dots,h_m)\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\varepsilon_{l_1}\cdots\varepsilon_{l_s}\otimes s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\otimes\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\right),
\end{aligned}$$

we obtain the desired identity. □

Proposition 3.12. *Let $x_1,\dots,x_m\in A$ and $h_1,\dots,h_m\in H$. Then we have the moment formula*

$$\tau(s(x_1,h_1)\cdots s(x_m,h_m))=\delta_{m\in2\mathbb{N}}\sum_{\sigma\in P_2(m)}q^{\mathrm{cr}(\sigma)}\prod_{\{l,r\}\in\sigma}\langle h_l,h_r\rangle\tau(\pi_{j_1^\sigma}(x_1)\cdots\pi_{j_m^\sigma}(x_m)),$$

as well as the B-valued moment formula

$$E_B(s(x_1,h_1)\cdots s(x_m,h_m))=\delta_{m\in2\mathbb{N}}\sum_{\sigma\in P_2(m)}q^{\mathrm{cr}(\sigma)}\prod_{\{l,r\}\in\sigma}\langle h_l,h_r\rangle E_B(\pi_{j_1^\sigma}(x_1)\cdots\pi_{j_m^\sigma}(x_m)),$$

where for every $\sigma\in P_2(m)$, the $j_1^\sigma,\dots,j_m^\sigma$ are chosen such that $(j_1^\sigma,\dots,j_m^\sigma)=\sigma$.

Proof. It's a straightforward application of the reduction formula. □

Remark 3.13. Proposition 3.1 shows that $M = \Gamma_q^0(B, S \otimes H)$ could be introduced abstractly as the tracial von Neumann algebra (M, τ) generated by elements $s(x, h),\ x \in BSB,\ h \in H$, which satisfy the above moment formula.

Proposition 3.14. *Let K be infinite-dimensional and $x_1, \dots, x_m \in BSB,\ h_1, \dots, h_m \in K,\ \sigma \in P_{1,2}(m)$. Then $x_\sigma(x_1, h_1, \dots, x_m, h_m) \in \Gamma_q^0(B, S \otimes K)$. For every Hilbert space H, all the Wick words $x_\sigma(x_1, h_1, \dots, x_m, h_m),\ x_i \in BSB,\ h_i \in H$, are in $M = \Gamma_q(B, S \otimes H)$. In particular, M is the ultraweakly closed linear span of the (reduced) Wick words and $L^2(M)$ is the $\|\cdot\|_2$-closed span of the (reduced) Wick words.*

Proof. We need a basic fact about infinite-dimensional Hilbert spaces.

Fact. Let K be an infinite-dimensional Hilbert space and $\lambda_1, \dots, \lambda_p \in \mathbb{C}$. Then there exist norm-bounded sequences $\xi_n^k, \eta_n^k \in K$ for $1 \le k \le p$ such that $\xi_n^k \to 0,\ \eta_n^k \to 0$ weakly and $\langle \xi_n^k, \eta_n^k \rangle = \lambda_k$ for all $1 \le k \le p$, and moreover $\xi_n^k, \eta_n^k \perp \xi_n^j, \eta_n^j$ for $k \ne j$. Indeed, let (e_n) be an orthonormal infinite sequence in K. Define

$$\xi_n^1 = \lambda_1 e_n, \quad \eta_n^1 = e_n, \quad \xi_n^2 = \lambda_2 e_{n+1}, \quad \eta_n^2 = e_{n+1}, \quad \dots, \quad \xi_n^p = \lambda_p e_{n+p-1}, \quad \eta_n^p = e_{n+p-1}.$$

To prove the proposition we will use induction on s, the numbers of singletons in σ. For $s = 0$, $x_\sigma(x_1, h_1, \dots, x_m, h_m) \in B$ due to the Wick word reduction formula, so the statement is trivial. For a given σ with pairs $B_1, \dots, B_p$ and $B = \{l, r\}$ we use the fact above to find uniformly norm-bounded vectors $h_{l,B}(k), h_{r,B}(k) \in K$ which converge to 0 weakly and such that $\langle h_{l,B}(k), h_{r,B}(k) \rangle = \langle h_l, h_r \rangle$ for all pairs $B = \{l, r\}$, and such that the $h_{l/r,B}(k)$'s are orthogonal for different pairs B. Let us define $\tilde{h}_i(k) = h_i$ for any singleton $\{i\} \in \sigma$ and $\tilde{h}_i(k) = h_{l/r,B}(k)$ if $i \in B$ and $i = l$ or $i = r$. For every other Wick word $x'_{\sigma'}(y_1, f_1, \dots, y_{m'}, f_{m'})$, with $y_j \in BSB,\ f_j \in K$, we have

$$\begin{aligned}\lim_{k\to\infty} \tau(s(x_1, \tilde{h}_1(k)) \cdots s(x_m, \tilde{h}_m(k)) x'_{\sigma'}) &\\ = \lim_{k\to\infty} \sum_{\theta \in P_{1,2}(m)} \tau(x_\theta(x_1, \tilde{h}_1(k), \dots, x_m, \tilde{h}_m(k)) x'_{\sigma'}) &\\ = \tau(x_\sigma x'_{\sigma'}) + \lim_{k\to\infty} \sum_{\theta_p \supset \sigma_p,\ |\theta_s| < s} \tau(x_\theta(x_1, \tilde{h}_1(k), \dots, x_m, \tilde{h}_m(k)) x'_{\sigma'}).&\end{aligned}$$

Indeed, for every $\theta \in P_{1,2}(m)$ which does not contain all the pairs of σ, we use the convolution and the moment formulas to obtain

$$\begin{aligned}\tau(x_\theta x'_{\sigma'}) &= \sum_{\nu \in P_2(m+m')} \tau(x_\nu(x_1, \tilde{h}_1(k), \dots, y_{m'}, f_{m'})) \\ &= \sum_{\nu \in P_2(m+m')} f_\nu(\tilde{h}_1(k), \dots, \tilde{h}_m(k), f_1, \dots, f_{m'})\, \tau(W_\nu(x_1, \tilde{h}_1(k), \dots, y_{m'}, f_{m'})),\end{aligned}$$

where the sum is taken over all ν that preserve the pairs of θ and σ' and additionally pair all the singletons of θ and σ'. Now since θ does not contain all the pairs of σ, there must be a leg l of a pair $\{l, r\} = B \in \sigma$ which is connected by θ to something other than its other leg in σ. There are three possibilities:

(1) θ connects l to a leg l' of another pair $B' = \{l', r'\} \in \sigma$. Then $\langle \tilde{h}_l(k), \tilde{h}_{l'}(k)\rangle = 0$; hence for every ν in the sum above we have $f_\nu(h_1, \ldots, f_{m'}) = 0$.

(2) θ connects l to a singleton $\{i\} \in \sigma$. Then, since $\tilde{h}_l(k) \to 0$ weakly, we have $\langle \tilde{h}_l(k), h_i\rangle \to 0$; hence for every ν we also have that $f_\nu(h_1, \ldots, f_{m'}) \to 0$ as $k \to \infty$.

(3) $\{l\}$ is a singleton of θ. In this case, every $\nu \in P_{1,2}(m+m')$ which appears in the sum has to connect l to a singleton $j \in \{1, \ldots, m'\}$. Thus, $\langle \tilde{h}_l(k), f_j\rangle \to 0$ and again $f_\nu(h_1, \ldots, f_{m'}) \to 0$ as $k \to \infty$.

Summing up, we see that for every θ such that $\sigma_p \not\subseteq \theta_p$, we have $\tau(x_\theta(k)x'_{\sigma'}) \to 0$ as $k \to \infty$. Thus, when letting $k \to \infty$, only those θ's containing the pairs of σ make a nonzero contribution. Among them, there is exactly one which has s singletons, namely σ; all the others have more pairs and hence less than s singletons. We deduce that

$$x_\sigma = w - \lim_{k\to\infty}\left(s(x_1, \tilde{h}_1(k)) \cdots s(x_m, \tilde{h}_m(k)) - \sum_{\theta_p \supset \sigma_p,\ |\theta_s| < s} x_\theta(x_1, \tilde{h}_1(k), \ldots, x_m, \tilde{h}_m(k))\right).$$

Since by the induction hypothesis all the x_θ's, with $|\theta_s| < s$, are in $\Gamma_q^0(B, S \otimes K)$, this proves the statement. For the second statement, let H be any Hilbert space and K an infinite-dimensional Hilbert space containing H. Let $x_i \in BSB$, $h_i \in H$ and $\sigma \in P_{1,2}(m)$. Then, by the first part, $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in \Gamma_q^0(B, S \otimes K)$. But $x_\sigma = (E_{\Gamma_q(\ell_n^2 \otimes H)} \otimes \mathrm{id})_n(x_\sigma)$; hence $x_\sigma \in \Gamma_q(B, S \otimes H)$. □

Remark 3.15. The reader can now better appreciate why we needed the "closure operation" in the definition of $\Gamma_q(B, S \otimes H)$. Indeed, Definition 3.4 ensures that the Wick words belong to $M = \Gamma_q(B, S \otimes H)$ for every Hilbert space H, finite- or infinite-dimensional. Also, Proposition 3.14 shows that $M = \Gamma_q(B, S \otimes H)$ could have been defined as the ultraweakly closed span of the Wick words.

In the following we use the notation $L_k^2(M)$ for the $\|\cdot\|_2$-closed span of the Wick words of degree k and $W_k(M)$ for the linear span of the Wick words of degree k.

Theorem 3.16. *Let (π_j, B, A, D) be a sequence of symmetric independent copies, $1 \in S = S^* \subset A$, H be a Hilbert space and $M = \Gamma_q(B, S \otimes H)$. Set $\widetilde{H} = H \oplus H$. Take an infinite-dimensional Hilbert space $K \supset H$ and set $\widetilde{K} = K \oplus K$:*

(1) *For every angle θ, let o_θ be the canonical rotation on $\widetilde{K}$. Then*

$$\theta \mapsto \alpha_\theta = (\Gamma_q(\mathrm{id} \otimes o_\theta) \otimes \mathrm{id})_n \in \mathrm{Aut}((\Gamma_q(\ell^2 \otimes \widetilde{K}) \bar{\otimes} D)^\omega)$$

defines by restriction a one-parameter group of automorphisms of $\widetilde{M} = \Gamma_q(B, S \otimes \widetilde{H})$. Moreover, for every Wick word $x_\sigma(x_1, \tilde{h}_1, \ldots, x_m, \tilde{h}_m) \in \widetilde{M}$ we have

$$\alpha_\theta(x_\sigma(x_1, \tilde{h}_1, \ldots, x_m, \tilde{h}_m)) = x_\sigma(x_1, o_\theta(\tilde{h}_1), \ldots, x_m, o_\theta(\tilde{h}_m)).$$

(2) *For every Wick word $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in M$, the following formula holds:*

$$(E_M \circ \alpha_\theta)(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = (\cos(\theta))^s x_\sigma(x_1, h_1, \ldots, x_m, h_m),$$

where $E_M : \widetilde{M} \to M$ is the conditional expectation and s is the number of singletons of σ.

(3) *For every* $\theta \in \left[0, \frac{\pi}{2}\right)$, *let* $t = -ln(\cos(\theta))$. *Then* $t \mapsto T_t = E_M \circ \alpha_\theta|_M$ *defines a one-parameter semigroup of normal, trace-preserving, u.c.p. maps on* M. *Moreover, for every Wick word* $x_\sigma \in M$ *we have* $T_t(x_\sigma) = e^{-ts}x_\sigma$, *where* s *is the number of singletons of* σ. *Hence, when viewed as a contraction on* $L^2(M)$, *we have* $T_t = \sum_{s\geq 0} e^{-ts}P_s$, *where* P_s *is the orthogonal projection of* $L^2(M)$ *on* $L^2_s(M)$ *and the series is* $\|\cdot\|_\infty$*-convergent for every* $t > 0$. *In particular, if* $L^2_s(M)$ *is finitely generated as a right* B*-module for every* s, *then* T_t *is compact over* B *for every* $t > 0$.

(4) *The generator* N *of* T_t *is a positive, self-adjoint, densely defined operator in* $L^2(M) = \bigoplus_{k=0}^\infty L^2_k(M)$, *acting by*

$$N(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) = k x_\sigma(x_1, h_1, \ldots, x_m, h_m)$$

for every $x_\sigma(x_1, h_1, \ldots, x_m, h_m) \in L^2_k(M)$. *The spectrum of* N *is the set of nonnegative integers* $\mathbb{N}$, *all of which are eigenvalues.* N *is called the number operator.*

Proof. The formula $\alpha_\theta(x_\sigma(x_1, \tilde{h}_1, \ldots, x_m, \tilde{h}_m)) = x_\sigma(x_1, o_\theta(\tilde{h}_1), \ldots, x_m, o_\theta(\tilde{h}_m))$ for $x_i \in BSB$, $\tilde{h}_i \in \widetilde{H}$ is easily checked, due to entrywise functoriality, and it shows that α_θ restricts to a one-parameter group of automorphisms on $\widetilde{M} = \Gamma_q(B, S \otimes \widetilde{H})$. This proves (1). Then, using the reduction formula and the functoriality in each entry, we see that

$$\begin{aligned}
&(E_M \circ \alpha_\theta)(x_\sigma(x_1, h_1, \ldots, x_m, h_m)) \\
&= f_\sigma(h_1, \ldots, h_m)(E_M \circ \alpha_\theta)\left(\left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} s(e_{l_1} \otimes h_{k_1}) \cdots s(e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1,\ldots,l_s}(F_\sigma(x_1, \ldots, x_m))\right)\right) \\
&= f_\sigma(h_1, \ldots, h_m)(E_M \circ \alpha_\theta)\left(\left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} W(e_{l_1} \otimes h_{k_1} \cdots e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1,\ldots,l_s}(F_\sigma(x_1, \ldots, x_m))\right)\right) \\
&= f_\sigma(h_1, \ldots, h_m)\left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} W(e_{l_1} \otimes P_H\alpha_\theta(h_{k_1}) \cdots e_{l_s} \otimes P_H\alpha_\theta(h_{k_s})) \otimes \alpha_{l_1,\ldots,l_s}(F_\sigma(x_1, \ldots, x_m))\right) \\
&= (\cos(\theta))^s f_\sigma(h_1, \ldots, h_m)\left(n^{-\frac{s}{2}} \sum_{(l_1,\ldots,l_s)=\dot{0}} s(e_{l_1} \otimes h_{k_1}) \cdots s(e_{l_s} \otimes h_{k_s}) \otimes \alpha_{l_1,\ldots,l_s}(F_\sigma(x_1, \ldots, x_m))\right) \\
&= (\cos(\theta))^s x_\sigma(x_1, h_1, \ldots, x_m, h_m),
\end{aligned}$$

which establishes (2). Part (3) is straightforward using (2). To obtain (4), we calculate

$$\lim_{t\to 0} \frac{1}{t}(T_t(x_\sigma) - x_\sigma) = \lim_{t\to 0} \frac{e^{-st} - 1}{t} x_\sigma = -s x_\sigma$$

for any Wick word x_σ of degree s. The rest of the statements are straightforward. □

Remark 3.17. Due to (4), we have that for every $x \in M$, the function

$$\theta \mapsto \|\alpha_\theta(x) - x\|_2, \quad \theta \in \left[0, \frac{\pi}{2}\right),$$

is increasing.

Definition 3.18. We denote by $D_k(S) \subset L^2(D)$ the $\|\cdot\|_2$-closed linear span of the expressions

$$F_\sigma(x_1, \ldots, x_m) = E_{1,\ldots,k}(\pi_{\phi(1)}(x_1) \cdots \pi_{\phi(m)}(x_m))$$

for all $m \geq 1$, $x_1, \ldots, x_m \in BSB$, $\sigma \in P_{1,2}(m)$ having k singletons and ϕ which encodes σ.

Lemma 3.19. *Let $y(j_1, \ldots, j_k) \in L^p(D)$ be such that*

$$\sup_{j_1,\ldots,j_k} \|y(j_1, \ldots, j_k)\|_p < \infty$$

and $h_1, \ldots, h_k \in H$. Then

$$\sup_n \left\| n^{-\frac{k}{2}} \sum_{(l_1,\ldots,l_k)=\dot{0}} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k) \right\|_p < \infty.$$

Proof. It suffices to consider

$$\left\| \sum_{l_1 \in C_1, \ldots, l_k \in C_k} s_{l_1}(h_1) \cdots s_{l_k}(h_1) \otimes y(l_1, \ldots, l_k) \right\|,$$

with $C_1 \cup \cdots \cup C_k = \{1, \ldots, n\}$. Using the martingale decomposition from Lemma 3.10 we deduce

$$\left\| \sum_{l_1 \in C_1, \ldots, l_k \in C_k} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k) \right\|_p \leq c(p)\sqrt{n} \sup_{l \in C_k} \left\| \sum_{l_1,\ldots,l_{k-1}} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k) \right\|_p.$$

Iterating this procedure we get

$$\left\| \sum_{l_1 \in C_1, \ldots, l_k \in C_k} s_{l_1}(h_1) \cdots s_{l_k}(h_k) \otimes y(l_1, \ldots, l_k) \right\|_p \leq c(p)^k n^{\frac{k}{2}} \sup_{l_1,\ldots,l_k} \|s_{l_1}(h_1) \cdots s_{l_k}(h_k)\|_p \, \|y(l_1, \ldots, l_k)\|_p.$$

Since the products $s_{l_1}(h_1) \cdots s_{l_k}(h_k)$ are uniformly bounded in the p-norm, we obtain the assertion. □

Proposition 3.20. *Let (π_j, B, A, D) be a sequence of independent symmetric copies, let H be a finite-dimensional Hilbert space and $1 \in S = S^* \subset A$, and assume that $D_s(S)$ is finitely generated as a right B-module. Then $L^2_s(M)$ is finitely generated as a right B-module. In particular, when $D_s(S)$ is finitely generated over B for every s, the maps T_t are compact over B for every $t > 0$.*

Proof. Let N be the dimension of D_s as a right B-module, and let $\{\xi_1, \ldots, \xi_N\}$ be a basis of D_s over B. Then, for every $\sigma \in P_{1,2}(m)$ having s singletons, and every $x_1, \ldots, x_m \in BSB$, we can find coefficients $b_k(\sigma, x_1, \ldots, x_m) \in B$ such that

$$F_\sigma(x_1, \ldots, x_m) = \sum_{k=1}^N \xi_k b_k(\sigma, x_1, \ldots, x_m).$$

For every $(l_1,\dots,l_s)=\dot{0}$ we have

$$\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))=\sum_{k=1}^{N}\alpha_{l_1,\dots,l_s}(\xi_k)\,b_k(\sigma,x_1,\dots,x_m).$$

Fix a finite basis $\mathcal{B}$ of H. Then, for every σ having s singletons, every $x_1,\dots,x_m\in BSB$ and every $h_1,\dots,h_m\in\mathcal{B}$ we have, due to the reduction formula,

$$\begin{aligned}x_\sigma(x_1,h_1,\dots,x_m,h_m)&=\Bigl(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}s_{l_1}(h_{i_1})\cdots s_{l_s}(h_{i_s})\otimes\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\Bigr)\\&=\sum_{k=1}^{N}\Bigl(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}s_{l_1}(h_{i_1})\cdots s_{l_s}(h_{i_s})\otimes\alpha_{l_1,\dots,l_s}(\xi_k)\Bigr)b_k(\sigma,x_1,\dots,x_m).\end{aligned}$$

Thus $L^2_s(M)$ is spanned over B by at most $N|\mathcal{B}|^s=N(\dim(H))^s$ elements, namely

$$\Bigl(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}s_{l_1}(h_{i_1})\cdots s_{l_s}(h_{i_s})\otimes\alpha_{l_1,\dots,l_s}(\xi_k)\Bigr),$$

with $h_i\in\mathcal{B}$ and $1\le k\le N$. These elements belong to $L^2_s(M)$ by the previous lemma, and this finishes the proof. □

Remark 3.21. Since the dimension of $D_s(S)$ over B is finite, the basis elements $\xi_k\in D_s\subset L^2(D)$ could be chosen in fact to be bounded, i.e., $\xi_k\in D$, due to [Paschke 1973; 1974]. This implies that $L^2_s(M)$ admits a basis over B consisting of elements in M.

Corollary 3.22. *Assume moreover that the dimension N_s of $D_s(S)$ over B has polynomial growth; i.e., there exist constants $d,C>0$ such that $N_s\le Cd^s$ for all s. Then the dimension of $L^2_s(M)$ over B is less than $C(\dim(H)d)^s$ for all s; i.e., the dimension of $L^2_s(M)$ over B also has polynomial growth.*

The following argument is essentially due to Śniady [2004] and Królak [2006].

Proposition 3.23. *Let $M=\Gamma_q(B,S\otimes H)$. There exists $d=d(q)$ such that for $\dim(H)\ge d$ we have $\mathcal{Z}(M)\subset\mathcal{Z}(B)$. In particular, M is a factor whenever B is.*

Proof. Let $\{e_i\}_{1\le i\le k}$ be an orthonormal set in H. We consider the operator $T:L^2(M)\to L^2(M)$ given by

$$T=\sum_{i=1}^{k}(L_{s(1,e_i)}-R_{s(1,e_i)})^2.$$

Here L_x and R_x, where $x\in M$, are the canonical left and right multiplication operators, respectively, on $L^2(M)$. We see that

$$T-2k\,\mathrm{id}=\sum_{i=1}^{k}(L_{s(1,e_i)^2-1}-R_{s(1,e_i)^2-1})-2\sum_{i=1}^{k}L_{s(1,e_i)}R_{s(1,e_i)}.$$

Since $s(1,e_i)^2-1$ is a mean-zero element, we deduce from [Nou 2004] that

$$\left\|\sum_{i=1}^{k} s(1,e_i)^2-1\right\|_\infty \le c_q\sqrt{k}.$$

Let us set $V=\sum_{i=1}^{k} L_{s(1,e_i)}R_{s(1,e_i)}$ and denote by $\iota: L^2(M)\to(\mathcal{F}_q(\ell^2\otimes H)\otimes L^2(D))^\omega$ the natural embedding given by the definition. Then we see that

$$\iota(V\xi)=(V_n\iota(\xi)_n)_n,\quad \xi\in L^2(M),$$

where

$$V_n=\frac{1}{n}\sum_{\substack{1\le i\le k\\ 1\le j,j'\le n}} L_{s(1,e_i\otimes e_j)}R_{s(1,e_i\otimes e_{j'})}.$$

Now we can easily modify the argument from [Królak 2006] to show that

(1) $\left\|\sum_{k,j,j'} l^+(e_i\otimes e_j)R_{s(1,e_i\otimes e_{j'})}\right\|\le c_q\sqrt{kn^2}$;

(2) $\left\|\sum_{k,j,j'} r^+(e_i\otimes e_j)L_{s(1,e_i\otimes e_{j'})}\right\|\le c_q\sqrt{kn^2}$;

(3) $\left\|\sum_{k,j\neq j'} l^-(e_i\otimes e_j)R_{s(1,e_i\otimes e_{j'})}\right\|\le c_q\sqrt{kn^2}$;

(4) $\left\|\sum_{k,j} l^-(e_i\otimes e_j)r^+(e_i\otimes e_j)|_{\mathbb{C}^\perp}\right\|\le q+c_q\sqrt{kn}$.

