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GENERAL EXPONENTIAL DICHOTOMIES: FROM FINITE TO INFINITE TIME

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ABSTRACT. We consider exponential dichotomies on finite intervals and show that if the constants in the notion of an exponential dichotomy are chosen appropriately and uniformly on those intervals, then there exists an exponential dichotomy on the whole line. We consider the general case of a nonautonomous dynamics that need not be invertible. Moreover, we consider both cases of discrete and continuous time.

1. Introduction

The notion of an *exponential dichotomy* or of an *exponential splitting*, together with its various extensions and generalizations, plays a central role in a large part of the stability and hyperbolicity theories of differential equations and dynamical systems. Essentially going back to Perron in [19], it is central in the study of stable and unstable invariant manifolds, of topological and smooth conjugacies to the linear part of the dynamics, of normal forms under appropriate resonance conditions, and of closing and shadowing properties, among many other developments. We refer the reader to the books [6, 7, 10, 11, 14, 25] for details and further references. A far reaching generalization of the hyperbolicity theory is the nonuniform hyperbolicity theory, which goes back to landmark works of Oseledets [16] and particularly Pesin [20, 21, 22]. For example, almost all trajectories of a dynamical system preserving a finite invariant measure with nonzero Lyapunov exponents are nonuniformly hyperbolic. The theory is an important

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part of the general theory of dynamical systems and a principal tool in the study of stochastic behavior, particularly in the context of smooth ergodic theory.

An important property of the notion of an exponential dichotomy is that it persists under sufficiently small linear perturbations, which naturally contributes to the abundance of the hyperbolic behavior. This is usually called the robustness or roughness property. The study of robustness has a long history. In particular, it was discussed by Massera and Schäffer [13] (building on earlier work of Perron [19]; see also [14]), Coppel [5] and Dalec'kiĭ and Kreĭn [7]. For more recent works we refer the reader to [4, 15, 23, 24] and the references therein. In a different direction, and as an example, we note that for a geodesic flow on a compact smooth Riemannian manifolds with strictly negative sectional curvature, the unit tangent bundle is a hyperbolic set (this essentially corresponds to the existence of an exponential splitting at all points). Furthermore, time changes and small C^1 perturbations of flows with a hyperbolic set also have a hyperbolic set. This last result is a natural version of the robustness property of an exponential dichotomy.

Here, in strong contrast with the usual stable and unstable exponential rates that are present in the notion of an exponential dichotomy, we allow asymptotic rates of the form $e^{c\rho(t)}$ determined by an arbitrary function ρ. The usual exponential behavior corresponds to take $\rho(t) = t$. This includes situations in which the Lyapunov exponents are all zero or are all infinite and to which one cannot apply, certainly without modifications, the existing hyperbolicity and stability theories. Incidentally, in [1] we showed that for a large class of rate functions ρ there exist many linear equations $x' = A(t)x$ exhibiting an asymptotic behavior that can be expressed in terms of the more general exponentials $e^{c\rho(t)}$. One can also consider the case of discrete time and the same observations apply without change.

In this work we consider exponential dichotomies on finite intervals. Certainly, without other requirements, the notion is trivially satisfied in this case. However, as first observed by Palmer in [18] (building on his related approach in [17]), if the constants in the notion of an exponential dichotomy are chosen appropriately and uniformly on intervals of sufficiently large length, then the existence of exponential dichotomies on these intervals allows one to deduce that there exists an exponential dichotomy on the whole line. More precisely, [18] considers the case of a nonautonomous dynamics defined by a sequence of invertible matrices. In [8], Ducrot, Magal, and Seydi extended Palmer's work to exponential trichotomies (which corresponds to consider stable, unstable, and central directions), again for discrete time, but without assuming that the dynamics is invertible, which causes several complications in the proof.

Our main aim is to obtain a version of Palmer's result in [18] for exponential dichotomies with arbitrary growth rates. As in [8], we also consider the general case of a nonautonomous dynamics that need not be invertible. In addition we consider both cases of discrete and continuous time. In order to formulate our main result, we first introduce the notion of a ρ-exponential dichotomy. Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators acting on a Banach space X.

For each $m, n \in \mathbb{Z}$ with $m \geq n$, let

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases}$$

Given an increasing function $\rho\colon \mathbb{Z} \to \mathbb{Z}$ with $\rho(0) = 0$ and an interval $I \subset \mathbb{Z}$ (that is, the intersection of $\mathbb{Z}$ with some interval in $\mathbb{R}$), we say that the sequence $(A_m)_{m\in\mathbb{Z}}$ has a *ρ-exponential dichotomy on I* if:

(1) there exist projections P_m for $m \in I$ satisfying

$$P_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n, \quad \text{for } m \geq n,$$

such that, writing $Q_n = \mathrm{Id} - P_n$, the map

$$\mathcal{A}(m,n) | \operatorname{im} Q_n \colon \operatorname{im} Q_n \to \operatorname{im} Q_m \tag{1.1}$$

is onto and injective;

(2) there exist $c, \kappa > 0$ such that, for all $m, n \in I$ with $m \geq n$, we have

$$\|\mathcal{A}(m,n) P_n\| \leq \kappa e^{-c(\rho(m)-\rho(n))}$$

and

$$\|\mathcal{A}(n,m) Q_m\| \leq \kappa e^{-c(\rho(m)-\rho(n))},$$

where $\mathcal{A}(n,m)$ denotes the inverse of the map in (1.1).