Here l^+,l^-,r^+,r^- are the left and right creation operators on the q-Fock space coming from the decomposition $L_{s(h)}=l^+(h)+l^-(h)$, $R_{s(h)}=r^+(h)+r^-(h)$. The main estimate is derived from

$$l^-(h)r^+(k)(\xi)=q^{|\xi|}\xi+l^-(h)(\xi)\otimes k.$$

The second part can then be estimated via the second item above. This yields

$$\|(T-2k\,\mathrm{id})((\mathrm{id}-E_B)(\xi))\|\le 2qk\|(\mathrm{id}-E_B)(\xi)\|+2c_q\sqrt{k}\|(\mathrm{id}-E_B)(\xi)\|.$$

Now take $z\in\mathcal{Z}(M)$ with $E_B(z)=0$. Thus $T(z)=0$ and also

$$0=\|T(z)\|=\|2kz-(T(z)-2kz)\|\ge 2k\|z\|-2qk\|z\|-C_q\sqrt{k}\|z\|=(2k(1-q)-C_q\sqrt{k})\|z\|.$$

Thus for $2k(1-q)-C_q\sqrt{k}>0$, i.e., $k>\sqrt{C_q/(2(1-q))}$, we have that $z=0$. This implies $z=E_B(z)$ for all $z\in\mathcal{Z}(M)$; hence $\mathcal{Z}(M)\subset B$ and also $\mathcal{Z}(M)\subset\mathcal{Z}(B)$. □

3A. ***H*-less generalized *q*-gaussians.** Finally, let us mention that there is an H-less version of the generalized q-gaussians, which can be described as follows: let (π_j,B,A,D) be a sequence of symmetric independent copies. For $1\in S=S^*\subset A$, define the von Neumann algebra $\Gamma_q(B,S)\subset(\Gamma_q(\ell^2)\bar{\otimes}D)^\omega$ as being generated by the elements $s_q(x)=\left(n^{-\frac{1}{2}}\sum_{j=1}^n s_q(e_j)\otimes\pi_j(x)\right)_n$ for $x\in BSB$. This is equivalent to taking H to be 1-dimensional in Definition 3.4 above; hence the H-less q-gaussians are a particular case of Definition 3.4. Surprisingly, the H generalized q-gaussians can also be obtained as a particular case of this construction. Indeed, let H be a (real) Hilbert space and (π_j,B,A,D) a sequence of symmetric independent copies. Let (X,μ) be a standard probability measure space and define a new sequence of

symmetric independent copies $(\tilde{\pi}_j, \widetilde{B}, \widetilde{A}, \widetilde{D})$ by taking $\widetilde{B} = B$, $\widetilde{A} = A \bar{\otimes} L^\infty(X)$, $\widetilde{D} = D \bar{\otimes} \big(\overline{\bigotimes}_1^\infty L^\infty(X)\big)$ and $\tilde{\pi}_j : \widetilde{A} \to \widetilde{D}$ by

$$\tilde{\pi}_j(a \otimes f) = \pi_j(a) \otimes (1 \otimes 1 \otimes \cdots \otimes \underbrace{f}_{j\text{-th position}} \otimes \cdots \otimes 1 \otimes \cdots), \quad \text{where } a \in A,\ f \in L^\infty(X).$$

Using Rademacher variables, we see that there exists a dense subspace $H_0 \subset H$ and an isometric embedding $\iota : H_0 \to L^\infty(X) \subset L^2(X)$. Take $\widetilde{S} = S \otimes \iota(H_0) = \{a \otimes \iota(h) : a \in S, h \in H_0\} \subset \widetilde{A}$. The reader can check that

$$\Gamma_q(B, S \otimes H) = \Gamma_q(\widetilde{B}, \widetilde{S}).$$

4. Examples

We will discuss several types of examples of generalized q-gaussian von Neumann algebras. The underlying idea in all these cases is that whenever we have a finite von Neumann algebra on which the symmetric group acts, we can construct a sequence of symmetric copies. In particular, countable tensor or (amalgamated) free products von Neumann algebras or the pure q-gaussian von Neumann algebras $\Gamma_q(H)$, for an infinite-dimensional H, constitute obvious candidates, since the symmetric group acts naturally on them.

4A. ***Tensor products.*** Let B and C be finite von Neumann algebras. Define $A = B \mathbin{\bar{\otimes}} C$ and $D = B \mathbin{\bar{\otimes}} C^{\mathbb{N}} = B \mathbin{\bar{\otimes}} \big(\overline{\bigotimes}_{\mathbb{N}} C\big)$. Define $\pi_j : A \to D$ by the formula

$$\pi_j(b \otimes a) = b \otimes 1 \otimes 1 \otimes \cdots \otimes \underbrace{a}_{j\text{-th position}} \otimes \cdots \otimes 1 \otimes \cdots .$$

Then it's easy to check that (π_j, B, A, D) is a sequence of symmetric independent copies. It's likewise easy to see that

$$\Gamma_q(B, A \otimes H) = B \mathbin{\bar{\otimes}} \Gamma_q(L^2_{\mathrm{sa}}(C) \otimes H).$$

For any finite subset $S \subset L^2_{\mathrm{sa}}(C) \otimes H$, the space $D_k(S)$ has finite dimension over B.

4B. ***Free products with amalgamation.*** Let $B \subset A$ be an inclusion of finite tracial von Neumann algebras. Take $D = *_B A_j$, the amalgamated free product of a countable number of copies A_j, $j \in \mathbb{N}$, of A. Define $\pi_j : A \to D$ by the formula

$$\pi_j(a) = 1 * 1 * \cdots * \underbrace{a}_{j\text{-th position}} * \cdots * 1 * \cdots .$$

Then (π_j, B, A, D) represents a sequence of independent symmetric copies. To see why this is true it suffices to consider elements a_i such that $E_B(a_i) = 0$. Then we have to calculate

$$\tau_\sigma(a_1, \ldots, a_m) = \tau(\pi_{j_1}(a_1) \cdots \pi_{j_m}(a_m))$$

such that $(j_1, \ldots, j_m) = \sigma$. If σ has no crossings, we can inductively replace neighboring pairs by $E_B(\pi_{j_i}(a_i)\pi_{j_i}(a_{i+1})) = E_B(a_i a_{i+1})$ and finally find an element in B. For a noncrossing pair partition

we can also join all the pairs, but then we find an expression of the form

$$\tau(b_1\pi_{j_{i_1}}(a_{i_1})b_2\pi_{j_{i_2}}(a_{i_2})\cdots\pi_{j_{i_k}}(a_{i_k}))=0.$$

Thus in the moment formula we only have to expand over noncrossing pair partitions. Now take $S=\{1,u,u^*\}$ for $u\in A$ a Haar unitary such that $E_B(u^n)=0$ for all $n\neq 0$. It's easy to see that $D_k(S)$ is the closed linear span of all the expressions $b_1\pi_1(u^{\varepsilon_1})b_2\cdots\pi_k(u^{\varepsilon_k})b_{k+1}$, with $b_i\in B$ and $\varepsilon_i\in\{0,1,*\}$. In particular, when $B=\mathbb{C}$ or is finite-dimensional, we have $\dim_B(D_k(S))\leq C2^{2k}$.

4C. *Group actions.*

4C1. *Second quantization.* Let $G\curvearrowright_\alpha C$ be a trace-preserving action of the discrete group G on the finite von Neumann algebra C. Also let $\nu: G\to\mathcal{O}(H_{\mathbb{R}})$ be an orthogonal representation of G on a real Hilbert space $H_{\mathbb{R}}$. Let (Ω,μ) be the gaussian construction associated to ν; see, e.g., [Peterson and Sinclair 2012]. We also denote the corresponding action $G\curvearrowright L^\infty(\Omega)$ by ν. Then define $B=C\rtimes_\alpha G$, $A=(C\mathbin{\bar\otimes}L^\infty(\Omega))\rtimes_\rho G$, $D=(C\mathbin{\bar\otimes}L^\infty(\Omega^{\mathbb{N}}))\rtimes_\rho G$, where the action ρ is given by $\rho_g(d\otimes f)=\alpha_g(d)\otimes\nu_g(f)$. Define the $*$-homomorphisms $\pi_j: A\to D$ by

$$\pi_j((d\otimes f)u_g)=(d\otimes 1\otimes 1\otimes\cdots\otimes\underbrace{f}_{j\text{-th position}}\otimes\cdots\otimes 1\otimes\cdots)u_g.$$

Then it is easy to see that the fixed points algebra is $C\rtimes_\alpha G$. Again the moments only depend on the inner product. Moreover, the gaussian functor yields a map $\mathrm{Br}: H\to L^2(\Omega)$. Then we find

$$M=\Gamma_q(C\rtimes G,\mathrm{Br}(H))=(C\mathbin{\bar\otimes}\Gamma_q(H))\rtimes G.$$

The spaces $D_k(S)$ are finite-dimensional modules over $B=C\rtimes G$ if $L^2_k(H)\rtimes G$ has a finite basis over G. For $k=1$ this means that H is finite-dimensional. In a forthcoming paper we will also analyze the case of profinite actions and/or representations, i.e., when H can be written as $H=\overline{\bigcup_i H_i}$ such that every H_i is a finite-dimensional G-invariant Hilbert subspace. However, discrete subgroups of $\mathcal{O}_n=\mathcal{O}(\mathbb{R}^n)$ provide a large class of nontrivial, nonamenable examples. The examples in [Junge, Longfield and Udrea 2014] are subalgebras of M.

4C2. *Symmetric group action.* Throughout this subsection Σ will denote the group of finite permutations on $\mathbb{N}$. Let us consider a countable discrete group G on which Σ acts by automorphisms. Examples for such a symmetric action are given by the natural action of Σ on the free group with countably many generators, or by the natural action of Σ on the direct product groups $\prod_{n\in\mathbb{N}}G$. More generally, let $R\subset\mathbb{F}_\infty$ be a set of generators which is invariant under the action of Σ, and assume that $\langle R\rangle\subset\mathbb{F}_\infty$ is a normal subgroup. Then $G=\mathbb{F}_\infty/\langle R\rangle$ is a group on which Σ acts. A perfect example is given by an amalgamated free product $*_H G_j$, where $G_j=G$. To make things more concrete, we may consider the discrete Heisenberg group $\mathcal{H}=\langle\mathbb{Z},\mathbb{Z}^\infty\rangle$ with generators $\{g_k\}_{k\geq 0}$ such that $\mathbb{Z}=\langle g_0\rangle$, $\mathbb{Z}^\infty=\langle g_k,k\geq 1\rangle$ and the following relations hold:

$$g_k^{-1}g_jg_k=g_0g_j,\quad k\neq j.$$

Then Σ acts on $\mathcal{H}$ by permuting the generators g_k for $k \geq 1$, and leaving g_0 fixed. Now we assume that such a G, with action $\Sigma \curvearrowright_\beta G$, acts trace-preservingly on a finite von Neumann algebra A and B is the fixed points algebra of this action α. Let $g \in G$ be an arbitrary element and $g_j = \beta_{(1j)}(g)$. We can then construct a sequence of symmetric copies (π_j, B, A, D) by defining $\pi_j : A \to A$, via $\pi_j(x) = \alpha_{g_j}(x)$. Working in the crossed product $(A \rtimes_\alpha G) \rtimes_\beta \Sigma$ it is easy to see that the π_j's are symmetric copies, and that B is the fixed points algebra for these symmetric copies. In fact we may and will always assume that G is generated by the g_j's and then $\pi_j(x) = x$ for all j is exactly the fixed points algebra of the action. In general $\pi_j(A) = A$ and hence we find an example of symmetric, but not necessarily independent, copies. In general independent copies are obtained from considering a suitable subalgebra $B \subset A_1 \subset A$. More generally for a subset $S \subset A$ we may however consider the algebras

$$A_j(S) = \{\pi_j(x) : x \in S,\ j \in A\}.$$

This is particularly interesting for a single self-adjoint x. Then independence depends on the mixing properties of the sequence $\pi_j(x)$, and has to be analyzed on a case by case basis. A more specific example can be constructed starting from a trace-preserving action α of $\mathbb{Z}$ on a finite von Neumann algebra N. Take $D = N \rtimes_\beta \mathcal{H}$ where the action β is obtained by lifting the action of $\mathbb{Z}$ via the group homomorphism $\pi : \mathcal{H} \to \mathbb{Z}$ given by $\pi(g_0) = 0$ and $\pi(g_j) = 1$ for $j \geq 1$. In other words,

$$\beta_g(x) = \alpha_{\pi(g)}(x), \quad g \in \mathcal{H},\ x \in N.$$

Let $\mathcal{H}_1$ be the group generated by g_0 and g_1 and take $B = N \rtimes \mathbb{Z} = N \mathbin{\bar{\otimes}} L(\mathbb{Z})$ and $A = N \rtimes \mathcal{H}_1$. Define $\pi_j : A \to D$ by

$$\pi_j(xu_{g_1}) = \alpha_{\pi(g_j)}(x)u_{g_j}, \quad \pi_j(xu_{g_0}) = xu_{g_0}, \quad x \in N,\ j,k \in \mathbb{N}.$$

Then (π_j, B, A, D) is a sequence of symmetric independent copies. In full generality the dimensions of the spaces $D_k(S)$ or $L^2_k(M)$, where $M = \Gamma_q(B, A \otimes H)$, cannot be controlled. If we restrict ourselves to a small set of generators, e.g., $S = \{1, g_1, g_1^{-1}\}$, then we get a more well-behaved example. The space $D_k(S)$ is the closed linear span of the expressions of the form

$$\pi_{j_1}(u_{g_1}) \cdots \pi_{j_k}(u_{g_k}) u_{g_0}^{l(\sigma)} \alpha_{n(\sigma)}(x).$$

Thus $\dim_B(D_k(S)) \leq (2\dim(H))^{2k}$. For more general group actions and $S \subset L(G)$, we find coefficients in $B = L([G,G]) \mathbin{\bar{\otimes}} N$ and finite dimension over B as long as we have finite generating sets. Note however, that $L([G,G])$ is in general not invariant under the action of Σ, and hence a more detailed case by case analysis is required. Again a particularly nice class of examples comes from one step nilpotent groups with commutators in the center, such as the Heisenberg groups.

4D. *Colored Brownian motion.*

4D1. *Top up q-gaussians.* Let H be a Hilbert space and $q_0 \in [-1, 1]$. Symmetric independent copies can be obtained from second quantization, or simply by defining $\pi(s_{q_0}(h)) = s_{q_0}(e_j \otimes h)$. This provides symmetric copies of $A = \Gamma_{q_0}(H)$ into $D = \Gamma_q(\ell_2(H))$. By looking at Wick words it is easy to

see that the fixed points algebra is $\mathbb{C}$. Moreover, independence follows from the moment formula for q_0-gaussian random variables. Let $S = \{x_1, \ldots, x_p\}$ be a finite, self-adjoint subset, where $x_i = s_{q_0}(h_1(i)) \cdots s_{q_0}(h_{l(i)}(i))$ for $1 \le i \le p$. Then we see that

$$\tau(s_q(k_1, x_1) \cdots s_q(k_m, x_m)) = \sum_{\sigma \in P_2(m)} q^{\mathrm{cr}(\sigma)} f_\sigma(k_1, \ldots, k_m)\, \tau(\pi_{j_1^\sigma}(x_1) \cdots \pi_{j_m^\sigma}(x_m)),$$

where for every σ, we choose an m-tuple $(j_1^\sigma, \ldots, j_m^\sigma)$ depending on σ such that $(j_1^\sigma, \ldots, j_m^\sigma) = \sigma$. Now we may use the formula for q_0-gaussians and find for $L = \sum_{i=1}^p l(i)$ that

$$\tau(\pi_{j_1^\sigma}(x_1) \cdots \pi_{j_m^\sigma}(x_m)) = \sum_{\sigma' \in P_2(L),\ \sigma' \le \phi(\sigma)} q_0^{\mathrm{cr}(\sigma')} f_{\sigma'}(h_1(1), \ldots, h_{l(1)}(1), \ldots, h_1(m), \ldots, h_{l(m)}(m)).$$

Here $\phi(\sigma)$ is the block partition which gives the same color to the union of two blocks in σ connected via pairs in σ'. This means

$$\tau(s_q(k_1, x_1) \cdots s_q(k_m, x_m)) = \sum_{\sigma \in P_2(m),\ \sigma' \le \phi(\sigma)} q^{\mathrm{cr}(\sigma')} q_0^{\mathrm{cr}(\sigma')} f_\sigma(k_1, \ldots, k_m) f_{\sigma'}(h_1, \ldots, h_L),$$

where σ' runs over the partitions of $\{1, \ldots, L\}$ and $\{h_1, \ldots, h_L\}$ is a relabeling of $\{h_j(i) : 1 \le i \le p,\ 1 \le j \le l(i)\}$. Note that $\Gamma_q(\mathbb{C}, \Gamma_{q_0}(H) \otimes K)$ contains both $\Gamma_q(K)$ and $\Gamma_{q_0}(H)$ if $s_{q_0}(H) \subset S$. Using a decomposition into minimal links, we deduce that the space $D_k(S)$ is the closed linear span of the elements

$$c(\sigma, x_1, \ldots, x_r)\, \pi_{j_{i_1}}(x_{i_1}) \cdots \pi_{j_{i_r}}(x_{i_r}),$$

where $c(\sigma, x_1, \ldots, x_r)$ is a scalar. This means for a finite set S of generators, the dimension of $D_k(S)$ over $B = \mathbb{C}$ is less than $(|S| \dim(K))^{2k}$. One could call these algebras "mixed" gaussian algebras, but the reader should not mistake them for the mixed q-gaussian algebras, introduced in [Junge and Zeng 2015], which we use in Section 6.

4D2. *Actions of Σ by conjugation.* Let us consider the finite permutations group $\Sigma_{\mathbb{Z}}$ acting on $\mathbb{Z}$ instead of $\mathbb{N} \setminus \{0\}$. For every subset $F \subset \mathbb{Z}$ we can identify Σ_F, the permutations group on F, with a subgroup of $\Sigma_{\mathbb{Z}}$ by viewing the elements of Σ_F as acting nontrivially only on F and acting as the identity on $\mathbb{Z} \setminus F$. For convenience, we use interval notation for the subsets of $\mathbb{Z}$. In particular we have $\Sigma = \Sigma_{[1,\infty)} \subset \Sigma_{\mathbb{Z}}$ in this way. Let Σ act on $\Sigma_{\mathbb{Z}}$ by conjugation. This gives rise to an action α of Σ on the von Neumann algebra $L(\Sigma_{\mathbb{Z}})$ (which is in fact isomorphic to the hyperfinite factor). We denote the canonical unitaries generating $L(\Sigma_{\mathbb{Z}})$ by u_σ, $\sigma \in \Sigma_{\mathbb{Z}}$. The fixed points algebra of this action is $B = L(\Sigma_{(-\infty,0]})$. Take $A = L(\Sigma_{(-\infty,1]}) = B \vee \{u_{(01)}\}''$, $D = L(\Sigma_{\mathbb{Z}})$ and define $\pi_j : A \to D$ by $\pi_j(a) = \alpha_{(j1)}(a)$ for $a \in A$ and $j \ge 2$, where $(j1)$ is the transposition interchanging j and 1, and $\pi_1 = \mathrm{id}$. Then (π_j, B, A, D) is a sequence of symmetric independent copies. Indeed, we recall that A is generated by transpositions $(k1)$, $k \le 0$, and that for $j \ge 2$ we have

$$(j1)(k1)(j1) = (kj).$$

This means $A_j = B \vee \{u_{(0j)}\}''$ and $A_{1,\ldots,j} = L(\Sigma_{(-\infty,j]})$. In particular, we have a coset representation $\sigma = \sigma'(j1)$ with $\sigma' \in \Sigma_{(-\infty,0]}$. The algebras A_I are generated by Σ_I, $\Sigma_{(-\infty,0]}$ and one generator $(j1)$ for

$j \in I$. This easily implies independence. We take $S = \{1, u_{(01)}\} \subset A$ and define $M = \Gamma_q(B, S \otimes H)$. Fix $\sigma \in P_{12}(m)$ having k singletons and p pairs, and take $\phi : \{1, \dots, m\} \to \{1, \dots, k+p\}$ which encodes σ. This means $\phi(j_t) = t$, where $\{j_t\},\ 1 \le t \le k$, are the singletons of σ, and $\phi(j_t') = \phi(j_t'') = k+t$, where $\{j_t', j_t''\},\ 1 \le t \le p$, are the pairs of σ. Then $D_k(S)$ is the closed span of elements of the form

$$\begin{aligned} &E_{1,\dots,k}(u_{(\phi(1)0)}u_{\gamma_1}u_{(\phi(2)0)}u_{\gamma_2}\cdots u_{\gamma_m}u_{(\phi(m)0)}u_{\gamma_{m+1}}) \\ &\qquad = E_{1,\dots,k}\big(u_{(\phi(1)0)}\mathrm{ad}(u_{\gamma_1})(u_{(\phi(2)0)})\cdots \mathrm{ad}(u_{\gamma_1\cdots\gamma_m})(u_{(\phi(m)0)})u_{\gamma_1\cdots\gamma_{m+1}}\big) \\ &\qquad = E_{1,\dots,k}(u_{(\phi(1)\gamma_1(0))}u_{(\phi(2)\gamma_1\gamma_2(0))}\cdots u_{(\phi(m)(\gamma_1\cdots\gamma_m)(0))}u_{\gamma_1\cdots\gamma_{m+1}}) \\ &\qquad = E_{1,\dots,k}(u_{(\phi(1)s_1)}u_{(\phi(2)s_2)}\cdots u_{(\phi(m)s_m)})u_{\gamma_1\cdots\gamma_{m+1}}, \end{aligned}$$

where $\gamma_1, \dots, \gamma_{m+1} \in \Sigma_{(-\infty,0]}$ are arbitrary. Here $s_1 = \gamma_1(0),\ s_2 = \gamma_1\gamma_2(0), \dots, s_m = \gamma_1\gamma_2\cdots\gamma_m(0)$ in $(-\infty, 0]$ depend only the γ_i's. In full generality the modules $D_k(S)$ do not have finite dimensions over B. If we however replace B by $B_d = L(\Sigma_{[-d,0]}) \cong L(\mathbb{S}_{d+1})$, A by $A_d = L(\Sigma_{[-d,1]}) = B_d \vee \{u_{(01)}\}''$ and D by $D_d = L(\Sigma_{[-d,\infty)})$ for a fixed $d \in \mathbb{N} \setminus \{0\}$, then we obtain a new sequence of symmetric independent copies (π_j, B_d, A_d, D_d) and in this case we have at most $(d+1)^k$ different choices for the s_j's. After repeated conjugation with the unitaries on the pair positions, the above expression becomes

$$u_{(s'_{j_1}1)}\cdots u_{(s'_{j_k}k)}E_{1,\dots,k}(u_{(s'_{j_{k+1}}k+1)}\cdots u_{(s'_{j_{k+p}}k+p)})$$

for some new indices $s_i' \in (-\infty, 0] \cap \mathbb{Z}$ which in general depend on the γ_i's and σ. Since for an inclusion of groups $H \subset G$ and $g \in G$ we have $E_{L(H)}(u_g) = \delta_{g\in H}u_g$, and the product $(s'_{j_{k+1}}k+1)\cdots(s'_{j_{k+p}}k+p)$ belongs to $\Sigma_{(-d,k]}$ only if it's equal to 1, we see that a spanning set of $D_k(S)$ over B_d is given by the elements

$$u_{(s'_{j_1}1)}u_{(s'_{j_2}2)}\cdots u_{(s'_{j_k}k)}$$

for all choices of $-d \le s_i' \le 0,\ 1 \le i \le k$, which in particular implies that the dimension of $D_k(S)$ over B_d is at most $(d+1)^{2k}$. Note that B_d and A_d are finite-dimensional von Neumann algebras. Thus, for the von Neumann algebras $M(d) = \Gamma_q(B_d, S \otimes H)$, the spaces $D_k(S)$ have polynomial growth of their dimensions over B_d. This remains true for any finite subset $1 \in S = S^* \subset A_d$.

4E. ***Operator-valued gaussians.*** This example is motivated by Shlyahktenko's A-valued semicircular algebras and derived from the tensor product construction. Let $x_k \in N$ be self-adjoint operators and $X = \sum_k g_k x_k$. We consider $A_1 = L^\infty(\mathbb{R})$ and the independent symmetric copies over N given by

$$\pi_j(f) = f\left(\sum_k g_{k,j}x_k\right),$$

where $g_{k,j}$ are i.i.d. gaussians (we could also work with q-gaussians). The copies are independent over N. Let D be the von Neumann algebra generated by the $\pi_j(f)$'s and B be the tail algebra

$$B = \bigcap_{m\ge 0}\ \bigvee_{j\ge m}\pi_j(L^\infty(\mathbb{R})).$$

One can show that the copies π_j are independent symmetric in the sense of our Definition 3.2. Note that N is invariant under the shift from the tensor product construction and hence $B \subset N$. Thus $M = \Gamma_q(B, S \otimes H)$ is a legitimate example where $S = \sum_k g_k x_k$ is obtained by approximating X with bounded functions. Since $X \in \bigcap_{1 \le p < \infty} L^p(\mathbb{R})$, one can actually directly work with one generator x. The dimension of $L^2_k(M)$ over B is in general hard to determine. The case of $N = M_m(\mathbb{C})$ and $X = \sum_{r,s} g_{rs}((e_{rs}+e_{sr})/2)$ has been considered by Avsec and Speicher.

Remark 4.1. The examples in Sections 4A, 4B, 4D1, 4D2 (for $d = 0$) and 4E are all factors if B is a factor and $\dim(H) \ge d(q)$.

5. Weak amenability produces approximately invariant states

Let (π, B, A, D) be a sequence of symmetric independent copies, $1 \in S = S^* \subset A$ and assume that $D_s(S)$ is finitely generated over B for all $s \ge 1$. Let $M = \Gamma_q(B, S \otimes H)$ for a finite-dimensional space H, $\mathcal{A} \subset M$ be a von Neumann subalgebra which is amenable relative to B inside M, and let $P = \mathcal{N}_M(\mathcal{A})''$. Define $\mathcal{M} = (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D) \vee M \subset (\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D)^\omega$, where $\Gamma_q(\ell^2 \otimes H) \bar{\otimes} D$ is embedded as constant sequences. Let

$$\mathcal{H} \subset \big((L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes H)\big)^\omega$$

be the $\|\cdot\|$-closed span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) y \otimes_{\mathcal{A}} z) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right)$$

for all $m \ge 1$, $\sigma \in P_{1,2}(m)$, $x_i \in BSB$, $y \in M$, $z \in P$ and $h_1, \dots, h_m \in H$. Define two $*$-representations $\pi : M \to B(\mathcal{H})$, $\theta : P^{\mathrm{op}} \to B(\mathcal{H})$ by

$$\pi(x_{\sigma'})\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) y \otimes_{\mathcal{A}} z) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right)$$
$$= \left(n^{-\frac{m+m'}{2}} \sum_{(i_k)=\sigma',(j_l)=\sigma} (\pi_{i_1}(y_1)\cdots\pi_{j_m}(x_m) y \otimes_{\mathcal{A}} z) \otimes s(e_{i_1} \otimes k_1)\cdots s(e_{j_m} \otimes h_m)\right)$$

and

$$\theta(w^{\mathrm{op}})\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) y \otimes_{\mathcal{A}} z) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right)$$
$$= \left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) y \otimes_{\mathcal{A}} zw) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right),$$

where

$$x_{\sigma'} = \left(n^{-\frac{m'}{2}} \sum_{(i_1,\dots,i_{m'})=\sigma'} \pi_{i_1}(y_1)\cdots\pi_{i_{m'}}(y_{m'}) \otimes s(e_{i_1} \otimes k_1)\cdots s(e_{i_{m'}} \otimes k_{m'})\right) \in M$$

is a Wick word in M and $w \in P$. Define $\mathcal{N} = \pi(M) \vee \theta(P^{\mathrm{op}}) \subset B(\mathcal{H})$. Note that $\pi(M)$ and $\theta(P^{\mathrm{op}})$ commute.

Theorem 5.1. *There exists a sequence of normal states $\omega_n \in \mathcal{N}_*$ satisfying the following properties*:

(1) $\omega_n(\pi(x)) \to \tau(x),\ x \in M$.

(2) $\omega_n(\pi(a)\theta(\bar{a})) \to 1,\ a \in \mathcal{U}(\mathcal{A})$.

(3) $\|\omega_n \circ \mathrm{Ad}(\pi(u)\theta(\bar{u})) - \omega_n\| \to 0,\ u \in \mathcal{N}_M(\mathcal{A})$.

Proof. Throughout the proof m_n will be the completely contractive finite-rank multipliers on $\Gamma_q(\ell^2 \otimes H)$ given by multiplication with a positive finitely supported function f_n constructed in [Avsec 2012] and $\varphi_n := (m_n \otimes \mathrm{id}) : M \to M$ the corresponding cb map on M. Take

$$\mathcal{K} \subset (L^2(\mathcal{M}) \otimes_D L^2(\mathcal{M}))^\omega$$

to be the $\|\cdot\|$-closed span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)) \otimes_D y\right) = (x_\sigma^n \otimes_D y),$$

where $x_i \in BSB$ and $y \in M$. Note that $\mathcal{K}$ is naturally an M-M bimodule with the actions

$$x_{\sigma'} \cdot \left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)) \otimes_D y\right) \cdot z$$
$$= \left(n^{-\frac{m+m'}{2}} \sum_{(i_k)=\sigma',(j_l)=\sigma} (\pi_{i_1}(y_1)\cdots\pi_{j_m}(x_m) \otimes s(e_{i_1} \otimes k_1)\cdots s(e_{j_m} \otimes h_m)) \otimes_D yz\right),$$

where $x_{\sigma'} = x_{\sigma'}(y_1, k_1, \dots, y_{m'}, k_{m'}) \in M$ and $z \in M$. Define $\mathcal{S}_\mathcal{A} = \lambda(M) \vee \rho(\mathcal{A}^{\mathrm{op}}) \subset B(\mathcal{K})$, where λ and ρ are the representations of M and M^{op} canonically associated to the left and right actions on $\mathcal{K}$, respectively.

Step 1. There exists a normal, unital, completely positive map $\mathcal{E} : \mathcal{N} \to \mathcal{S}_\mathcal{A}$ such that

$$\mathcal{E}(\pi(x)\theta(y^{\mathrm{op}})) = \lambda(x)\rho(E_\mathcal{A}(y)^{\mathrm{op}}), \quad x \in M,\ y \in P.$$

Indeed, define an isometry $V : \mathcal{K} \to \mathcal{H}$ by

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)) \otimes_D y\right)$$
$$\mapsto \left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y \otimes_\mathcal{A} 1) \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right).$$

Then $\mathcal{E}$ can be defined by $\mathcal{E}(z) = V^* z V,\ z \in \mathcal{N}$.

Step 2. There exist normal functionals $\mu_n^\mathcal{A} : \mathcal{S}_\mathcal{A} \to \mathbb{C}$ such that

$$\mu_n^\mathcal{A}(\lambda(x)\rho(a^{\mathrm{op}})) = \tau(\varphi_n(x)a), \quad x \in M,\ a \in \mathcal{A}.$$

We need two lemmas. Recall the formulas for Wick words and reduced Wick words introduced in Theorem 3.11.

Lemma 5.2. *$L^2(M)\otimes_B L^2(M)$ embeds as an M-M bimodule into $\mathcal{K}$.*

Proof. The map

$$L^2(M)\otimes_B L^2(M)\to\mathcal{K},$$
$$\left(n^{-\frac{m}{2}}\sum_{(j_1,\dots,j_m)=\sigma}\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\right)\otimes_B y$$
$$\mapsto\left(n^{-\frac{m}{2}}\sum_{(j_1,\dots,j_m)=\sigma}(\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m))\otimes_D y\right),$$

or in other words $(x_\sigma^n)\otimes_B y\mapsto(x_\sigma^n\otimes_D y)$, is an M-M bimodular isometry. The bimodularity is obvious, so it remains to check that it preserves inner products, in other words that

$$\langle(x_n)\otimes_B y,(x_n')\otimes_B y'\rangle=\langle(x_n\otimes_D y),(x_n'\otimes_D y')\rangle.$$

Let's denote by $E_D:\mathcal{M}\to D$ and by $E_{D\otimes 1}:\Gamma_q(\ell^2\otimes H)\bar{\otimes}D\to D\otimes 1$ the canonical conditional expectations. Since $D=D\otimes 1\subset\mathcal{M}\subset(\Gamma_q(\ell^2\otimes H)\bar{\otimes}D)^\omega$ is embedded as constant sequences, for every $(x_n)\in\mathcal{M}$ we have

$$E_D((x_n))=w-\lim_{n\to\omega}E_{D\otimes 1}(x_n).$$

We now claim that for any $(x_n)\in M\subset\mathcal{M}$ we have $E_B((x_n))=E_D((x_n))$. It suffices to prove this for $(x_n)=W_\sigma\in M$ a reduced Wick word. Let s be the number of singletons in σ. Let

$$W_\sigma=\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\otimes s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\right).$$

We have two possibilities. If $s=0$, then $W_\sigma=F_\sigma(x_1,\dots,x_m)=E_B(\pi_{\phi(1)}(x_1)\cdots\pi_{\phi(m)}(x_m))\in B$; hence $E_D(W_\sigma)=W_\sigma=E_B(W_\sigma)$. If $s>0$, then $E_B(W_\sigma)=0$. On the other hand, according to our previous remark, we have

$$\begin{aligned}E_D(W_\sigma)&=w-\lim_n E_{D\otimes 1}\left(n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\otimes s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s})\right)\\&=w-\lim_n n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\tau(s(e_{l_1}\otimes h_{k_1})\cdots s(e_{l_s}\otimes h_{k_s}))\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))\\&=w-\lim_n n^{-\frac{s}{2}}\sum_{(l_1,\dots,l_s)=\dot{0}}\tau(W(e_{l_1}\otimes h_{k_1}\cdots e_{l_s}\otimes h_{k_s}))\alpha_{l_1,\dots,l_s}(F_\sigma(x_1,\dots,x_m))=0.\end{aligned}$$

This proves our claim. Now, for $(x_n),(x_n'),y,y'\in M$ we have

$$\begin{aligned}\langle(x_n)\otimes_B y,(x_n')\otimes_B y'\rangle&=\tau_M(E_B((x_n'^*x_n))yy'^*)=\tau_{\mathcal{M}}(E_D((x_n'^*x_n))yy'^*)\\&=\lim_n\tau_{\mathcal{M}}(E_{1\otimes D}(x_n'^*x_n)yy'^*)=\lim_n\langle x_n\otimes_D y,x_n'\otimes_D y'\rangle\\&=\langle(x_n\otimes_D y),(x_n'\otimes_D y')\rangle,\end{aligned}$$

which finishes the proof of the lemma. □

Lemma 5.3. *There exists an orthonormal basis Y_α of $L^2(M)$ over B such that for every n, $f_n(Y_\alpha) = 0$ for all but finitely many α's, where we denote somewhat abusively $f_n(Y_\alpha) = f_n(s)$, $s =$ the degree of Y_α.*

Proof. Since D_s is finitely generated over B for all s, according to Proposition 3.20, for every $s \geq 0$ we can find a finite orthonormal basis (Y_β^s) of $L^2_s(M)$ over B. The union (Y_α) of all the Y_β^s's is a basis of $L^2(M)$ over B. For a fixed n, there exists $s = s(n)$ such that $f_n(\xi) = 0$ for all $\xi \in H^{\otimes k}$, for $k > s(n)$. For any $t \geq 0$ and $Y_\alpha \in L^2_t(M)$ we have $f_n(Y_\alpha) = f_n(t)$ and also for every $Y_\alpha \in \bigoplus_{k>s(n)} L^2_k(M)$ we have $f_n(Y_\alpha) = 0$, both due to the reduction formula. On the other hand, the set of those $Y_\alpha \in \bigoplus_{k=0}^{s(n)} L^2_k(M)$ is finite, which finishes the proof. □

Denote by ι the M-bimodular embedding in Lemma 5.2 and define

$$\mu_n^{\mathcal{A}}(T) = \sum_\alpha f_n(Y_\alpha)\langle T\iota(1 \otimes_B 1), \iota(Y_\alpha^* \otimes_B Y_\alpha^*)\rangle, \quad T \in \mathcal{S}_{\mathcal{A}}.$$

Then $\mu_n^{\mathcal{A}} \in (\mathcal{S}_{\mathcal{A}})_*$ satisfies all the required properties.

Step 3. Set $\gamma_n = \mu_n^{\mathcal{A}} \circ \mathcal{E} \in \mathcal{N}_*$, and $\omega_n = \|\gamma_n\|^{-1}|\gamma_n|$. We will prove that the ω_n's satisfy all the required properties. First note that, by construction,

$$\gamma_n(\pi(x)\theta(y^{\mathrm{op}})) = \tau(\varphi_n(x)E_{\mathcal{A}}(y)), \quad x \in M,\ y \in P.$$

Toward proving the required properties of the ω_n's, we will first establish the following two claims:

Claim 1. $\limsup_n \|\mu_n^{\mathcal{A}}\| = 1$.

Claim 2. $\lim_n \|\mu_n^{\mathcal{A}} \circ \mathrm{Ad}(\lambda(u)\rho(\bar{u})) - \mu_n^{\mathcal{A}}\| = 0,\ u \in \mathcal{N}_M(\mathcal{A})$.

Proof of Claim 1. Fix a von Neumann subalgebra $Q \subset P$ which is amenable over B. Just as in Step 2 above one can construct normal functionals μ_n^Q on $\mathcal{S}_Q = \lambda(M) \vee \rho(Q^{\mathrm{op}}) \subset B(\mathcal{K})$ satisfying $\mu_n^Q(\lambda(x)\rho(y^{\mathrm{op}})) = \tau(\varphi_n(x)y)$ for $x \in M,\ y \in Q$. We will show that $\limsup \|\mu_n^Q\| = 1$, and this will help us establish both claims. Since μ_n^Q is normal, it suffices to estimate its norm on an ultraweakly dense C^*-subalgebra of $\mathcal{S}_Q$. Denote by S_Q the ultraweakly dense C^*-subalgebra of $\mathcal{S}_Q$ generated by $\lambda(x_\sigma)$ for $x_\sigma \in M$ the Wick words and $\rho(Q^{\mathrm{op}})$. First we note that there exist cb maps $\tilde{\varphi}_n : S_Q \to S_Q$ such that

$$\tilde{\varphi}_n(\lambda(x_\sigma)\rho(y^{\mathrm{op}})) = \lambda(\varphi_n(x_\sigma))\rho(y^{\mathrm{op}}), \quad x_\sigma \in M,\ y \in Q,$$

and $\|\tilde{\varphi}_n\|_{\mathrm{cb}} = \|\varphi_n\|_{\mathrm{cb}}$. To prove this take $\widetilde{\mathcal{K}} \subset L^2((\mathcal{M} \bar{\otimes} \Gamma_q(\ell^2 \otimes H))^\omega)$ to be the $\|\cdot\|_2$-closed linear span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m)=\sigma} \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y \otimes s(e_{j_1} \otimes h_1)\cdots s(e_{j_m} \otimes h_m)\right) = (x_\sigma^n(y \otimes 1))$$

for all $x_i \in BSB,\ h_i \in H,\ y \in M$. Now define an unitary operator $U : \mathcal{K} \to \widetilde{\mathcal{K}}$ by

$$(x_\sigma^n \otimes_D y) \mapsto (x_\sigma^n(y \otimes 1)).$$

We can then define

$$\tilde{\varphi}_n(z) = U^*(\mathrm{id} \otimes m_n)^\omega(UzU^*)U, \quad z \in S_Q.$$

Then the maps $\tilde{\varphi}_n$ satisfy all the required properties. The complete boundedness of the $\tilde{\varphi}_n$ is a delicate matter and it will be addressed in Section 5A below. On the other hand, since Q is amenable relative to B, we see that the M-Q bimodule $L^2(M)$ is weakly contained in $L^2(M)\otimes_B L^2(M)$, which in turn is contained in $\mathcal{K}$. This produces a $*$-homomorphism $\Theta : S_Q \to B(L^2(M))$ such that $\Theta(\lambda(x)\rho(y^{\rm op})) = \lambda_M(x)\rho_M(y^{\rm op})$, where λ_M, ρ_M are the natural actions of M on $L^2(M)$. But then

$$\mu_n^Q(z) = \langle(\Theta(\tilde{\varphi}_n(z))1, 1\rangle, \quad z \in S_Q,$$

and this implies that $\limsup \|\mu_n^Q\| = 1$. Then by taking $Q = \mathcal{A}$ we get $\limsup \|\mu_n^{\mathcal{A}}\| = 1$, which finishes the proof of the first claim. □

Proof of Claim 2. Fix a unitary $u \in \mathcal{N}_M(\mathcal{A})$. The algebra $Q = \langle \mathcal{A}, u\rangle \subset P$ is amenable relative to B, so by the proof of Claim 1, $\limsup \|\mu_n^Q\| = 1$. Now since $\mu_n^Q(1) = \tau(\phi_n(1)) \to 1$ and $\mu_n^Q(\lambda(u)\rho(\bar{u})) = \tau(\phi_n(u)u^*) \to 1$, we see that $\|\mu_n^Q \circ \mathrm{Ad}(\lambda(u)\rho(\bar{u})) - \mu_n^Q\| \to 0$; hence by restricting to $S_{\mathcal{A}}$ we get $\|\mu_n^{\mathcal{A}} \circ \mathrm{Ad}(\lambda(u)\rho(\bar{u})) - \mu_n^{\mathcal{A}}\| \to 0$. Using the fact that $\mathrm{Ad}(\lambda(u)\rho(\bar{u}))\circ\mathcal{E} = \mathcal{E}\circ\mathrm{Ad}(\pi(u)\theta(\bar{u}))$ and the fact that $\gamma_n = \mu_n^{\mathcal{A}}\circ\mathcal{E}$, we see at once that $\|\gamma_n \circ \mathrm{Ad}(\pi(u)\theta(\bar{u})) - \gamma_n\| \to 0$. But since $\gamma_n(1) = \tau(\phi_n(1)) \to 1$ and $\limsup \|\gamma_n\| = 1$, we see that $\|\gamma_n - \omega_n\| \to 0$. This further implies $\|\omega_n \circ \mathrm{Ad}(\pi(u)\theta(\bar{u})) - \omega_n\| \to 0$, which establishes the third required property, and the other two follow easily. □

This completes the proof of Theorem 5.1. □

5A. ***cb-estimates for the multipliers.*** Here we will prove that some multipliers defined on certain C^*-algebras or von Neumann algebras are completely bounded. The first case is that of the maps $\tilde{\varphi}_n$ which were used in the proof of Theorem 5.1 above. In the second case we prove the complete boundedness of some normal multipliers on the von Neumann algebra $\mathcal{N}$ introduced above, which are needed to construct a concrete standard form for $\mathcal{N}$. We recall some notation.

Notation. $\mathcal{M} = (\Gamma_q(\ell^2\otimes H)\,\bar{\otimes}\, D)\vee M \subset (\Gamma_q(\ell^2\otimes H)\,\bar{\otimes}\, D)^\omega$, where we regard $\Gamma_q(\ell^2\otimes H)$ and D as constant sequences. Let $K = L^2(\mathcal{M})$ or $K = L^2(\mathcal{M})\otimes_{\mathcal{A}} L^2(P)$. We introduce the subspace

$$\mathcal{L} \subset (K \otimes \mathcal{F}_q(\ell_2\otimes H))^\omega$$

as the $\|\cdot\|$-closed linear span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\right) = (x_\sigma^n(y\otimes 1)) \in (K\bar{\otimes}\Gamma_q(\ell^2\otimes H))^\omega,$$

for $m \geq 1$, $\sigma \in P_{12}(m)$, $x_i \in BSB$, $h_i \in H$, $y \in M$. Let's define the extended Wick words $x_\sigma = x_\sigma(x_1, h_1, \dots, x_m, h_m, y^{\rm op})$ by

$$x_\sigma = \left(n^{-\frac{m}{2}} \sum_{(j_1,\dots,j_m)=\sigma} \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y^{\rm op}\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\right),$$

where $m \geq 1$, $\sigma \in P_{1,2}(m)$, $x_i \in BSB$, $h_i \in H$, $y \in P$, viewed as operators in $B(\mathcal{K})$, i.e., acting naturally on sequences in $\mathcal{L}$. The reader can check that:

(a) $\mathcal{L}$ is invariant to the natural action of the extended Wick words.

(b) $\mathcal{L} = \overline{\text{span}}\{\lambda(x_\sigma)\rho(y^{\text{op}})(1\otimes 1) : x_\sigma \in M,\ y \in M\}$ when $K = L^2(\mathcal{M})$ and

$$\mathcal{L} = \overline{\text{span}}\{\pi(x_\sigma)(1\otimes y)\theta(z^{\text{op}})((1\otimes_{\mathcal{A}} 1)\otimes 1) : x_\sigma \in M,\ y \in M,\ z \in P\}$$

when $K = L^2(\mathcal{M})\otimes_{\mathcal{A}} L^2(P)$.

(c) $\mathcal{L}$ is invariant to the natural action by orthogonal transformations of H given by

$$\mathcal{O}(H) \to \text{Aut}((\mathcal{M}\otimes\Gamma_q(\ell^2\otimes H))^\omega), \quad o \to \alpha_o = (\text{id}\otimes\Gamma_q(\text{id}\otimes o)).$$

Let $C(H)\subset B(\mathcal{L})$ be the C^*-algebra generated by the elements

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m)=\sigma} \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y^{\text{op}}\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\right) = (x_\sigma^n(y^{\text{op}}\otimes 1)),$$

where $x_i \in BSB,\ h_i \in H,\ y \in M,\ \sigma \in P(m)$. Also let $\widehat{C}(H)\subset (B(K)\otimes_{\min}\Gamma_q(\ell^2\otimes H))^\omega$ be the C^*-algebra generated by the elements

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m)=\sigma} \pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y^{\text{op}}\otimes s(e_{j_1}\otimes h_1)\cdots s(e_{j_m}\otimes h_m)\right) = (x_\sigma^n(y^{\text{op}}\otimes 1)),$$

where $x_i \in BSB,\ y \in M,\ h_i \in H,\ \sigma \in P(m)$, the ultraproduct being the C^*-algebra ultraproduct.

Remark 5.4. Let m_α be the multipliers on $\Gamma_q(H)$ associated to the nonnegative finite support functions $f_\alpha : \mathbb{N}\to\mathbb{R}$:

(1) One may assume that for every k, $f_\alpha(k) = 1$ for α large enough and that $\limsup_\alpha \|m_\alpha\|_{\text{cb}} = 1$.

(2) $(\text{id}\otimes m_\alpha) : \widehat{C}(H) \to (B(K)\otimes_{\min}\Gamma_q(\ell^2\otimes H))^\omega$ are completely bounded, and are the restrictions of normal maps.

Lemma 5.5. *Let $\widehat{C}(H)$, $C(H)$ and m_α be defined as above*:

(1) *Let $\rho : (B(K)\otimes_{\min}\Gamma_q(\ell^2\otimes H))^\omega \to B((K\otimes\mathcal{F}_q(\ell^2\otimes H))^\omega)$ be the $*$-homomorphism defined by $\rho((T_n))(\xi_n) = (T_n\xi_n)$. Then $\rho(\widehat{C}(H))(\mathcal{L})\subset\mathcal{L}$, so $[\rho(\widehat{C}(H)), P_{\mathcal{L}}] = 0$.*

(2) *The map $\Phi : \widehat{C}(H) \to C(H)$ defined by $\Phi(T) = \rho(T)P_{\mathcal{L}}$ is a surjective $*$-homomorphism.*

(3) *If $\sigma \notin P_{1,2}(m)$, then $\Phi(x_\sigma^n(y^{\text{op}}\otimes 1)) = 0$. In particular, $C(H) = \Phi(\widehat{C}(H))$ is spanned by the elements $\Phi((x_\sigma^n(y^{\text{op}}\otimes 1)))$ for $m \geq 1$, $\sigma \in P_{1,2}(m)$.*

(4) *If $(x_n) = (x_n') \in M$, then $\Phi((x_n(y^{\text{op}}\otimes 1))) = \Phi((x_n'(y^{\text{op}}\otimes 1)))$. In particular, $C(H)$ is spanned by the elements $\Phi((W_\sigma(y^{\text{op}}\otimes 1)))$, where $W_\sigma \in M$, $\sigma \in P_{1,2}(m)$ are the reduced Wick words.*

Proof. Take $(x_\sigma^n(y^{\text{op}}\otimes 1)) \in \widehat{C}(H)$, $(x_{\sigma'}^n(z\otimes 1)) \in \mathcal{L}$. Due to the convolution rule we have

$$\Phi((x_\sigma^n(y^{\text{op}}\otimes 1)))(x_{\sigma'}^n(z\otimes 1)) = (x_\sigma^n x_{\sigma'}^n(zy\otimes 1)) = \sum_{\gamma\in P(m+m')} (x_\gamma^n(zy\otimes 1)),$$

where the summation is taken over all those γ's which preserve the connections of both σ and σ'; i.e., if some indices are connected by σ or σ', they will remain connected in γ. Now for all $\gamma \notin P_{1,2}(m+m')$,

the corresponding term vanishes, because $\|x_\gamma^n(zy\otimes 1)\|_2 \le \|zy\|_\infty \|x_\gamma^n\|_2 \to 0$. Thus

$$\Phi((x_\sigma^n(y^{\rm op}\otimes 1)))(x_{\sigma'}^n(z\otimes 1)) = \sum_{\gamma\in P_{1,2}(m+m')} (x_\gamma^n(zy\otimes 1)) \in \mathcal{L},$$

which proves the first statement. Also, if $\sigma \notin P_{1,2}(m)$ to begin with, every γ in the sum will also not be in $P_{1,2}(m+m')$; hence the whole sum vanishes, which proves the third statement. The second statement is trivial. If $(x_n), (x_n') \in M$ such that $\lim \|x_n - x_n'\|_2 = 0$, then for every $(y_\sigma^n(z\otimes 1)) \in \mathcal{K}$, we have $\|x_n y_\sigma^n(zy\otimes 1) - x_n' y_\sigma^n(zy\otimes 1)\|_2 \le \|y_\sigma^n\|_\infty \|zy\|_\infty \|x_n - x_n'\|_2 \to 0$, i.e., $\Phi((x_n(y^{\rm op}\otimes 1))) = \Phi((x_n'(y^{\rm op}\otimes 1)))$. The last statement then follows from the reduction formula. □

Our goal is to prove that under certain conditions the maps $({\rm id}\otimes m_\alpha)$ descend to a multiplier on the quotient algebra, namely $C(H)$. This is done via a careful analysis of Φ_*.

Lemma 5.6. *There exists a complete contraction*

$$\psi : (\overline{K\otimes L^2(\Gamma_q(\ell^2\otimes H))})^r \otimes_h (K\otimes L^2(\Gamma_q(\ell^2\otimes H)))^c \to \overline{L^1(B(K)\otimes \Gamma_q(\ell^2\otimes H))}$$

such that

$$\psi((h\otimes a)\otimes(k\otimes b)) = (h\otimes \bar{k})\otimes ab^*$$

and ${\rm tr}((S\otimes T)(\psi((h\otimes a)\otimes(k\otimes b))^*) = ((S\otimes T)(k\otimes b), h\otimes a)$. *Here* $(k\otimes h)$ *is the rank-*1 *operator with entries* $(k_i h_j)$ *in a given basis and* $\otimes_h$ *denotes the Haagerup tensor product of operator spaces.*

Proof. We recall that for a semifinite von Neumann algebra M the space $M = \overline{L^1(M,{\rm tr})}^*$ is the antilinear dual with respect to the trace $\langle T,\rho\rangle = {\rm tr}(T\rho^*)$. Moreover, for $M = B(H)$ one usually considers linear duality with respect to the transposed ρ^t of a density ρ:

$$\langle\langle T,\rho\rangle\rangle_{B(H),S_1(H)} = {\rm tr}(T\rho^t) = {\rm tr}(T\dot{\rho}^*) = \langle T,\bar{\rho}\rangle_{B(H),\overline{S_1(H)}}.$$

Using the description of $S_1(H) = H^r \otimes_h H^c$ as a Haagerup tensor product, we find a natural map $\omega : H^r\otimes_h H^c \to B(H)^*$ given by

$$\omega(h\otimes k)(T) = {\rm tr}\left(T\left(\sum_{ij} h_i k_j e_{ij}\right)^t\right) = {\rm tr}\left(T\left(\sum_{ij} h_i k_j e_{ji}\right)\right) = \sum_{ij} T_{ij} h_i k_j = (T(k),\bar{h}).$$

Let M be a semifinite von Neumann algebra and (ξ_j) be an orthonormal basis. Then we may define the antilinear map $v(a) = \sum_j \langle \xi_j, a\rangle \xi_j$ and observe that

$$(b,\overline{v(a)}) = \sum_j \langle b,\xi_j\rangle\langle v(a),\xi_j\rangle = \sum_j \langle b,\xi_j\rangle\langle \xi_j,a\rangle = \tau(ba^*).$$

Therefore $\bar{m} = \omega(v\otimes{\rm id}) : \overline{L_r^2(M)}\otimes_h L_c^2(M) \to B(L^2(M))^*$ satisfies

$$m(a\otimes b)(T) = (T(b),\overline{v(a)}) = \tau(Tba^*) = \tau(T(ab^*)^*)) = \langle T,(ab^*)\rangle$$

for all $T\in M$. This shows that $m(a\otimes b) = ab^*$ is a complete contraction from $\overline{L_r^2}(M)\otimes_h L_c^2(M) \to \overline{L^1(M)}$. Now we repeat the argument for $H = K\otimes L^2(M)$ and $V(h\otimes b) = \bar{k}\otimes v(b)$. Then we obtain a

complete contraction $\psi = \omega(V \otimes \mathrm{id}) : \overline{(K \otimes L^2(M))}^r \otimes_h K \otimes L^2(M) \to B(K \otimes L^2(M))^*$ such that for $S \in B(K)$ and $T \in M$

$$\psi((h \otimes a) \otimes (k \otimes b))(S \otimes T)(S(k), \bar{\bar{h}})\tau(Tba^*) = (S(k), h)(Tb, a) = ((S \otimes T)(k \otimes b), h \otimes a)$$
$$= (tr \otimes \tau)(S \otimes T)(k \otimes \bar{h}).$$

Here $(\alpha \otimes \beta) = \sum_{ij} e_{ij}\alpha_i\beta_j$ is the density of the corresponding rank-1 operator. Therefore the map $\psi((k \otimes a) \otimes (h \otimes b)) = (h \otimes \bar{k}) \otimes ab^*$ does the job. □

Corollary 5.7. *Let $\mathcal{L} \subset (K \otimes L^2(\Gamma_q(\ell^2 \otimes H)))^\omega$ be defined as above. Then there exists a completely contractive map*

$$\Psi : \bar{\mathcal{L}}^r \otimes_h \mathcal{L}^c \to \overline{(L_1(B(K) \otimes \Gamma_q(\ell^2 \otimes H)))}^\omega$$

and a complete contraction $q : \overline{(L_1(B(K) \otimes \Gamma_q(\ell^2 \otimes H)))}^\omega \to [(B(K) \bar{\otimes} \Gamma_q(\ell^2 \otimes H))^\omega]^*$ such that

$$(q \circ \Psi)(k \otimes h)(T) = \langle T(k), h\rangle.$$

In particular $\Psi^*|_{\widehat{C}(H)} = \Phi$.

Proof. For $\xi, \eta \in \mathcal{L}$ given by $\xi = (\xi_n)_n$, $\eta = (\eta_n)_n$ we may define

$$\Psi(\xi \otimes \eta) = (\psi(\xi_n \otimes \eta_n))_n,$$

where ψ is the map from Lemma 5.6. Now Ψ obviously extends by linearity, thanks to the definition of the Haagerup tensor product and the well-known fact that $M_m((X_n)^\omega) = (M_m(X_n))^\omega$; see [Pisier 2003]. The map q is given by the limit

$$q((\bar{\zeta}_n)_n)(T_n)_n = \lim_{n\to\omega} (\mathrm{tr} \otimes \tau)(T_n\zeta_n^*).$$

Now the assertion follows from Lemma 5.6 and the fact that the duality pairing is given by the limit along the ultraproduct. □

Remark 5.8. Let H be an infinite Hilbert space and $H \subset H'$. Thanks to the definition of the C^*-algebra $\widehat{C}(H)$ as a subalgebra of the ultraproduct, we clearly have an isometric inclusion $\widehat{C}(H) \subset \widehat{C}(H')$. The C^*-algebra $C(H) \subset B(\mathcal{L}(H))$ depends on our minimalistic definition of $\mathcal{L}(H)$. Certainly, $\mathcal{L}(H) \subset \mathcal{L}(H')$ and hence the tautological map $\iota(x_\sigma) = x_\sigma$, $\iota(y^{\mathrm{op}}) = y$ produces a larger norm on $\mathcal{L}(H')$ than on $\mathcal{L}(H)$. Let us consider a noncommutative polynomial p in a finite number of x_σ's and y^{op}'s, and we may assume that the x_σ only contain vectors from a finite-dimensional subspace $H_0 \subset H$. Then we can find norm attaining vectors $\xi, \eta \in \mathcal{L}(H')$ for p. Then we write $H' = H_0 \oplus H_0^\perp$ and may also assume that the ξ and η are linear combination of elements in $\mathcal{L}(H_0)$ and $\mathcal{L}(H_1)$, where $H_1 \subset H_0^\perp$ is a finite-dimensional subspace. Using the moment formula, we see that the inner product remains unchanged after applying an orthogonal transformation o which sends H_1 to a finite-dimensional subspace of H orthogonal to H_0 and leaves H_0 invariant. This implies

$$\|p\|_{C(H')} = \sup_{\|\xi\|\le 1, \|\eta\|\le 1} |\langle \xi, p\eta\rangle| = \sup_{\|\xi\|\le 1, \|\eta\|\le 1} |\langle \alpha_o(\xi), p\alpha_o(\eta)\rangle| \le \|p\|_{C(H)}.$$

Let us denote by $q_H = \Phi|_{\widehat{C}(H)} : \widehat{C}(H) \to C(H)$ the quotient map. Then we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{C}(H) & \xrightarrow{q_H} & C(H) \\ \downarrow & & \downarrow \\ \widehat{C}(H') & \xrightarrow{q_{H'}} & C(H') \end{array}$$

where the left-hand downward arrow is the natural ultraproduct inclusion and the right-hand downward arrow is the tautological inclusion (which is well-defined and injective). This allows us to identify elements in the kernel of q_H by considering $q_{H'}$.

We recall that thanks to Avsec's result, the orthogonal projection $P_k : \Gamma_q(H) \to \Gamma_q(H)$ onto Wick words of length k is a normal completely bounded map. We use the same notation

$$\begin{aligned} P_k &: L^1(\Gamma_q(H)) \to L^1(\Gamma_q(H)), \\ \mathrm{id} \otimes P_k &: \overline{L^1(B(K) \otimes \Gamma_q(H))} \to \overline{L^1(B(K) \otimes \Gamma_q(H))}. \end{aligned}$$

Let us note that one can take $P_k^\omega : \prod \overline{L^1(B(K) \otimes \Gamma_q(H))}$, the extension to the ultraproduct of L^1 spaces, which satisfies

$$\langle (\mathrm{id} \otimes P_k)((T_n)), (\xi_n) \rangle = \langle (T_n), (\mathrm{id} \otimes P_k)((\xi_n)) \rangle$$

with respect to the antilinear bracket given by the ultraproduct trace; see also [Raynaud 2002].

Lemma 5.9. *The kernel of $\Phi \circ P_k^\omega$ contains the kernel of q_H.*

Proof. The map $\Phi \circ P_k^\omega$ is normal. According to Remark 5.8 it therefore suffices to show that for $\xi, \eta \in \mathcal{L}$ we have

$$(\mathrm{id} \otimes P_k^\omega)(\Psi(\xi \otimes \eta)) \in \mathrm{Im}(\psi_{H'})$$

for some potentially larger Hilbert space H'. Let us now consider Wick words $(x_\sigma^n)_n$, $\tilde{x}_\sigma^n$, and y^{op}, $\tilde{y}^{\mathrm{op}}$. We have to consider

$$\Psi((\tilde{x}_\sigma \tilde{y}^{\mathrm{op}})_n \otimes (x_\sigma y^{\mathrm{op}})_n) = (\psi(\tilde{x}_\sigma^n y \tilde{y}^{\mathrm{op}} \otimes x_\sigma^n y^{\mathrm{op}}))_n.$$

For fixed $n \in \mathbb{N}$ we see that

$$\begin{aligned} \Psi((\tilde{x}_{\tilde{\sigma}}^n y) \otimes (x_\sigma^n y^{\mathrm{op}})) &= n^{-\frac{m+\tilde{m}}{2}} \sum_{(\tilde{j}_k)=\tilde{\sigma},\ (j_k)=\sigma} (\overline{\vec{\pi}(a) y^{\mathrm{op}}} \otimes \vec{\pi}_{\tilde{j}_1}(\tilde{a}) \tilde{y}^{\mathrm{op}}) \otimes \vec{s}_{\tilde{j}}(\tilde{h}) \vec{s}_j^{\,*} \\ &= \sum_{\sigma' \in P(m+m')} \Psi^{\sigma'}(\tilde{x}_{\tilde{\sigma}} \tilde{y}^{\mathrm{op}} \otimes x_\sigma y^{\mathrm{op}}), \end{aligned}$$

where

$$\Psi^{\sigma'}(\tilde{x}_{\tilde{\sigma}} \tilde{y}^{\mathrm{op}} \otimes x_\sigma y^{\mathrm{op}}) = \sum_{(\tilde{j}_1, \ldots, \tilde{j}_{\tilde{m}}, j_m, \ldots, j_1) = \sigma'} (\overline{\vec{\pi}(a) y^{\mathrm{op}}} \otimes \vec{\pi}_{\tilde{j}_1}(\tilde{a}) \tilde{y}^{\mathrm{op}}) \otimes \vec{s}_{\tilde{j}}(\tilde{h}) \vec{s}_j^{\,*}.$$

Note also that σ' has to be obtained by joining singletons from $\tilde{\sigma}$ and σ. In this context we observe again that is enough to consider $\sigma' \in P_{1,2}(\tilde{m}+m)$. In the following example we see that

$$\begin{aligned}\Big\|\sum_{j_1}(\overline{\pi_{j_1}(a_1)\pi_{j_1}(a_2)y^{\mathrm{op}}}\otimes\pi_{j_1}(\tilde{a}_1)\tilde{y}^{\mathrm{op}})\otimes s_{j_1}^2 s_{j_1}\Big\|_1 &\\ \leq \Big\|\sum_{j_1}\pi_{j_1}(a_1)\pi_{j_1}(a_2)y^{\mathrm{op}}\otimes s_{j_1}^2\otimes e_{1,j_1}\Big\|\,\Big\|\sum_{j_1}\pi_{j_1}(\tilde{a}_1)\tilde{y}^{\mathrm{op}}\otimes s_{j_1}e_{j_1,1}\Big\| &\\ \leq c_q(a,\tilde{a})n \ll n^{\frac{3}{2}} &\end{aligned}$$

is much smaller than $n^{\frac{3}{2}}$ and hence vanishes in the limit. For more complicated configurations, we may assume that σ and $\tilde{\sigma}$ are pair/singleton partitions, and that new links in $\sigma' \in P_{1,2}(m+\tilde{m})$ are obtained from joining pairs or singletons in σ with pairs in $\tilde{\sigma}$ (or the other way around). All the joint pairings can be estimated using the definition of the Haagerup tensor product as above which yields the bound

$$\Big\|\sum_{\substack{(\tilde{j}_k)=\tilde{\sigma},\ (j_k)=\sigma\\(\tilde{j}_k,j_k)=\sigma'}}(\overline{\vec{\pi}_j(a)y^{\mathrm{op}}}\otimes\vec{\pi}_j(\tilde{a})\tilde{y}^{\mathrm{op}})\otimes\vec{s}_{\tilde{j}}(\tilde{h})^*\vec{s}_j(h)\Big\| \leq c_q n^{f(\sigma,\tilde{\sigma},\sigma')}\sup_j\|a_j\|\sup_j\|\tilde{a}_j\|\,\|y^{\mathrm{op}}\|\,\|\tilde{y}^{\mathrm{op}}\|.$$

The function f is obtained as follows. Let α be the number of pairs in σ being linked to either a pair or singleton in $\tilde{\sigma}$, and similarly let β be the number of linked pairs. Then we find

$$f(\sigma,\tilde{\sigma},\sigma') = \frac{|\sigma_s|}{2}+|\sigma_p|-\alpha+\frac{\alpha}{2}+\frac{|\tilde{\sigma}_s|}{2}+|\tilde{\sigma}_p|-\beta+\frac{\beta}{2} = \frac{m+\tilde{m}}{2}-\frac{\alpha+\beta}{2}$$

using row and column vectors $e_{1,i_1,\ldots,i_l}$, $e_{i_1,\ldots,i_l,1}$ for the number l of links in σ'. Thus for $\alpha+\beta>0$ we obtain 0 in the limit and therefore only those σ' which link singletons to singletons give a contribution in the limit. Now we use Pisier's version [2000, Sublemma 3.3] of the Möbius transform. Let σ' be a fixed partition with pairs $\{\{l_1,r_1\},\ldots,\{l_p,r_p\}\}$. Then there are unitaries $\lambda_j^{\sigma'}$ in a product of free group factors such that

$$S_j(h) = s_j\otimes\lambda_j^{\sigma'}$$

satisfies

$$\begin{aligned}a(\sigma') &:= \sum_{\sigma''\geq\sigma'}\Psi^{\sigma'}(\tilde{x}_{\tilde{\sigma}}\tilde{y}^{\mathrm{op}}\otimes x_\sigma y^{\mathrm{op}}) = \sum_{\tilde{j}_k,j_k}(\overline{\vec{\pi}_j(a)y^{\mathrm{op}}}\otimes\vec{\pi}_j(\tilde{a})\tilde{y}^{\mathrm{op}})\otimes(\mathrm{id}\otimes E)(\vec{S}_j(\tilde{h})^*\vec{S}(h))\\ &= (\mathrm{id}\otimes E)\psi(\widetilde{X}_{\tilde{\sigma}}^{\sigma'}\otimes X_\sigma^{\sigma'}).\end{aligned}$$

Here

$$X_\sigma^{\sigma'} = \Big(n^{-\frac{m}{2}}\sum_{(j_1,\ldots,j_m)\leq\sigma}\pi_{j_1}(a_1)\cdots\pi_{j_m}(a_m)\otimes S_{j_1}(h_1)\cdots S_{j_m}(h_m)\Big)$$

and the corresponding expression for $\widetilde{X}_{\tilde{\sigma}}^{\sigma'}$ depends on σ'. Moreover, there exists a Möbius function $\mu(\cdot,\cdot)$ such that, see [Pisier 2000, Proposition 1],

$$\Psi^{\sigma'}(\tilde{x}_{\tilde{\sigma}}\tilde{y}^{\mathrm{op}}\otimes x_\sigma y^{\mathrm{op}}) = \sum_{\pi\geq\sigma'}\mu(\sigma',\pi)a(\pi).$$

The advantage of this representation comes from the fact that we can actually calculate P_k for such a fixed σ'. Recall that we may assume that σ' is a pair/singleton partition. For fixed $n \in \mathbb{N}$ and an element $\eta_n = S \otimes W_n$, W_n a Wick word of length k, we obtain

$$\mathrm{tr}(S((\overline{\vec{\pi}_j(a)y^{\mathrm{op}}} \otimes \vec{\pi}_j(\tilde{a})\tilde{y}^{\mathrm{op}}))\,\tau(W_n s_{\tilde{j}_1} \cdots s_{\tilde{j}_{\tilde{m}}} s_{j_m} \cdots s_{j_1})$$

such that $(\tilde{j}_1, \ldots, \tilde{j}_{\tilde{m}}, j_m, \ldots, j_1) = \sigma'$. Then we obtain a nonzero term only if $|\sigma'_s| = k$ has exactly k singletons. Hence we find that

$$(\mathrm{id} \otimes P_k)(\psi(\xi \otimes \eta)) = \sum_{|\sigma'_s|=k} \Psi^{\sigma'}(\tilde{x}_{\tilde{\sigma}}\tilde{y}^{\mathrm{op}} \otimes x_\sigma y^{\mathrm{op}}) = \sum_{|\sigma'_s|=k} \sum_{\pi \geq \sigma'} \mu(\sigma, \pi) a(\pi).$$

Therefore we are left to consider

$$a(\pi) = (\mathrm{id} \otimes E)(\Psi(\widetilde{X}^\pi_{\tilde{\sigma}} \otimes X^\pi_\sigma)).$$

In order to use Remark 5.8 we have to modify the variables X^π_σ. Indeed, for every pair $p = \{l, r\}$ in π we introduce a label e_p and replace $s(e_{j_l} \otimes h_l)$ by $s(e_{j_l} \otimes e_p \otimes h_l)$, and $s(e_{\tilde{j}_r} \otimes \tilde{h}_r)$ by $s(e_{\tilde{j}_r} \otimes e_p \otimes h_r)$. For the remaining singletons we replace $s(e_j \otimes h_j)$ by $S_j = s(e_j \otimes e_0)$ and work in the Hilbert space $H' = H \otimes \ell_2$. Using the so modified X^π_σ's we still have

$$a(\pi) = (\mathrm{id} \otimes E_{\Gamma_q(\ell_2 \otimes H \otimes e_0)})_n \Psi(\widetilde{X}^\pi_{\tilde{\sigma}} \otimes X^\pi_\sigma) = \lim_{j \to \infty} \Psi(\alpha_{o_j}(\widetilde{X}^\pi_{\tilde{\sigma}}) \otimes \alpha_{o_j}(X^\pi_\sigma))$$

for any sequence (o_j) of orthogonal transformations such that $o_j(e_0) = e_0$, which converges weakly to $e_0^\perp$. For elements in $\widehat{C}(H)$ the limit for $j \to \infty$ converges, and hence this remains true for the norm closure. Thus for an element $x \in \widehat{C}(H)$ in the kernel of q_H we find $q_{H'}(x) = 0$ and hence

$$\langle x, a(\pi)\rangle = \lim_{j \to \infty} \langle x, \Psi(\alpha_{o_j}(\widetilde{X}^{\sigma'}_{\tilde{\sigma}}) \otimes \alpha_{o_j}(X^{\sigma'}_\sigma)\rangle = 0.$$

Using linear combinations we deduce indeed that $\langle P_k(x), \Psi(\tilde{x}_\sigma \tilde{y}^{\mathrm{op}} \otimes x_\sigma y^{\mathrm{op}})\rangle = 0$. □

Corollary 5.10. *Let m_α be multipliers given by the cb-approximation property for $\Gamma_q(H)$:*

(i) *Then $(\mathrm{id} \otimes m_\alpha)_n$ extend to completely bounded maps on $C(H)$ with $\limsup_\alpha \|(\mathrm{id} \otimes m_\alpha)_n\|_{\mathrm{cb}} = 1$, and $\lim_\alpha f_\alpha(k) = 1$, where f_α are the associated scalar finitely supported functions. In particular, the maps $\tilde{\varphi}_n$ used in the proof of Theorem 5.1 above are completely bounded with $\limsup_n \|\tilde{\varphi}_n\|_{\mathrm{cb}} = 1$.*

(ii) *Let $L(H) = \overline{C(H)}^{\mathrm{so}} \subset B(\mathcal{L})$ and note that $L(H)$ is spanned by "extended Wick words" (i.e., images of extended Wick words through Φ) such that $L^2_k(L(H))$ (i.e., the $\|\cdot\|_2$-closed linear span of the extended Wick words of degree k) is finitely generated over B. Then there exists a modified family $f_\alpha(N)^* : L(H)_* \to L(H)_*$ converging in the point-norm topology.*

Proof. Since $(\mathrm{id} \otimes m_\alpha)(T) = \sum_k f_\alpha(k) P_k(T)$, we see that $\|\Phi_H \circ (\mathrm{id} \otimes m_\alpha)\|_{\mathrm{cb}} \leq 1 + \varepsilon_\alpha$ and also $\ker(q_H) \subset \ker(\Phi_H \circ (\mathrm{id} \otimes m_\alpha))$. But that means that there is a unique map $\tilde{m}_\alpha : \widehat{C}(H)/\ker(q_H) \to B(\mathcal{K})$ such that

$$\|\tilde{m}_\alpha\|_{\mathrm{cb}} = \|m_\alpha\|_{\mathrm{cb}} \leq 1 + \varepsilon_\alpha.$$

However, $\widehat{C}(H)/\ker(q_H) = C(H)$ completely isometrically, and hence $\tilde{m}_\alpha = (\mathrm{id} \otimes m_\alpha)$ coincides with the densely defined map $(\mathrm{id} \otimes m_\alpha)W(\sigma, \xi, a, y) = f_\alpha(|\sigma_s|)W(\sigma, \xi, a, y)$. Let us now consider a finite-dimensional subspace $H_0 \subset H$. Since $L^2_k(L(H))$ is finitely generated over B, we deduce that the projection P_d is normal on $L(H_0)$. Hence the maps m_α are also normal and restricted to the weakly dense subspace $C(H_0)$ we know that

$$\|m_\alpha\|_{\mathrm{cb}} \leq (1 + \varepsilon_\alpha).$$

Since a weakly dense subspace is norming for $L(H_0)_*$ we deduce that $\|(m_\alpha)_* : L(H_0)_* \to L(H_0)_*\|_{\mathrm{cb}} \leq (1 + \varepsilon_\alpha)$. Hence the normal map m_α coincides with the normal map $((m_\alpha)_*)^*$ and satisfies the same cb-norm estimate. Moreover, since we have normal conditional expectations $\mathcal{E}_{H_0} : L(H) \to L(H_0)$ so that $\bigcup_{H_i} \mathcal{E}_{H_i}(L(H_i))_*$ is norm dense in $L(H)_*$, we deduce that $(m_\alpha)_*$ extends to a completely bounded map of cb-norm at most $1 + \varepsilon_\alpha$ and hence $m_\alpha = ((m_\alpha)_*)^*$ is indeed a normal extension of the map $m_\alpha : C(H) \to C(H)$ with the same cb-norm estimate. This concludes the proof of (ii). □

The remainder of the subsection is devoted to proving some auxiliary results which will help us construct a standard form for the von Neumann algebra $\mathcal{N}$ which was used in the proof of Theorem 5.1. This standard form will be crucial in the proof of the main technical theorem.

Lemma 5.11. *There exists an action by $*$-automorphisms $\alpha : \mathcal{O}(H) \to \mathrm{Aut}(\mathcal{N})$ such that*

$$\alpha_o(\pi(x)\theta(y^{\mathrm{op}})) = \pi(\alpha_o(x))\theta(y^{\mathrm{op}}), \quad o \in \mathcal{O}(H),\ x \in M,\ y^{\mathrm{op}} \in P^{\mathrm{op}}.$$

Moreover, let E_0 be the orthogonal projection of $\mathcal{L}$ onto the closed linear span of the extended Wick words of degree zero. For $T \in \mathcal{N}$ the condition

$$\alpha_o(T) = T \quad \textit{for all } o \in \mathcal{O}(H)$$

implies that $[T, E_0] = 0$.

Proof. Let us recall that $\mathcal{N}$ acts on

$$\begin{aligned}\mathcal{H} &= \mathrm{span}\{\pi(x_\sigma)(y \otimes 1)\theta(z^{\mathrm{op}})((1 \otimes_{\mathcal{A}} 1) \otimes 1) : x_\sigma \in M,\ y \in M,\ z \in P\} \\ &\subset \big((L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(P)) \otimes L_2(\Gamma_q(\ell^2 \otimes H))\big)^\omega.\end{aligned}$$

Recall here that H is infinite-dimensional, and thanks to second quantization, $u_o = (\mathrm{id} \otimes \alpha_o)_n$ acts on $\mathcal{H}$ as a unitary. By normality, we deduce that $\alpha_o(x) = u_o x u_o^*$ extends to a $*$-automorphism of $\mathcal{N}$ and moreover, $\alpha_o(\theta(y^{\mathrm{op}})) = \theta(y^{\mathrm{op}})$. Let $o_i \in \mathcal{O}(H)$ be a family of orthogonal transformations of H such that $o_i(h)$ goes to 0 weakly in H. Let $\xi = \pi(x_\sigma)(y \otimes_{\mathcal{A}} z \otimes 1)$ and $\eta = \pi(x'_{\sigma'})(y' \otimes_{\mathcal{A}} z' \otimes 1)$. Then we obtain

$$\begin{aligned}\lim_i (u_{o_i}(\xi), \eta) &= \lim_i \lim_{n\to\omega} n^{-\frac{m+m'}{2}} \sum_{(j_k)=\sigma,\ (j'_{k'})=\sigma'} (\vec{\pi}_j(x)(y \otimes_{\mathcal{A}} z), \vec{\pi}_{j'}(x')(y' \otimes_{\mathcal{A}} z') \\ &\qquad \tau(s_{j_1}(o_i(h_1)) \cdots s_{j_m}(o_i(h_m)) s_{j'_{m'}}(h'_{m'}) \cdots s_{j'_1}(h'_1)) \\ &= 0.\end{aligned}$$

Indeed, we expand the sum into the summation over $\sigma'' \in P_{1,2}(m+m')$ and execute the limit over n. Then we observe that the coefficients remain uniformly bounded. However, $o_i(h_k)$ is eventually orthogonal to

every $h'_{k'}$ and then the moment formula for the q-gaussian yields 0 in the limit. We have therefore shown that u_{o_i} converges weakly to E_0, the projection onto words of length 0 in the second component. By taking convex combinations we find a net such that

$$\mathrm{SOT}-\lim_s \sum_i \alpha_i^s u_{o_i} = E_0.$$

Thus for $T \in N$ with $\alpha_o(T) = T$ for all o, we deduce that $[u_o, T] = 0$ and hence

$$E_0(T(\xi)) = \lim_s \sum_i \alpha_i^s u_{o_i} T(\xi)) = T\left(\lim_s \sum_i \alpha_i^s u_{o_i}(\xi)\right) = T(E_0(\xi)).$$

This means $E_0 T = T E_0$ as desired. □

Lemma 5.12. *Let $B \vee P^{\mathrm{op}} \subset B(L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(P))$. Then the natural inclusion map*

$$\pi : B \vee P^{\mathrm{op}} \to \mathcal{N}$$

is normal.

Proof. By density it suffices to consider $\xi_n = \pi(x_\sigma^n)(y \otimes_A z)$ and $\eta_n = \pi(\tilde{x}_{\tilde{\sigma}}^n)(\tilde{y} \otimes_A \tilde{z})$. We may assume that x_σ and $\tilde{x}_\sigma$ is a Wick word. Our goal is to analyze

$$\phi(T) = \lim_{n\to\omega} \langle T\xi_n, \eta_n\rangle.$$

Let us first fix $n \in \mathbb{N}$. Then $\omega_n(T) = \langle T\xi_n, \eta_n\rangle$ is normal, and hence it suffices to assume $T = b\theta(p^{\mathrm{op}})$. It turns out that we need $|\sigma| = |\tilde{\sigma}| = k$ and then

$$\omega_n(T) = \frac{n\cdots(n-k+1)}{n^k} \sum_{\gamma\in S_k} q^{\mathrm{inv}(\gamma)} \tau\big(\tilde{z}^* E_A(\tilde{y}^* \pi_{\gamma(k)}(\tilde{x}_k)\cdots\pi_{\gamma(1)}(\tilde{x}_1) b \pi_1(x_1)\cdots\pi_k(x_k) y) z p\big).$$

Thanks to Lemma 5.2, we may replace $L^2(\mathcal{M})$ by $L^2(D) \otimes_B L^2(\mathcal{M})$ in the definition of $\mathcal{H}$. For fixed γ we may now define

$$x_\gamma = \alpha_{1,\dots,k}(x) \otimes_B y \otimes_{\mathcal{A}} z, \quad \tilde{x}_\gamma = \alpha_{\gamma(1),\dots,\gamma(k)}(\tilde{x}) \otimes_B \otimes \tilde{y} \otimes_{\mathcal{A}} \tilde{z}.$$

Since ω_n is normal we deduce that

$$\omega_n(T) = \sum_\gamma q^{\mathrm{inv}(\gamma)} \frac{n\cdots(n-k+1)}{n^k} \langle T(x_\gamma), \tilde{x}_\gamma\rangle$$

for all $T \in B \vee P^{\mathrm{op}}$. Since the summation is finite and the scalar coefficients converge, the limit exists for all $T \in B \vee P^{\mathrm{op}}$ and results in a normal functional $\phi(T)$ given by the same sum but with coefficient 1 instead of $n\cdots(n-k+1)/n^k$. □

Proposition 5.13. *Assume that for every finite-dimensional Hilbert space H, $L^2_k(M(H))$ is finitely generated as a right B-module (note that in particular this is the case if $\dim_B(D_k(S)) < \infty$ for all k).*

Then:

(i) *There exists a faithful normal conditional expectation* $\mathcal{E}:\mathcal{N}\to B_P=\pi(B)\vee\theta(P^{\mathrm{op}})$.

(ii) *The action* α *is implemented by an sot-continuous family of unitary operators* $(V_o)_{o\in\mathcal{O}(H)}$ *on* $L^2(\mathcal{N})$.

(iii) $L^2(\mathcal{N})=\overline{\bigoplus_{k\geq 0}W_k(M)L^2(B_P)}$ *and* $V_o(\pi(x_\sigma)\xi)=\pi(\alpha_o(x_\sigma))\xi$ *for* $x_\sigma\in M$, $\xi\in L^2(B_P)$. *Moreover,* $\mathcal{E}|_{\pi(M)}=E_B$, *where* $E_B:\pi(M)\to\pi(B)$ *is the conditional expectation.*

Proof. For a subspace $H'\subset H$ we use the notation

$$\mathcal{H}(H')=\{\pi(x_\sigma)((y\otimes_{\mathcal{A}}z)\otimes 1):y\in M,\ z\in P,\ x_\sigma=x_\sigma(x_1,\dots,x_m,h_1,\dots,h_m),\ h_i\in H'\}$$

for the subspace generated by H'-Wick words. Let $\iota_{H'}:\mathcal{H}(H')\subset\mathcal{H}$ be the canonical inclusion map and $F_{H'}(T)=\iota_{H'}^*T\iota_{H'}$ the induced completely positive map. Certainly, we have $F_{H'}(\theta(y^{\mathrm{op}}))=\theta(y^{\mathrm{op}})$ and

$$F_{H'}(x_\sigma)=E_{H'}(x_\sigma).$$

Indeed, if a Wick word x_σ contains a singleton $h_i\in(H')^\perp$, then $F_{H'}(x_\sigma)=0$. Using $h_i\in H'\cup(H')^\perp$ we deduce the assertion by linearity. Thus $F_{H'}(\mathcal{N}(H))=\mathcal{N}(H_i)\subset B(\mathcal{H}(H'))$ defines a normal surjective conditional expectation $F_{H'}$. Let $e_{H'}$ be the support of $F_{H'}$. We observe that $\pi(M(H'))$ and $\theta(P^{\mathrm{op}})$ belong to the multiplicative domain of $F_{H'}$. Let $\widetilde{N}(H')\subset\mathcal{N}(H)$ be the von Neumann algebra generated by $\pi(M(H'))$ and $\theta(P^{\mathrm{op}})$ inside $\mathcal{N}_P(H)$. According to Remark 5.8 and Kaplansky's density theorem, we deduce that $F_{H'}$ induces the same weak* topology on the unit ball of $\widetilde{N}(H')$. This means that the tautological embedding $\sigma_{H'H}:\mathcal{N}(H_i)\to\mathcal{N}(H)$ given by $\sigma_{H'H}(x_\sigma)=x_\sigma$ and $\sigma_{H'H}(\theta(y^{\mathrm{op}}))=\theta(y^{\mathrm{op}})$ satisfies $F_{H'}\sigma=\mathrm{id}_{\mathcal{N}(H_i)}$ and $F_{H'}$ is an isomorphism when restricted to $\widetilde{N}(H')$. We denote by $\mathcal{E}_{H'}=\sigma_{H_iH}F_{H'}:\mathcal{N}\to\mathcal{N}$ the resulting, not necessarily faithful, conditional expectation. Let H_i be an increasing net of finite-dimensional spaces whose union is dense. Since $\bigcup_i\mathcal{H}(H_i)$ is norm dense, we deduce that $\widehat{\mathcal{E}}_{H_i}(x)$ converges weakly to x as i goes to infinity along the net of finite-dimensional subspaces. Recall that the multiplier maps m_α are normal and commute with every $\mathcal{E}_{H_i}$. Adding convex combination we may find a new completely contractive net, still denoted by m_α, converging in the strong, strong* operator topology. Thus we may assume that

$$\lim_i\lim_\alpha(\widehat{E}_{H_i}(m_\alpha x))=x$$

converges strongly for all $x\in\mathcal{N}$. In our next step we consider $H'=0$, i.e., the map $\iota:L^2(M)\otimes_{\mathcal{A}}L^2(P)\to\mathcal{H}$, given by $\iota(y\otimes_{\mathcal{A}}z)=(y\otimes_{\mathcal{A}}z)\otimes 1$. This yields a completely positive map $\Phi(T)=\iota^*T\iota$ such that $\Phi(\theta(y^{\mathrm{op}}))=\theta(y^{\mathrm{op}})$ and $\Phi(\pi(b))=\pi(b)$. On the other hand, for a Wick word $x=W_\sigma$, we see that

$$\langle\pi(x)\iota(y\otimes_{\mathcal{A}}z),\iota(y'\otimes_{\mathcal{A}}z')\rangle=\lim_{n\to\omega}n^{-\frac{m}{2}}\sum_{(i_j)=\sigma}\langle\vec{\pi}(x)(y\otimes_{\mathcal{A}}z),y'\otimes_{\mathcal{A}}z'\rangle\tau(s_{j_1}(h_1)\cdots s_{j_m}(h_m))=0.$$

By normality, we deduce that $\Phi(\mathcal{N})=B\vee P^{\mathrm{op}}\subset B(L^2(\mathcal{M})\otimes_{\mathcal{A}}L^2(P))$. Let us denote by $B_P=\Phi(\mathcal{N})$ the resulting von Neumann algebra and by e_{B_P} the support of $\mathcal{E}=\Phi|_{\mathcal{N}_P}$. Since the Wick words of order 0 are obviously invariant under α_o for all $o\in\mathcal{O}(H)$ and

$$\mathcal{E}\alpha_o(x)=\alpha_o(\mathcal{E}(x))=\mathcal{E}(x),$$

we must have $\alpha_o(e_{B_P}) = e_{B_P}$ for every $o \in \mathcal{O}(H)$. More precisely, $1 - e_{B_P}$ is the projection of the ideal $I = \{x : \mathcal{E}(x^*x) = 0\}$ and we certainly have $\alpha_o(I) = I$. This implies $\alpha_o(1 - e_{B_P}) = 1 - \alpha_o(e_{B_P})$. We deduce that for all α we have $\alpha_o(m_\alpha e_{B_P}) = m_\alpha e_{B_P}$ and hence, thanks to Lemma 5.11 we know that $[E_0, m_\alpha(e_{B_P}))] = 0$. Now, we fix α and consider $x_{i,\alpha} = \mathcal{F}_{H_i}(m_\alpha(e_{B_P})) = m_\alpha F_{H_i}(e_{B_P})$. This means

$$x_{i,\alpha} = \sum_{k \le k(\alpha)} x_k,$$

where $x_k = P_k(x)$. However, we have a finite basis $\xi_{k,s}$ of $L^2_k(M(H))$ over B made of elements in $W_k(H_i)$ and hence for all $z = \pi(x_{\sigma'})\theta(y^{\rm op})$ we find

$$P_k(z) = \sum_s \pi(\xi_{k,s}) E_B(\xi_s^* x_{\sigma'})\theta(y^{\rm op}).$$

Since P_k is normal we deduce that there are coefficients $a_s \in \pi(B) \vee \theta(P^{\rm op})$ such that

$$x_k = \sum_s \pi(\xi_{k,s}) a_{s,k} \in \mathcal{N}(H_i).$$

Note here that we have rewritten m_α as normal map, because the maps $T_{k,s}(x) = \pi(\xi_{ks})\sigma(\mathcal{E}(\xi_{ks}^* x))$ are normal, thanks to Lemma 5.12. Note also that due to Lemma 5.12, $\sigma(B \vee P^{\rm op}) = \pi(B) \vee \theta(P^{\rm op}) \subset \mathcal{N}$. On the other hand the projection P_{H_i} onto the range of ι_{H_i} contains the range of ι and hence

$$[E_0, \iota_{H_i}^* \hat{m}_\alpha(e_{B_P})\iota_{H_i}] = \iota_{H_i}^* [E_0, \hat{m}_\alpha(e_{B_P})]\iota_{H_i} = 0.$$

Thus we have $[E_0, x_{i,\alpha}] = 0$. Let us consider $\eta = (y \otimes_{\mathcal{A}} z) \otimes 1$. We deduce that

$$x_{i,\alpha}(\eta) = \sum_{k \le k(\alpha)} \sum_s \pi(\xi_{k,s}) a_{k,s}(\eta).$$

Moreover, we see that

$$E_B(\xi_{s,k}^* x_{i,\alpha}(\eta)) = E_B(\xi_{s,k}^* \xi_{s,k}) a_{k,s}(\eta).$$

We may assume that $f_{k,s} = E_B(\xi_{s,k}^* \xi_{s,k})$ is a projection in B and $a_{k,s} = f_{ks} a_{ks}$. Since the conditional expectation can be calculated using vectors in the Hilbert space, we deduce that

$$a_{k,s}(\eta) = E_B(\xi_{s,k}^* x_{i,\alpha}(\eta)) = E_B(\xi_{s,k}^* E_0(x_{i,\alpha}(\eta))) = 0$$

for all $k > 0$. Thus only the coefficient for $k = 0$ survives and hence $x_{i,\alpha} \in \sigma(B \vee P^{\rm op})$. This remains true for the limit along α; i.e., $x_i = F_{H_i}(e_{B_P}) \in \sigma(B \vee P^{\rm op})$. Since $\bigcup_i \iota_{H_i}$ is norm dense we find that

$$e_{B_P} = w^* - \lim_i F_{H_i}(e_{B_P}) \in \sigma(B \vee P^{\rm op}).$$

The restriction of the normal map $\sigma \circ \mathcal{E}$ to $\sigma(B \vee P^{\rm op})$ is the identity. This implies

$$1 - e_{B_P} = \sigma \circ \mathcal{E}(1 - e_{B_P}) = \sigma \circ \mathcal{E}(e_{B_P}(1 - e_{B_P})e_{B_P}) = 0.$$

Thus $e_{B_P} = 1$ and $\mathcal{E}$ is indeed a faithful normal expectation. Now it is easy to conclude the proof of the crucial assertion (iii). Indeed, we may assume that $\pi(B)$ and $\theta(P^{\rm op})$ both admit weakly dense separable C^*-subalgebras and hence fix a faithful normal state ϕ on B_P. Then $\psi = \phi \circ \mathcal{E}$ satisfies Connes'

commutativity relation for the modular group $\mathcal{E}(\sigma_t^{\psi}(x)) = \sigma_t^{\phi}(\mathcal{E}(x))$. We refer to [Haagerup, Junge and Xu 2010] for the fact that we have a natural embedding of the Haagerup spaces $L^p(B_P) \to L^p(\mathcal{N})$ given by

$$\iota_p(xd_{\phi}^{\frac{1}{p}}) = xd_{\psi}^{\frac{1}{p}}$$

for the densities $d_\phi \in L^1(B \vee P^{\mathrm{op}})$, $d_\psi \in L^1(\mathcal{N}_P)$ associated with the states. Moreover, the support of d_ψ is 1. This implies that $L^2(\mathcal{N}) = \mathcal{N}L^2(B_P)$. By approximation in the C^*-algebra generated by $\pi(M)$ and $\theta(P^{\mathrm{op}})$, we see that the span of elements of the form

$$\pi(x_\sigma)\theta(y^{\mathrm{op}})d_{\psi}^{\frac{1}{2}}$$

is dense in $L^2(\mathcal{N})$. However, we have

$$\begin{aligned}
\mathrm{tr}\big((\pi(x_\sigma)\theta(y^{\mathrm{op}})d_{\psi}^{\frac{1}{2}})^*\pi(x_\nu)\theta(z^{\mathrm{op}})d_{\psi}^{\frac{1}{2}}\big) &= \mathrm{tr}\big(\theta(y^{\mathrm{op}})^*\pi(x_\sigma)^*\pi(x_\nu)\theta(z^{\mathrm{op}})d_\psi\big)\\
&= \mathrm{tr}\big(\theta(y^{\mathrm{op}})^*\theta(z^{\mathrm{op}})\pi(x_\sigma)^*\pi(x_\nu)d_\psi\big)\\
&= \psi\big(\theta(y^{\mathrm{op}})^*\theta(z^{\mathrm{op}})\pi(x_\sigma)^*\pi(x_\nu)\big)\\
&= \phi\big(\mathcal{E}(\theta(y^{\mathrm{op}})^*\theta(z^{\mathrm{op}})\pi(x_\sigma)^*\pi(x_\nu))\big)\\
&= \phi\big(\theta(y^{\mathrm{op}})^*\theta(z^{\mathrm{op}})\mathcal{E}(\pi(x_\sigma)^*\pi(x_\nu))\big)\\
&= \phi\big(\theta(y^{\mathrm{op}})^*\theta(z^{\mathrm{op}})E_B\pi(x_\sigma)^*\pi(x_\nu)\big). \qquad (5\text{-}1)
\end{aligned}$$

For the proof of the last equality, we may assume that x_σ and x_ν are reduced Wick words. As in Lemma 5.12, we see that

$$\begin{aligned}
&\langle \pi(x_\xi)(y \otimes_{\mathcal{A}} z), \pi(x_\nu)(\tilde{y} \otimes_{\mathcal{A}} \tilde{z})\rangle\\
&= \lim_n n^{-\frac{|\sigma|+|\nu|}{2}} \sum_{(j_k)=\sigma,\ (\tilde{j}_{\tilde{k}})=\tilde{\sigma}} \tau\big(\tilde{z}^*E_{\mathcal{A}}(\tilde{y}^*\vec{\pi}_{\tilde{j}}(\tilde{x})^*\vec{\pi}_j(x)y)z\big)\,\tau(s_{\tilde{j}_m}(\tilde{h}_m)\cdots s_{\tilde{j}_1}(\tilde{h}_1)s_{j_1}(h_1)\cdots s_{j_m}(h_m))\\
&= \delta_{|\sigma|,|\nu|}\sum_{\gamma\in S_k} q^{\mathrm{inv}(\sigma)}n^{-|\sigma|}\sum_{(j_1,\ldots,j_k)} \tau\big((\alpha_{j_{\gamma(1)},\ldots,j_{\gamma(k)}}(\tilde{x}))^*\alpha_{j_1,\ldots,j_k}(x)yE_{\mathcal{A}}(z\tilde{z}^*)\tilde{y}^*\big)\\
&= \delta_{|\sigma|,|\nu|}\sum_{\gamma\in S_k} q^{\mathrm{inv}(\sigma)}\tau(b(x,\tilde{x},\gamma)yE_{\mathcal{A}}(z\tilde{z}^*)\tilde{y}^*).
\end{aligned}$$

The limit $b(x,\tilde{x},\gamma) \in B$ only depends on x and $\tilde{x}$ and the permutation γ. Placing the summation inside we find indeed $E_B(x_\nu^*x_\sigma)$. Thus we have shown that $\mathcal{E}|_{\pi(M)} = E_B$. We deduce that the spaces $W_k(M)L_2(B_P)$ are mutually orthogonal. Finally, we have to discuss the action $\alpha : \mathcal{O}(H) \to \mathrm{Aut}(\mathcal{N})$. For an arbitrary $*$-automorphism α of $\mathcal{N}$, we may define the action on $L^2(\mathcal{N})$ via

$$\alpha(xd_{\psi}^{\frac{1}{2}}) = \alpha(x)(d_\psi \circ \alpha^{-1})^{\frac{1}{2}}.$$

It is easy to show that this action is independent of the choice of a normal faithful density d associated with state ψ. Here $d \circ \alpha^{-1}$ is the density of $\psi \circ \alpha^{-1}$. Thus we deduce from $\alpha_o(\theta(y^{\mathrm{op}})) = \theta(y^{\mathrm{op}})$ and

the fact that $\psi \circ \alpha_o = \psi$, that

$$\alpha_o(\pi(x_\sigma)\theta(y^{\rm op})d_\psi^{\frac{1}{2}}) = \alpha_o(\pi(x_\sigma))\theta(y^{\rm op})d_\psi^{\frac{1}{2}},$$

as expected. □

Remark 5.14. A posteriori, we deduce that under the assumptions above, $F_{H'}$ is faithful for every subspace $H' \subset H$ because $\mathcal{E} = \mathcal{E}F_{H'}$.

6. The deformation bimodules are weakly contained in $L^2(M) \otimes_B L^2(M)$ for polynomial dimensions of $D_k(S)$ over B

6A. ***Norm estimates for decomposable maps.*** Let H be an M-N bimodule over finite von Neumann algebras M and N. We will introduce some norms which will enable us to show that the M-N bimodules associated to certain maps $\Phi : M \to L^1(N) = N_*^{\rm op}$ are weakly contained in H. To be more precise define

$$\|\Phi\|_H = \inf\left\{ \sum_j \|\xi_j\| \|\eta_j\| : \tau(\Phi(x)y) = \sum_j \langle (x \otimes y^{\rm op})\xi_j, \eta_j \rangle \right\}.$$

The infimum is taken over elements $\xi_j, \eta_j \in H$.

Lemma 6.1. *Let K be an M-N bimodule such that for a total set of vectors $\xi \in K$ the map $\Phi_\xi : M \to L^1(N)$ defined by*

$$\tau(\Phi_\xi(x)(y)) = \langle (x \otimes y^{\rm op})\xi, \xi \rangle = \langle x\xi y, \xi \rangle$$

satisfies $\|\Phi_\xi\|_H < \infty$. Then K is weakly contained in H.

Proof. Let us recall that $K \prec H$ if and only if we have the relation between the kernels

$$\ker(\pi_H) \subset \ker(\pi_K),$$

where $\pi_H : M \otimes_{\rm bin} N^{\rm op} \to B(H)$, respectively $\pi_K : M \otimes_{\rm bin} N^{\rm op} \to B(K)$ are the canonical representations. Let $z = \lim z_j$ be a limit of norm-1 elementary tensors which converges to an element $z \in \ker(\pi_H)$ with respect to the max norm. Let $\xi \in K$ such that $\|\Phi_\xi\|_H < \infty$. This means we may assume that

$$\tau(\Phi_\xi(x)y) = \sum_l \alpha_l \langle \xi_l, x\eta_l y \rangle, \quad \|\xi_l\| \|\eta_l\| \le 1,$$

and $\sum_l |\alpha_l|$ is finite. Using $\|z_j\|_{\rm bin} \le 1$ and uniform convergence, we may interchange limits and deduce

$$\langle z\xi, z\xi \rangle = \lim_j \langle \xi, z_j^* z_j \xi \rangle = \sum_l \alpha_l \lim_j \langle \xi_l, z_j^* z_j \eta_l \rangle = \sum_l \alpha_l \langle z\xi_l, z\eta_l \rangle = 0.$$

Thus for any linear combination $\xi = \sum_k \xi_k$ of elements such that the Φ_{ξ_k}'s have finite H norm, we still have $\pi_K(z)\xi = 0$. By density this holds for all $\xi \in K$. □

As an illustration for the norm estimates let us prove the following result.

Lemma 6.2. *Let $H_B = L_2(M) \otimes_B L_2(M)$, and assume that $L_k^2(M)$ has dimension d_k over B. Let $P_k : L^2(M) \to L_k^2(M)$ be the orthogonal projection. Then*

$$\|P_k\|_{H_B} \leq d_k.$$

Proof. We recall that

$$\langle x \otimes y^{\mathrm{op}}(c \otimes d), a \otimes b\rangle = \tau(b^* E_B(a^* xc)dy) = \tau(E_B(a^* xc)E_B(dyb^*)). \tag{6-1}$$

Assuming that ξ_j is a basis with $E_B(\xi_j^* \xi_i) = \delta_{ij} e_i$, e_i a projection, we see that

$$\tau(yP_k(x)) = \sum_j \tau(y\xi_j E_B(\xi_j^* x)) = \sum_j \langle x \otimes y^{\mathrm{op}}(1 \otimes 1), \xi_j \otimes \xi_j^*\rangle.$$

Since $\langle \xi_j \otimes \xi_j^*, \xi_j \otimes \xi_j^*\rangle = \tau(E(\xi_j^* \xi_j)E(\xi_j^* \xi_j)) \leq \tau(e_j)$, we deduce the assertion. □

6B. ***Configurations.*** Our main goal here is to analyze the operators $\Phi_{\xi,\eta} : M \to L_1(M)$ given by $\Phi_{\xi,\eta}(x) = E_M(\xi x \eta)$, where ξ, η are elements in $\Gamma_q(B, A \otimes (H \oplus H))$. We will start with monomials

$$\xi = s(x_1, h_1) \cdots s(x_m, h_m), \quad \eta = s(x'_{m'}, h'_{m'}) \cdots s(x'_1, h'_1),$$

where $h_i, h'_{i'} \in H \times \{0\} \cup \{0\} \times H$. Although our goal is to obtain estimates for arbitrary x, we will first assume that $x = \zeta$ is a reduced Wick word from M and only contains singletons from $H \times \{0\}$. By considering the moment formula we can reorganize the trace using configurations

$$\tau(\zeta' \xi \zeta \eta) = \sum_{\alpha \text{ configuration}} \tau(\zeta' \Phi_\alpha(\zeta))$$

whenever ζ' is another reduced Wick word. Here a configuration $\alpha = (\sigma_{0\times H}, \sigma_{H\times 0}, I_{\xi,\xi}, I_{\zeta,\eta})$ is given by

(i) a pair partition $\sigma_{0\times H}$ of $\{1, \ldots, m\} \dot\cup \{m', \ldots, 1\}$ so that all the pairs $\{l, r\}$ have indices in $0 \times H$;

(ii) a pair partition $\sigma_{H\times 0}$ of $\{1, \ldots, m\} \dot\cup \{m', \ldots, 1\}$ so that all the pairs $\{l, r\}$ have indices from $H \times 0$;

(iii) subsets $I_{\xi,\xi} \subset \{1, \ldots, m\}$, $I_{\zeta,\eta} \subset \{m', \ldots, 1\}$ disjoint from the support $\bigcup \sigma_{0\times H} \cup \bigcup \sigma_{H\times 0}$ of the partitions above.

Indeed, using the moment formula for $\tau(\zeta' \xi \zeta \eta)$ we know that we have to take the sum over all pair partitions of length $m + m' + k + k'$, $k = |\zeta|$, $k' = |\zeta'|$. Every such pair partition has to respect the pairs of $0 \times H$ and that defines our $\sigma_{0\times H}$. Some pairs can combine elements from ξ and η with coefficients in $H \times 0$. This defines $\sigma_{H\times 0}$. Some partitions connect ξ and ζ and some ζ with η. The left-hand sides of the pairs between ξ and η define the set $I_{\xi,\xi}$ and the right-hand sides of the pairs from ζ, η define $I_{\zeta,\eta}$. All the remaining pairs will connect ζ' and ζ. Since ζ and ζ' are themselves Wick words, there are no pairs connecting elements from ζ (ζ') with itself. We see that indeed, the sum over all partitions can be regrouped into first summing over all configurations (which only depend on ξ and η), and then summing over all partitions supported by these configurations. Let us note that once a configuration α is known we can determine exactly how many crossings will be produced by pairs in $0 \times H$. Indeed, we know that

$|I_{\xi,\zeta}|+|I_{\zeta,\eta}|$ many singletons will be removed from ζ. According to the position of the left legs in $\sigma_{0\times H}$ some extra crossing will be produced from the set $I_{\xi,\zeta}$. The same applies for $I_{\xi,\zeta}$. Here is an example:

$$a_1\ a_2\ b_1\ a_3\ b_3\,|\,c_1\ c_2\ a_3\ d_1\ c_3\ c_4\,|\,d_2\ b_3\ d_1\ a_1\ b_1\ .$$

Here $\sigma_{0\times H}$ are given by the positions of b_1 and b_3. The set $I_{\xi,\zeta}$ is given by the position of a_3 and $\sigma_{H\times 0}$ is given by the positions of a_1. The b's are responsible for $8+1+1+1$ crossings: eight crossings with c's, one crossing among themselves, one crossing coming from a's and b's, one crossing from the b's and d's. Thus $k(\alpha)=2\times(6-2)+1+2$.

In our next step we replace the monomials ξ and η by Wick words. This means we only have to sum over those configurations such that $\sigma_{0\times H}$ and $\sigma_{H\times 0}$ connect ξ and η and no pair ξ and η with itself. In addition the reduction procedure produces scalars and new operator-valued expression $\alpha_{j_1,\dots,j_l}(\beta)$ with $\beta\in D_k(S)$. We have proved the following simple combinatorial fact:

Lemma 6.3. *Let ξ and η be Wick words obtained by reduction and $\zeta\in M$ be a Wick word of length $k=|\zeta|$. For a fixed configuration α there is a number $k(\alpha)$ such that for all $-1\le q\le 1$*

$$\Phi_\alpha(\zeta)=q^{k(\alpha)}\tilde{\xi},$$

where $\tilde{\xi}$ is a linear combination of reduced words with smaller length $k-|I_{\xi,\zeta}|-|I_{\zeta,\eta}|$. Moreover, if $k\ge m+m'$ is the length of ζ, and L is the cardinality of $\sigma_{0\times H}$, then

$$k(\alpha)\ge(k-m-m')L.$$

We will give more precise information about $\tilde{\zeta}$ in the next paragraph.

6C. ***Generalized q-gaussians.*** As a tool we will use a slight generalization of the von Neumann algebra $\Gamma_q(B,A\otimes H)$. This generalization is based on matrix models of the ordinary q-gaussian von Neumann algebras. This approach was invented by Speicher [1992; 1993] and has been applied in many situations; see, e.g., [Biane 1997; Junge 2006; Junge, Palazuelos, Parcet, Perrin and Ricard 2015; Junge and Zeng 2015; Nou 2004; 2006]. Let $\mathrm{Br}: H\to\bigcap_p L^p(\Omega,\Sigma,\mu)$ the standard brownian motion so that $\mathrm{Br}(h)$ is a normal random variable and $(\mathrm{Br}(h),\mathrm{Br}(h'))=(h,h')$. The σ-algebra is chosen minimal. This construction is well known as the gaussian measure space construction. Given a self-adjoint matrix ε_{ij} with values $\{-1,1\}$, there are symmetries $v_j\in M_{2^n}(\mathbb{C})$ such that

$$v_iv_j=\varepsilon_{ij}v_jv_i.$$

Speicher's important idea is to choose the matrix ε_{ij} independently at random for all pairs. We will work with double indices $\varepsilon_{(j,t),(k,s)}$, which are independent as functions of the pairs $\{(j,t),(k,s)\}$ whenever $t\ne s$ or $j\ne k$ and satisfy

$$P(\varepsilon_{(jt),(ks)}=1)=\tfrac{1}{2}(1-Q_{s,t})$$

as long as $(j,t)\ne(ks)$ for a given matrix $Q_{s,t}$. This allows us to construct matrix models

$$u(t,h)=\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}v_{j,t}\otimes g_j(h)\right)_n\in\prod_{n,\omega}(M_{2^n}(\mathbb{C})\otimes L^\infty(\Omega))_n$$

which satisfy

$$\tau(u(t_1,h_1)\cdots u(t_m,h_m)) = \sum_{\sigma\in P_2(m)} \prod_{\substack{\{a,b\}\in\sigma,\ \{c,d\}\in\sigma\\ a<c<b<d}} Q_{t_at_c} \prod_{\{a,b\}\in\sigma} \langle h_a,h_b\rangle.$$

In other words the constant term $q^{\mathrm{inv}(\sigma)}$ is replaced by the product of the crossing inversions weighted according to Q. Indeed, by independence

$$\prod_{\substack{\{a,b\}\in\sigma,\ \{c,d\}\in\sigma\\ a<c<b<d}} Q_{t_at_c} = \mathbb{E}\tau(v_{j_1,t_1}\cdots v_{j_m,t_m})$$

for $(j_1,\dots,j_m)\le\sigma$. In particular for a fixed t and $\|h\|=1$ the random variable $u(t,h)$ is just an ordinary q-gaussian. This central limit theorem is well known and goes back to [Speicher 1992; 1993]; see also [Junge 2006; Junge and Zeng 2015].

We may easily generalize this to the A-valued situation by considering a sequence of symmetric independent copies (π_j, B, A, D) and defining

$$u(t,h,a) = \left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} v_{j,t}\otimes g_j(h)\otimes\pi_j(a)\right)_n \in \prod_{n,\omega}(M_{2^n}(\mathbb{C})\otimes L^\infty(\Omega)\,\bar\otimes\, D)_n.$$

For a subset $1\in S=S^*\subset A$, we denote the von Neumann algebra generated by the elements $u(t,h,a)$, $t\in Q$, $h\in H$, $a\in S$ by $\Gamma_Q^0(B,S\otimes H)$. Then define the von Neumann algebra $\Gamma_Q(B,S\otimes H)$ by the same procedure as in Definition 3.4. A look at the moment formula allows us to state the following fact.

Lemma 6.4. *Let $T_0\subset T$ be a nonempty subset such that $Q_{st}=q$ for all $s,t\in T_0$. Then $\Gamma_q(B,S\otimes H)$ embeds into $\Gamma_Q(B,S\otimes H)$ in a trace-preserving way.*

Remark 6.5. As observed in [Junge and Zeng 2015] the reduction procedure still works in the generalized q-gaussian setting.

Let us return to a configuration α as in Section 6B above. We replace the Wick words $W_q(\vec a,\vec h)$ and $W_q(\vec{a'},\vec{h'})$ by new Wick words $W_Q(\vec a,\vec h)$ $W_Q(\vec{a'},\vec{h'})$ as follows. For a configuration α with a partition $\sigma_{0\times H}$ of the indices labeled with $0\times H$, we define a new matrix

$$Q_{st}(\tilde q) = \begin{cases}\tilde q & \text{if } h_s \text{ and } k_s \text{ are both in } 0\times H,\\ q & \text{else.}\end{cases}$$

Note that the matrix only depends on the first component $\sigma_{0\times H}$ of a configuration. For every pair $p=\{l,r\}\in\sigma_{0\times H}$ we introduce a label e_p and replace h_l and h'_r by $h_l\otimes e_p$ and $h'_r\otimes e_p$ to avoid over-counting. We denote by H_s, H'_t the modified vectors. Starting from

$$\xi_{Q(\tilde q)} = s_{Q(\tilde q)}(H_1,a_1)\cdots s_{Q(\tilde q)}(H_m),\quad \eta_{Q(\tilde q)} = s_{Q(\tilde q)}(H'_{m'},a'_{m'})\cdots s_{Q(\tilde q)}(H'_1,a'_1)$$

we apply the same reduction procedure (eliminating all the pairs from the nonreduced words $X_\sigma(\vec h,\vec a)$) for the W_q's and obtain the reduced Wick words $W_{Q(\tilde q)}(\vec H,\vec a)$, $W_{Q(\tilde q)}(\vec{H'},\vec{a'})$.

Lemma 6.6. *Fix $\sigma_{0\times H}$. The function*

$$F(\tilde q)=\sum_{\alpha,\alpha_1=\sigma_{0\times H}} E_M(W_{Q(\tilde q)}(\vec H,\vec a)\zeta W_{Q(\tilde q)}(\vec H',\vec a'))$$

is a polynomial in $\tilde q$ with lowest degree at least $(|\zeta|-m+m')L$ and largest degree at most $(|\zeta|+m+m')L$.

Proof. Let α be a configuration which contains $\sigma_{0\times H}$. Comparing the terms in the moment formula for

$$\tau(\zeta' W_q(\vec h,\vec a)\zeta W_q(\vec h',\vec a')) \quad\text{and}\quad \tau(\zeta' W_{Q(\tilde q)}(\vec h,\vec a)\zeta W_{Q(\tilde q)}(\vec h',\vec a'))$$

we see that they differ by the factor $(\tilde q/q)^{k(\alpha)}$ number of pairs. Note however, that $k(\alpha)$ only depends on α. This implies the assertion. □

6D. ***Weak containment.*** We need a simple fact about polynomials:

Lemma 6.7. *Let $[a,b]$ be an interval, $\mathcal{P}_d(a,b)$ the set of polynomials of degree d, and $a<t_0<t_1<\cdots<t_d<b$ distinct points. Then the map $\Phi: P_d(a,b)\to\mathbb{C}^{d+1}$, $\phi(f)=f(t_j)$, is injective. Moreover, there exists a matrix $a_{i,j}$ such that for every polynomial*

$$p(t)=\sum_{0\le k\le d}\alpha_k t^k$$

of degree $\le d$ we have

$$\alpha_k=\sum_j a_{k,j}f(t_j).$$

Proof. For $0\le j\le d$ we define the polynomial $p_j(t)=\big(\prod_{i\ne j}(t_j-t_i)\big)^{-1}\prod_{i\ne j}(t-t_i)$, which has degree d. Then we see that $p_j(t_j)=1$ and $p_j(t_i)=0$ for $i\ne j$. In particular, the polynomials $(p_j)_{0\le j\le d}$ are linearly independent and hence $P_d(a,b)=\operatorname{span}\{p_j:0\le j\le d\}$. This implies

$$p(t)=\sum_{0\le j\le d}p(t_j)p_j(t)$$

and in particular Φ is injective. Since moreover, the monomials are linearly independent in $C_\infty(a,b)$, we see that the linear map $\Psi(\alpha_0,\dots,\alpha_d)=\Phi\big(\sum_k\alpha_k t_j^k\big)$ is invertible and can be represented by the matrix $C_{j,k}=t_j^k$, the well-known Vandermonde matrix. Then $A=C^{-1}$ does the job. □

From now on we fix $\sigma=\sigma_{0\times H}$, and Wick words $\xi=W_q(\vec H,\vec a)$, $\eta=W_q(\vec H',\vec a')$ which are obtained after reduction from possible longer terms $s_q(h_1,a_1)\cdots s_q(h_m,a_m)$ and $s_q(h'_{m'},a'_{m'})\cdots s_q(h'_1,a'_1)$. This allows us to define

$$F_\sigma(t)=E_M\big(W_{Q(t)}(\vec H,\vec a)\zeta W_{Q(t)}(\vec H,\vec a)\big).$$

As in Section 6B, we assume that at least L labels of ξ and η are of the form $(0,h_i)$.

Corollary 6.8. *Fix m,m' and L. Then there exists a degree $D=D(m,m',L)$ such that for $q\in[a,b]$ and $a\le t_1<\cdots<t_D\le b<1$ there are coefficients γ_l such that*

$$E_M(\xi\zeta\eta)=\sum_\sigma\sum_l\left(\frac{q}{t_l}\right)^{(k-m-m')L}\gamma_l F_\sigma(t_l)$$

holds for $k = |\zeta| \geq 2(m+m')$. Moreover, for some possibly different coefficients $\tilde{\gamma}_l$

$$E_M(\xi\zeta\eta) = \sum_\sigma \sum_l \tilde{\gamma}_l F_\sigma(t_l)$$

holds for $|\zeta| \leq 2(m+m')$.

Proof. We fix σ and $k \geq m+m'$. Let $[a,b] \subset (-1,1)$ be an interval and $a = q$. The t_i's are all chosen bigger than a. We define the polynomial $p_k(t) = t^{-(k-m-m')L} F(t)$ which has degree at most $(k+m'+m-(k-m-m'))L \leq (2m+2m')L$ and hence

$$p_k(t) = \sum_{0 \leq j \leq (2m+2m')L} a_j t^j \quad \text{and} \quad a_j = \sum_i c_{ij}\, p_k(t_i)$$

hold for mutually different points $a \leq t_1, \ldots, t_d \leq b$, where $d \leq (2m+2m')L+1$ are independent of k. Hence we get

$$\begin{aligned} F_\sigma(q) &= q^{(k-m-m')L} p_k(q) = q^{(k-m-m')L} \sum_{j,i} c_{ij} q^j\, p_k(t_i) \\ &= q^{(k-m-m')L} \sum_{j,i} c_{ij} q^j t_i^{-(k-m-m')L} F_\sigma(t_i) = \sum_i \left(\sum_j c_{ij} q^j \right) \left(\frac{q}{t_i} \right)^{(k-m-m')L} F_\sigma(t_i). \end{aligned}$$

This defines the coefficients γ_i. For $k \leq 2(m+m')$ we work directly with the polynomial $F(t)$ of degree at most $2(m+m')L$. □

Let $M = \Gamma_q(B, S \otimes H)$ and $\widetilde{M} = \Gamma_q(B, S \otimes (H \oplus H))$. Define the M-M bimodule $\mathcal{F}_m \subset L^2(\widetilde{M})$ as the $\|\cdot\|_2$-closed linear span of the reduced Wick words $W_\sigma(x_1, \ldots, x_t, h_1, \ldots, h_N)$, $N \geq 1$, such that $h_i \in H \times \{0\} \cup \{0\} \times H$ for all i and at least m of the vectors h_i belong to $\{0\} \times H$. This bimodule will play a crucial role in our deformation-rigidity arguments in the next section.

Theorem 6.9. *Let $M = \Gamma_q(B, S \otimes H)$ and let $C > 0$, $d > 0$ be two constants such that the dimension of $L^2_k(M)$ over B is smaller than Cd^k for all k. Let $|q| < 1$. Then there exists an $L_0 \in \mathbb{N}$ and a B-M bimodule $\mathcal{K}$ such that $\mathcal{F}_l$ is weakly contained in $L_2(M) \otimes_B \mathcal{K}$ for all $l \geq L_0$.*

Proof. Let us recall that

$$\langle \zeta \otimes (\zeta')^{\mathrm{op}} (a \otimes_B b), \alpha \otimes_B \beta \rangle = \tau(\beta^* E_B(\alpha^* \zeta a) b \zeta') = \tau(E_B(\alpha^* \zeta a) E_B(b \zeta' \beta^*)).$$

Now we may assume that $(\xi_i)_{i \in I_k}$ is a basis of dimension d_k over B so that

$$P_k(\zeta) = \sum_{i \in I_k} \xi_i E_B(\xi_i^* \zeta) \quad \text{and} \quad E_B(\xi_i^* \xi) \leq 1.$$

This implies

$$\begin{aligned} \tau(\zeta' \xi \zeta \eta) &= \sum_{i \in I_k} \tau(\zeta' \xi \xi_i E_B(\xi_i^* \zeta) \eta) = \sum_{i \in I_k} \tau(\xi_i E_B(\xi_i^* \zeta) \eta \zeta') \\ &= \sum_{i \in I_k} \langle \zeta \otimes (\zeta')^{\mathrm{op}} (1 \otimes_B \eta), \xi_i \otimes_B (\xi \xi_i)^* \rangle. \end{aligned}$$

Let $q_0 < 1$ so that $q/q_0 < 1$. Then we define the B-M bimodule

$$\mathcal{K} = \bigoplus_{q/q_0 \le t < 1} L^2(\Gamma_{Q(t)}(B, S \otimes H))$$

with the natural left and right actions. For fixed ξ, η we choose $a = \pm q$ and $|q|/q_0 \le t_0 < \cdots < t_D < b$ for some $b < 1$. This allows us to define $W_{Q(t_i)}(\vec{h}, \vec{a})$ and $W_{Q(t_i)}(\vec{h}', \vec{a}')$ in $\mathcal{K}$. With the help of Corollary 6.8 we deduce that the map $\Phi^+(\zeta) = \sum_{k \ge 2(m+m')} E_M(\xi P_k(\zeta)\eta)$ satisfies

$$\|\Phi^+\|_{L_2(M)\otimes_B \mathcal{K}} \le \sum_\sigma \sum_l |\gamma_l| \sum_{k \ge 2(m+m')} q_0^{(k-m-m')L} \sum_{i \in I_k} \|1 \otimes_B W_{Q(t_l)}(\vec{H}', \vec{a}')\| \, \|\xi_i \otimes_B (W_{Q(t_l)}(\vec{H}, \vec{a})\xi_i)^*\|.$$

Now we note that

$$\|1 \otimes_B W_{Q(t_l)}(\vec{H}', \vec{a}')\| = \|W_{Q(t_l)}(\vec{H}', \vec{a}')\|_{L_2(\Gamma_{Q(t_l)})} \le c(t_l)$$

and

$$\begin{aligned} \|\xi_i \otimes_B (W_{Q(t_l)}(\vec{H}, \vec{a})\xi_i)^*\| &= \tau(W_{Q(t_l)}(\vec{H}, \vec{a})\xi_i E_B(\xi_i^* \xi_i)(W_{Q(t_l)}(\vec{H}, \vec{a})\xi_i)^*) \\ &\le \tau(\xi_i \xi_i^* W_{Q(t_l)}(\vec{H}, \vec{a})^* W_{Q(t_l)}(\vec{H}, \vec{a})) \\ &\le \|W_{Q(t_l)}(\vec{H}, \vec{a})\|^2_{\Gamma_{Q(t_l)}} \le c(t_l). \end{aligned}$$

Thus it suffices to know that $\sum_k q_0^{(k-m-m')L} C d^k$ is finite. Note here that m and m' depend on the Wick word and that we may assume $l \ge L_0$. Thus $q_0^{L_0} d < 1$ and $b < 1$ is enough to achieve summability. Using the second part of Corollary 6.8 we also have summability for $k \le 2(m + m')$. Lemma 6.1 then yields the assertion. □

Corollary 6.10. *Let $M = \Gamma_q(B, S \otimes H)$ and assume that H is finite-dimensional and $\dim_B(D_k(S)) \le C d^k$ for some constants $C, d > 0$. Then there exists an B-M bimodule $\mathcal{K}$ such that for $m \ge 1$ large enough we have $\mathcal{F}_m \prec L^2(M) \otimes_B \mathcal{K}$. In particular, for m large enough, $\mathcal{F}_m$ is weakly contained into $L^2(M) \otimes_B L^2(M)$.*

7. The proof of the main theorem and its applications

We first need some preliminaries. Throughout this section we use the notation $M = M(H) = \Gamma_q(B, S \otimes H)$ and $\widetilde{M} = \Gamma_q(B, S \otimes (H \oplus H)) = M(H \oplus H)$. Let

$$\mathcal{M} = (D \bar{\otimes} \Gamma_q(\ell^2 \otimes H)) \vee M \subset (D \bar{\otimes} \Gamma_q(\ell^2 \otimes H))^\omega.$$

As in Section 5, let

$$\mathcal{H} \subset \left((L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes H)\right)^\omega$$

be the norm-closed linear span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1, \ldots, j_m) = \sigma} (\pi_{j_1}(x_1) \cdots \pi_{j_m}(x_m) y \otimes_{\mathcal{A}} z) \otimes s_{j_1}(h_1) \cdots s_{j_m}(h_m)\right)$$

for $m \geq 1,\ \sigma \in P_{1,2}(m),\ x_i \in BSB,\ h_i \in H,\ y \in M,\ z \in P$. Take the representations

$$\pi : M \to B(\mathcal{H}), \quad \theta : P^{\mathrm{op}} \to B(\mathcal{H})$$

introduced in Section 5 and define $\mathcal{N} = \pi(M) \vee \theta(P^{\mathrm{op}}) \subset B(\mathcal{H})$. As seen in Section 5, we choose a normal faithful state ϕ on $B_P = \pi(B) \vee \theta(P^{\mathrm{op}}) \subset B(\mathcal{H})$ and then define a normal faithful state ψ on $\mathcal{N}$ by $\psi = \phi \circ E_{B_P}$, where $E_{B_P} : \mathcal{N} \to B_P$ is the normal faithful conditional expectation. Let $d_\psi \in L^1(\mathcal{N})$ be the density of ψ and $\xi_0 = d_\psi^{\frac{1}{2}}$. Then $L^2(\mathcal{N})$ is the norm-closed span of the elements $\pi(x_\sigma)\theta(y^{\mathrm{op}})\xi_0$ for $x_\sigma \in M$ a Wick word and $y \in P$. Let $W_k(M)$ be the linear span of the Wick words of degree k in M and $L^2(B_P) = L^2(B_P, \phi)$ be the standard form for $B_P \subset B(\mathcal{H})$. Then $\mathcal{N}$ is standardly represented on

$$L^2(\mathcal{N}) = \overline{\bigoplus_{k \geq 0} W_k(M) L^2(B_P)}$$

by the formulas

$$\pi(x_\sigma)\theta(y^{\mathrm{op}})(\pi(x_\nu)\theta(z^{\mathrm{op}})\xi_0) = \pi(x_\sigma x_\nu)\theta((zy)^{\mathrm{op}})\xi_0, \quad x_\sigma, x_\nu \in M,\ y, z \in P.$$

The conjugation $\mathcal{J} : L^2(\mathcal{N}) \to L^2(\mathcal{N})$ associated to the standard representation of $\mathcal{N}$ is given by

$$\mathcal{J}(\pi(x_\sigma)\theta(y^{\mathrm{op}})\xi_0) = \sigma^\psi_{-\frac{i}{2}}(\pi(x_\sigma^*)\theta(\bar{y}))\xi_0, \quad x_\sigma \in M,\ y \in P,$$

where σ_t^ψ is the modular group on $\mathcal{N}$ associated to ψ. We will also consider $\widetilde{\mathcal{N}} = \mathcal{N}(\widetilde{H})$ constructed in the same way as $\mathcal{N}$ by using $\widetilde{H} = H \oplus H$ instead of H. Thus take

$$\widetilde{\mathcal{H}} \subset \left((L^2(\mathcal{M}) \otimes_{\mathcal{A}} L^2(P)) \otimes \mathcal{F}_q(\ell^2 \otimes \widetilde{H})\right)^\omega$$

to be the norm-closed linear span of the sequences

$$\left(n^{-\frac{m}{2}} \sum_{(j_1,\ldots,j_m)=\sigma} (\pi_{j_1}(x_1)\cdots\pi_{j_m}(x_m)y \otimes_{\mathcal{A}} z) \otimes s_{j_1}(\tilde{h}_1)\cdots s_{j_m}(\tilde{h}_m)\right)$$

for $m \geq 1,\ \sigma \in P_{1,2}(m),\ x_i \in BSB,\ \tilde{h}_i \in \widetilde{H}$. Exactly as in Section 5, define the $*$-representations

$$\pi : \widetilde{M} \to B(\widetilde{\mathcal{H}}), \quad \theta : P^{\mathrm{op}} \to B(\widetilde{\mathcal{H}})$$

and then define $\widetilde{\mathcal{N}} = \pi(\widetilde{M}) \vee \theta(P^{\mathrm{op}})$. Then $\widetilde{\mathcal{N}}$ is standardly represented on

$$L^2(\widetilde{\mathcal{N}}) = \overline{\bigoplus_{k \geq 0} W_k(\widetilde{M}) L^2(B_P)},$$

and the associated conjugation $\widetilde{\mathcal{J}} : L^2(\widetilde{\mathcal{N}}) \to L^2(\widetilde{\mathcal{N}})$ is given by the formula

$$\widetilde{\mathcal{J}}(\pi(x_\sigma)\theta(y^{\mathrm{op}})\xi_0) = \sigma^\psi_{-\frac{i}{2}}(\pi(x_\sigma^*)\theta(\bar{y}))\xi_0, \quad x_\sigma \in \widetilde{M},\ y \in P,$$

where σ_t^ψ is the modular automorphism group on $\widetilde{\mathcal{N}}$ associated to ψ. For every angle t define the unitary V_t on $L^2(\widetilde{\mathcal{N}})$ by

$$\pi(x_\sigma(x_1, \tilde{h}_1, \ldots, x_m, \tilde{h}_m))\theta(y^{\mathrm{op}})\xi_0 \mapsto \pi(x_\sigma(x_1, o_t(\tilde{h}_1), \ldots, x_m, o_t(\tilde{h}_m)))\theta(y^{\mathrm{op}})\xi_0.$$

Then the one-parameter group of $*$-automorphisms $\mathrm{Ad}(V_t)$ of $B(L^2(\widetilde{\mathcal{N}}))$ restricts to a group α_t of $*$-automorphisms of $\widetilde{\mathcal{N}}$, acting according to the formula

$$\alpha_t(\pi(x_\sigma(x_1,\tilde{h}_1,\dots,x_m,\tilde{h}_m))\theta(y^{\mathrm{op}})) = \pi(x_\sigma(x_1,o_t(\tilde{h}_1),\dots,x_m,o_t(\tilde{h}_m)))\theta(y^{\mathrm{op}}).$$

When further restricted to $\widetilde{M} = \Gamma_q(B, S\otimes\widetilde{H})$ this group of $*$-automorphisms coincides with the one introduced in Theorem 3.16 and we have the identity

$$T_t(x) = E_M(\alpha_s(x)), \quad x\in M,\ 0\le s<\tfrac{\pi}{2},$$

where T_t is the heat semigroup introduced in Theorem 3.16 and $t=-\ln(\cos(s))$. We finally introduce the bimodules needed in the deformation argument. To do this, recall that $\widetilde{M}=M(H\oplus H)$ is the generalized q-gaussian algebra generated by B, $s_q(a,h,0)$ and $s_q(a,0,h)$, where $a\in S$ runs through the generating set and $h\in H$ are unit vectors. Let $F\subset H$ be an orthonormal basis. Then we define an M-M bimodule $\mathcal{F}_{=m}\subset L^2(\widetilde{M})$ by

$$\mathcal{F}_{=m} = \overline{\mathrm{span}}^{\|\cdot\|_2}\big\{W_\sigma(k_1,\dots,k_N,a_1,\dots,a_{N'}) : k_i\in F\times\{0\}\cup\{0\}\times F,\ \#\{i\,|k_i\in\{0\}\times F\}=m\big\}.$$

Note that we use reduced Wick words. This means $N=|\sigma_s|$ and the vectors $(k_1,\dots,k_N)$ are the ones obtained after contracting the pairs. Here $\sigma\in P_{1,2}(N')$ and $a_1,\dots,a_{N'}$ are the original coefficients from S. One can see that $\mathcal{F}_{=m}$ is exactly the eigenspace of vectors $\xi\in L^2(\widetilde{M})$ such that $E_{M(0\oplus H)}(\alpha_t(\xi))=e^{-tm}\xi$ for all (some) $t>0$. Likewise we define the M-M bimodule $\mathcal{F}^P_{=m}\subset L^2(\widetilde{\mathcal{N}})$ as the $\|\cdot\|_2$-closed span of the elements

$$\pi(W_\sigma(x_1,h_1,\dots,x_m,h_m))\theta(y^{\mathrm{op}})\xi_0, \quad x_i\in BSB,\ h_i\in F\times\{0\}\cup\{0\}\times F,$$

such that exactly m of the vectors h_i belong to $\{0\}\times F$. It's easy to see that $\mathcal{F}^P_{=m}$ can be described by

$$\mathcal{F}^P_{=m} = \{\xi\in L_2(\widetilde{\mathcal{N}}) : E_{\mathcal{N}(0\oplus H)}(\alpha_t(\xi)) = e^{-tm}\xi \text{ for all } t>0\}.$$

Finally, we set

$$\mathcal{F}_m = \bigoplus_{m'\ge m}\mathcal{F}_{=m'}\subset L^2(\widetilde{M}), \quad \mathcal{F}^P_m = \bigoplus_{m'\ge m}\mathcal{F}^P_{=m'}\subset L^2(\widetilde{\mathcal{N}}).$$

Let's remark that we have the following transversality property, whose proof is virtually the same as that of Proposition 5.1 in [Avsec 2012].

Lemma 7.1. *There exists a constant $C=C(m)>0$ such that for $0<t<2^{-m-1}$ we have*

$$\|V_{t^{m+1}}(\xi)-\xi\|\le C\|e^{\perp}V_t(\xi)\| \quad \textit{for all } \xi\in\bigoplus_{k\ge m+1}L^2_k(\mathcal{N})\subset L^2(\widetilde{\mathcal{N}}).$$

Theorem 7.2. *Let $M=\Gamma_q(B,S\otimes H)$ associated to a sequence of symmetric independent copies (π_j,B,A,D) and assume that the dimension of $D_k(S)$ over B is finite for every k and that H is finite-dimensional. Let $\mathcal{A}\subset M$ be a von Neumann subalgebra which is amenable relative to B and define $P=\mathcal{N}_M(\mathcal{A})''$. Let $m\ge 1$ be fixed. Then at least one of the following statements holds*:

(1) *The M-M bimodule $\mathcal{F}_m$ is left P-amenable.*

(2) *There exist $t,\delta>0$ such that $\inf_{a\in\mathcal{U}(\mathcal{A})}\|T_t(a)\|_2\ge\delta$.*

Proof. The approximately invariant states $\omega_n \in \mathcal{N}_*$ constructed in Theorem 5.1 are implemented by unit vectors $\xi_n \in L^2(\mathcal{N}) \subset L^2(\widetilde{\mathcal{N}})$. Using the Powers–Stormer inequalities we see that the vectors ξ_n have the following properties:

(1) $\langle \pi(x)\xi_n, \xi_n\rangle \to \tau(x),\ x \in M.$

(2) $\|\pi(a)\theta(\bar{a})\xi_n - \xi_n\| \to 0,\ a \in \mathcal{U}(\mathcal{A}).$

(3) $\|\pi(u)\theta(\bar{u})\mathcal{J}\pi(u)\theta(\bar{u})\mathcal{J}\xi_n - \xi_n\| \to 0,\ u \in \mathcal{N}_M(\mathcal{A}).$

Let $e^{\perp} : L^2(\widetilde{\mathcal{N}}) \to \mathcal{F}_m^P$ be the orthogonal projection. We have the following alternative:

Case 1. For every nonzero projection $p \in \mathcal{Z}(P)$ and for every $t > 0$ we have

$$\limsup_n \|e^{\perp} V_t \pi(p)\xi_n\| > \frac{\|p\|_2}{8C}.$$

We will prove that in this case the M-M bimodule $\mathcal{F}_m$ is left P-amenable.

Lemma 7.3. *Let X be the strong operator topology completion of $\mathcal{F}_m$ as a right M-module with respect to the M-valued inner product $\langle x, y\rangle = E_M(x^*y)$, $x, y \in \mathcal{F}_m$. Let $\mathcal{L}(X)$ be the von Neumann algebra of adjointable operators on X. Then there exists a normal $*$-homomorphism $\Psi : \mathcal{L}(X) \to B(L^2(\mathcal{F}_m^P))$ such that $\Psi(\mathcal{L}(X)) \subset B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^{\mathrm{op}})' \cap (\theta(P^{\mathrm{op}}))'$.*

Proof. The condition 5.13(iii) implies that $\mathcal{F}_m^P = X \otimes_M L^2(\mathcal{N})$, where the left action on $\mathcal{N}$ is that of $\pi(M)$. Therefore the map $\Psi : \mathcal{L}(X) \to B(\mathcal{F}_m^P)$ given by

$$T \mapsto T \otimes_M \mathrm{id}$$

is a well-defined normal $*$-homomorphism. Let us consider a rank-1 operator $\xi \otimes \bar{\eta} \in \mathcal{L}(X)$ with ξ, η Wick words in $\widetilde{M}$. Then we calculate

$$\Psi(\xi \otimes \bar{\eta})(\pi(x_\sigma)\theta(y^{\mathrm{op}})\xi_0) = \pi(\xi)\pi(E_M(\eta^* x_\sigma))\theta(y^{\mathrm{op}})\xi_0.$$

Let $e_{\mathcal{N}}$ be the orthogonal projection of $\widetilde{\mathcal{N}}$ onto the closure of $\mathcal{N}\xi_0$, which exists thanks to the fact that $E_{B_P}^{\widetilde{\mathcal{N}}}$ is faithful, see Remark 5.14. Then we note that for $\tilde{x}_{\tilde{\sigma}} \in M$ we have

$$\begin{aligned}\langle \pi(E_M(\eta^* x_\sigma))\theta(y^{\mathrm{op}})\xi_0, \tilde{x}_{\tilde{\sigma}}\tilde{y}^{\mathrm{op}}\xi_0\rangle &= \psi(\theta(\tilde{y}^{\mathrm{op}})^*\theta(y^{\mathrm{op}})E_{B_P}^{\widetilde{\mathcal{N}}}(\tilde{x}_{\tilde{\sigma}}^* x_\sigma)) \\ &= \psi(\theta(\tilde{y}^{\mathrm{op}})^*\theta(y^{\mathrm{op}})(E_{\pi(B)}^{\pi(M)} \circ E_{\pi(M)}^{\widetilde{\mathcal{N}}})(\tilde{x}_{\tilde{\sigma}}^* x_\sigma)) \\ &= \langle \pi(E_M(\eta^* x_\sigma))\theta(y^{\mathrm{op}})\xi_0, \pi(\tilde{x}_{\tilde{\sigma}})\theta(\tilde{y}^{\mathrm{op}})\xi_0\rangle.\end{aligned}$$

This shows that

$$\pi(E_M(\eta^* x_\sigma))\theta(y^{\mathrm{op}})\xi_0 = e_{\mathcal{N}}(\pi(\eta^* x_\sigma)\theta(y^{\mathrm{op}})\xi_0).$$

Thus we deduce that for all the rank-1 operators $\xi \otimes \bar{\eta} \in \mathcal{L}(X)$

$$\Psi(\xi \otimes \bar{\eta}) = L_{\pi(\xi)} e_{\mathcal{N}} L_{\pi(\eta^*)}$$

is a right $\mathcal{N}$-module map, and hence belongs to $B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^{\mathrm{op}})'$. It's also trivial to check that $\Psi(\xi \otimes \bar{\eta})$ commutes with the operators $L_{\theta(y^{\mathrm{op}})}$ for all $y \in P$. Since Ψ is normal and the linear span

of the rank-1 operators is so-dense in $\mathcal{L}(X)$, we have $\Psi(\mathcal{L}(X)) \subset B(L^2(\mathcal{F}_m^P)) \cap (\mathcal{N}^{\mathrm{op}})' \cap (\theta(P^{\mathrm{op}}))'$, as desired. □

The lemma provides a normal $*$-homomorphism

$$\Psi : B(\mathcal{F}_m) \cap (M^{\mathrm{op}})' \to B(\mathcal{F}_m^P) \cap (\theta(P^{\mathrm{op}}) \vee \widetilde{\mathcal{J}}\pi(M)\widetilde{\mathcal{J}} \vee \widetilde{\mathcal{J}}\theta(P^{\mathrm{op}})\widetilde{\mathcal{J}})'$$

such that $\Psi(\lambda(x)) = \pi(x)$ for $x \in M$, where λ is the natural left action of M on $L^2(\widetilde{M})$. From this point on, the proof proceeds verbatim as in [Popa and Vaes 2014a, proof of Case 1 in Theorem 3.1].

Case 2. There exist a nonzero central projection $p \in \mathcal{Z}(P)$ and $t > 0$ such that

$$\limsup_n \|e^{\perp} V_t \pi(p)\xi_n\| \leq \frac{\|p\|_2}{8C}.$$

In this case we prove that there exist $s, \delta > 0$ such $\|T_s(a)\|_2 \geq \delta$ for all $a \in \mathcal{U}(\mathcal{A})$. Write $\pi(p)\xi_n = \zeta_n + \eta_n$, where $\zeta_n \in \bigoplus_{k \leq m} L_k^2(\mathcal{N})$, $\eta_n \in \bigoplus_{k \geq m+1} L_k^2(\mathcal{N})$. Note that $\|\zeta_n\| \leq 1$, $\|\eta_n\| \leq 1$. Since V_t converges uniformly on $\left(\bigoplus_{k \leq m} L_k^2(\mathcal{N})\right)_1$, there exists a $t_0 > 0$ such that for $0 < s < t_0$ we have

$$\|V_s\xi - \xi\| \leq \min\left\{\frac{\|p\|_2}{8}, \frac{\|p\|_2}{8C}\right\} \quad \text{for } \xi \in \left(\bigoplus_{k \leq m} L_k^2(\mathcal{N})\right)_1.$$

Fix $0 < s < \min\{t^{m+1}, t_0, t_0^{m+1}, 2^{-(m+1)^2}\}$. For every $n \geq 1$ we have the estimate

$$\begin{aligned}
\|V_s\pi(p)\xi_n - \pi(p)\xi_n\| &\leq \|V_s\zeta_n - \zeta_n\| + \|V_s\eta_n - \eta_n\| \\
&\leq \frac{\|p\|_2}{8} + \|V_s\eta_n - \eta_n\| \leq \frac{\|p\|_2}{8} + C\|e^{\perp}V_{m+\frac{1}{\sqrt{s}}}\eta_n\| \\
&\leq \frac{\|p\|_2}{8} + C\|e^{\perp}V_{m+\frac{1}{\sqrt{s}}}\pi(p)\xi_n\| + C\|e^{\perp}V_{m+\frac{1}{\sqrt{s}}}\zeta_n\| \\
&\leq \frac{\|p\|_2}{8} + C\|e^{\perp}V_{m+\frac{1}{\sqrt{s}}}\pi(p)\xi_n\| + C\|e^{\perp}(V_{m+\frac{1}{\sqrt{s}}}\zeta_n - \zeta_n)\| + C\|e^{\perp}\zeta_n\| \\
&\leq \frac{\|p\|_2}{8} + C\|e^{\perp}V_{m+\frac{1}{\sqrt{s}}}\pi(p)\xi_n\| + C\|V_{m+\frac{1}{\sqrt{s}}}\zeta_n - \zeta_n\| \\
&\leq \frac{\|p\|_2}{4} + C\|e^{\perp}V_t\pi(p)\xi_n\|.
\end{aligned}$$

Taking the limsup with respect to n we obtain

$$\limsup_n \|V_s\pi(p)\xi_n - \pi(p)\xi_n\| \leq \frac{3\|p\|_2}{8}.$$

From this point on, the proof proceeds verbatim as in [Popa and Vaes 2014a, proof of Case 2 in Theorem 3.1]. □

Proof of Theorem A. For the first alternative, we use the second item in Theorem 7.2, Proposition 3.20 and Proposition 2.3. For the second alternative, we use the first item in Theorem 7.2, Corollary 6.10 and Remark 2.7. □

Proof of Corollaries B, C, D. These follow immediately from Theorem A. □

References

[Avsec 2012] S. T. Avsec, *Gaussian-like von Neumann algebras and noncommutative Brownian motion*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2012, available at https://search.proquest.com/docview/1448260344.

[Biane 1997] P. Biane, "Free hypercontractivity", *Comm. Math. Phys.* **184**:2 (1997), 457–474. MR Zbl

[Boutonnet, Houdayer and Vaes 2018] R. Boutonnet, C. Houdayer, and S. Vaes, "Strong solidity of free Araki–Woods factors", *Amer. J. Math.* **140**:5 (2018), 1231–1252. MR Zbl

[Bożejko 1991] M. Bożejko, "A q-deformed probability, Nelson's inequality and central limit theorems", pp. 312–335 in *Nonlinear fields: classical, random, semiclassical* (Karpacz, Poland, 1991), edited by P. Garbaczewski and Z. Popowicz, World Sci., River Edge, NJ, 1991. MR

[Bożejko and Speicher 1991] M. Bożejko and R. Speicher, "An example of a generalized Brownian motion", *Comm. Math. Phys.* **137**:3 (1991), 519–531. MR Zbl

[Bożejko and Speicher 1992] M. Bożejko and R. Speicher, "An example of a generalized Brownian motion, II", pp. 67–77 in *Quantum probability and related topics*, edited by L. Accardi, QP-PQ **7**, World Sci., River Edge, NJ, 1992. MR Zbl

[Bożejko and Speicher 1994] M. Bożejko and R. Speicher, "Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces", *Math. Ann.* **300**:1 (1994), 97–120. MR Zbl

[Bożejko and Speicher 1996] M. Bożejko and R. Speicher, "Interpolations between bosonic and fermionic relations given by generalized Brownian motions", *Math. Z.* **222**:1 (1996), 135–159. MR Zbl

[Bożejko, Kümmerer and Speicher 1997] M. Bożejko, B. Kümmerer, and R. Speicher, "q-Gaussian processes: non-commutative and classical aspects", *Comm. Math. Phys.* **185**:1 (1997), 129–154. MR Zbl

[Caspers 2018] M. Caspers, "Gradient forms and strong solidity of free quantum groups", preprint, 2018. arXiv

[Chifan and Sinclair 2013] I. Chifan and T. Sinclair, "On the structural theory of II_1 factors of negatively curved groups", *Ann. Sci. Éc. Norm. Sup.* (4) **46**:1 (2013), 1–33. MR Zbl

[Chifan, Sinclair and Udrea 2013] I. Chifan, T. Sinclair, and B. Udrea, "On the structural theory of II_1 factors of negatively curved groups, II: Actions by product groups", *Adv. Math.* **245** (2013), 208–236. MR Zbl

[Connes and Jones 1982] A. Connes and V. Jones, "A II_1 factor with two nonconjugate Cartan subalgebras", *Bull. Amer. Math. Soc.* (*N.S.*) **6**:2 (1982), 211–212. MR Zbl

[Connes, Feldman and Weiss 1981] A. Connes, J. Feldman, and B. Weiss, "An amenable equivalence relation is generated by a single transformation", *Ergodic Theory Dynam. Systems* **1**:4 (1981), 431–450. MR Zbl

[Dabrowski 2014] Y. Dabrowski, "A free stochastic partial differential equation", *Ann. Inst. Henri Poincaré Probab. Stat.* **50**:4 (2014), 1404–1455. MR Zbl

[Dykema and Nica 1993] K. Dykema and A. Nica, "On the Fock representation of the q-commutation relations", *J. Reine Angew. Math.* **440** (1993), 201–212. MR Zbl

[Haagerup 1975] U. Haagerup, "The standard form of von Neumann algebras", *Math. Scand.* **37**:2 (1975), 271–283. MR Zbl

[Haagerup 1979] U. Haagerup, "L^p-spaces associated with an arbitrary von Neumann algebra", pp. 175–184 in *Algèbres d'opérateurs et leurs applications en physique mathématique* (Marseille, 1977), edited by M. A. Connes et al., Colloq. Internat. CNRS **274**, CNRS, Paris, 1979. MR Zbl

[Haagerup, Junge and Xu 2010] U. Haagerup, M. Junge, and Q. Xu, "A reduction method for noncommutative L_p-spaces and applications", *Trans. Amer. Math. Soc.* **362**:4 (2010), 2125–2165. MR Zbl

[Houdayer and Vaes 2013] C. Houdayer and S. Vaes, "Type III factors with unique Cartan decomposition", *J. Math. Pures Appl.* (9) **100**:4 (2013), 564–590. MR Zbl

[Ioana 2015] A. Ioana, "Cartan subalgebras of amalgamated free product II_1 factors", *Ann. Sci. Éc. Norm. Sup.* (4) **48**:1 (2015), 71–130. MR Zbl

[Isono 2015a] Y. Isono, "Examples of factors which have no Cartan subalgebras", *Trans. Amer. Math. Soc.* **367**:11 (2015), 7917–7937. MR Zbl

[Isono 2015b] Y. Isono, "On bi-exactness of discrete quantum groups", *Int. Math. Res. Not.* **2015**:11 (2015), 3619–3650. MR Zbl

[Jones and Popa 1982] V. Jones and S. Popa, "Some properties of MASA's [sic] in factors", pp. 89–102 in *Invariant subspaces and other topics* (Timişoara/Herculane, Romania, 1981), edited by C. Apostol et al., Operator Theory: Adv. Appl. **6**, Birkhäuser, Basel, 1982. MR Zbl

[Junge 2006] M. Junge, "Operator spaces and Araki–Woods factors: a quantum probabilistic approach", *Int. Math. Res. Pap.* **2006** (2006), art. id. 76978. MR Zbl

[Junge and Sherman 2005] M. Junge and D. Sherman, "Noncommutative L^p modules", *J. Operator Theory* **53**:1 (2005), 3–34. MR Zbl

[Junge and Zeng 2015] M. Junge and Q. Zeng, "Ultraproduct methods for mixed q-Gaussian algebras", preprint, 2015. arXiv

[Junge, Longfield and Udrea 2014] M. Junge, S. Longfield, and B. Udrea, "Some classification results for generalized q-Gaussian algebras", 2014. To appear in *Doc. Math.* arXiv

[Junge, Palazuelos, Parcet, Perrin and Ricard 2015] M. Junge, C. Palazuelos, J. Parcet, M. Perrin, and É. Ricard, "Hypercontractivity for free products", *Ann. Sci. Éc. Norm. Sup.* (4) **48**:4 (2015), 861–889. MR Zbl

[Królak 2000] I. Królak, "Wick product for commutation relations connected with Yang–Baxter operators and new constructions of factors", *Comm. Math. Phys.* **210**:3 (2000), 685–701. MR Zbl

[Królak 2006] I. Królak, "Factoriality of von Neumann algebras connected with general commutation relations: finite dimensional case", pp. 277–284 in *Quantum probability* (Będlewo, Poland, 2004), edited by M. Bożejko et al., Banach Center Publ. **73**, Polish Acad. Sci. Inst. Math., Warsaw, 2006. MR Zbl

[Nou 2004] A. Nou, "Non injectivity of the q-deformed von Neumann algebra", *Math. Ann.* **330**:1 (2004), 17–38. MR Zbl

[Nou 2006] A. Nou, "Asymptotic matricial models and QWEP property for q-Araki–Woods algebras", *J. Funct. Anal.* **232**:2 (2006), 295–327. MR Zbl

[Ozawa 2004] N. Ozawa, "Solid von Neumann algebras", *Acta Math.* **192**:1 (2004), 111–117. MR Zbl

[Ozawa and Popa 2010a] N. Ozawa and S. Popa, "On a class of II_1 factors with at most one Cartan subalgebra", *Ann. of Math.* (2) **172**:1 (2010), 713–749. MR Zbl

[Ozawa and Popa 2010b] N. Ozawa and S. Popa, "On a class of II_1 factors with at most one Cartan subalgebra, II", *Amer. J. Math.* **132**:3 (2010), 841–866. MR Zbl

[Paschke 1973] W. L. Paschke, "Inner product modules over B^*-algebras", *Trans. Amer. Math. Soc.* **182** (1973), 443–468. MR Zbl

[Paschke 1974] W. L. Paschke, "The double B-dual of an inner product module over a C^*-algebra B", *Canad. J. Math.* **26**:5 (1974), 1272–1280. MR Zbl

[Peterson and Sinclair 2012] J. Peterson and T. Sinclair, "On cocycle superrigidity for Gaussian actions", *Ergodic Theory Dynam. Systems* **32**:1 (2012), 249–272. MR Zbl

[Pisier 2000] G. Pisier, "An inequality for p-orthogonal sums in non-commutative L_p", *Illinois J. Math.* **44**:4 (2000), 901–923. MR Zbl

[Pisier 2003] G. Pisier, *Introduction to operator space theory*, Lond. Math. Soc. Lect. Note Series **294**, Cambridge Univ. Press, 2003. MR Zbl

[Popa 2006a] S. Popa, "On a class of type II_1 factors with Betti numbers invariants", *Ann. of Math.* (2) **163**:3 (2006), 809–899. MR Zbl

[Popa 2006b] S. Popa, "Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I", *Invent. Math.* **165**:2 (2006), 369–408. MR Zbl

[Popa and Vaes 2014a] S. Popa and S. Vaes, "Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups", *Acta Math.* **212**:1 (2014), 141–198. MR Zbl

[Popa and Vaes 2014b] S. Popa and S. Vaes, "Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups", *J. Reine Angew. Math.* **694** (2014), 215–239. MR Zbl

[Raynaud 2002] Y. Raynaud, "On ultrapowers of non commutative L_p spaces", *J. Operator Theory* **48**:1 (2002), 41–68. MR Zbl

[Ricard 2005] É. Ricard, "Factoriality of q-Gaussian von Neumann algebras", *Comm. Math. Phys.* **257**:3 (2005), 659–665. MR Zbl

[Shlyakhtenko 1999] D. Shlyakhtenko, "A-valued semicircular systems", *J. Funct. Anal.* **166**:1 (1999), 1–47. MR Zbl

[Shlyakhtenko 2004] D. Shlyakhtenko, "Some estimates for non-microstates free entropy dimension with applications to q-semicircular families", *Int. Math. Res. Not.* **2004**:51 (2004), 2757–2772. MR Zbl

[Shlyakhtenko 2009] D. Shlyakhtenko, "Lower estimates on microstates free entropy dimension", *Anal. PDE* **2**:2 (2009), 119–146. MR Zbl

[Sinclair 2011] T. Sinclair, "Strong solidity of group factors from lattices in SO$(n,1)$ and SU$(n,1)$", *J. Funct. Anal.* **260**:11 (2011), 3209–3221. MR Zbl

[Śniady 2001] P. Śniady, "Gaussian random matrix models for q-deformed Gaussian variables", *Comm. Math. Phys.* **216**:3 (2001), 515–537. MR Zbl

[Śniady 2004] P. Śniady, "Factoriality of Bożejko–Speicher von Neumann algebras", *Comm. Math. Phys.* **246**:3 (2004), 561–567. MR Zbl

[Speicher 1992] R. Speicher, "A noncommutative central limit theorem", *Math. Z.* **209**:1 (1992), 55–66. MR Zbl

[Speicher 1993] R. Speicher, "Generalized statistics of macroscopic fields", *Lett. Math. Phys.* **27**:2 (1993), 97–104. MR Zbl

[Terp 1982] M. Terp, "Interpolation spaces between a von Neumann algebra and its predual", *J. Operator Theory* **8**:2 (1982), 327–360. MR Zbl

[Voiculescu 1985] D. Voiculescu, "Symmetries of some reduced free product C^*-algebras", pp. 556–588 in *Operator algebras and their connections with topology and ergodic theory* (Buşteni, Romania, 1983), edited by H. Araki et al., Lecture Notes in Math. **1132**, Springer, 1985. MR Zbl

[Voiculescu 1991] D. Voiculescu, "Limit laws for random matrices and free products", *Invent. Math.* **104**:1 (1991), 201–220. MR Zbl

[Voiculescu 1996] D. Voiculescu, "The analogues of entropy and of Fisher's information measure in free probability theory, III: The absence of Cartan subalgebras", *Geom. Funct. Anal.* **6**:1 (1996), 172–199. MR Zbl

[Voiculescu, Dykema and Nica 1992] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series **1**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl

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