The vector spaces $\operatorname{im} P_n$ and $\operatorname{im} Q_n$ are then called, respectively, *stable* and *unstable* spaces at time n. For example, when $\rho(n) = n$ we recover the classical notion of an exponential dichotomy. By considering other functions ρ one can consider other types of asymptotic behavior.

The following theorem is our main result. Given $\tau \in \mathbb{N}$, we say that an increasing sequence $(\alpha_i)_{i\in\mathbb{Z}}$ in $\mathbb{Z}$ is *τ-dense* if $\alpha_{i+1} \leq \alpha_i + \tau$ for every $i \in \mathbb{Z}$ (in other words, each interval of length τ contains at least one element of the sequence).

Theorem 1.1. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators such that*

$$K := \sup_{m\in\mathbb{Z}} \left(\|A_m\| e^{c(\rho(m+1)-\rho(m))} \right) < +\infty, \tag{1.2}$$

for some $c > 0$, and let $(\alpha_i)_{i\in\mathbb{Z}}$ be a τ-dense sequence. Given $c > \overline{c} > 0$, there exist $\ell = \ell(\tau, c, \overline{c}, \kappa, K)$ and $\overline{\kappa} = \overline{\kappa}(\kappa, K)$ such that if $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy on $[\alpha_i, \alpha_i + \ell]$ for each $i \in \mathbb{Z}$ with constants c and κ, then it has a ρ-exponential dichotomy on $\mathbb{Z}$ with constants $\overline{c}$ and $\overline{\kappa}$.

See Section 2 for the proof of Theorem 1.1. The method of proof follows the strategy devised by Palmer in [17, 18] and also used in [8]. Namely, there are two main elements:

(1) In a first step we show that if there are exponential dichotomies in sufficiently large intervals intersecting only at their endpoints, and the projections of two intervals at the intersection endpoint are sufficiently close, then there is an exponential dichotomy on the whole line. An important element of this part of the proof is the use of the robustness property of

an exponential dichotomy to deduce that the original dynamics has an exponential dichotomy by showing that this happens for a sufficiently close related dynamics.

(2) In a second step we show that if the original finite intervals in which we assume that there are exponential dichotomies have a sufficiently large intersection, then the projections associated with any two intersecting intervals are uniformly close. This is assured by assuming that the original intervals are sufficiently close, while the closeness of the projections allows one to apply the first step to the original intervals (and not only to those that intersected at the endpoints; recall that in the first step the projections are assumed to be sufficiently close).

The corresponding result for continuous time (see Theorem 3.1) is obtained by reducing the statement to the case of discrete time and then applying a result of Henry in [12] that shows the existence of an exponential dichotomy on the whole line for continuous time if it exists for the discretization of the dynamics (under some appropriate growth assumptions).

2. Proof of Theorem 1.1

We first consider the case when there exists $q \in \mathbb{N}$ such that the sequence $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy on $[(i-1)q, iq]$ for each $i \in \mathbb{Z}$, with constants c and κ (independent of i). Later on the general case will be reduced to this one. Thus, we assume that

(1) there exist projections P_n^i for $i \in \mathbb{Z}$ and $n \in [(i-1)q, iq]$ and satisfying

$$P_m^i \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n^i \quad \text{for } m \geq n$$

such that, writing $Q_n^i = \mathrm{Id} - P_n^i$, the map

$$\mathcal{A}(m,n)|\operatorname{im} Q_n^i \colon \operatorname{im} Q_n^i \to \operatorname{im} Q_m^i \tag{2.1}$$

is onto and injective;

(2) there exist $c, \kappa > 0$ such that for all $i \in \mathbb{Z}$ and $m, n \in [(i-1)q, iq]$ with $m \geq n$ we have

$$\|\mathcal{A}(m,n) P_n^i\| \leq \kappa e^{-c(\rho(m)-\rho(n))} \tag{2.2}$$

and

$$\|\mathcal{A}(n,m) Q_m^i\| \leq \kappa e^{-c(\rho(m)-\rho(n))}, \tag{2.3}$$

where $\mathcal{A}(n,m)$ denotes the inverse of the map in (2.1).

Note that

$$\|P_n^i\| \leq \kappa \quad \text{and} \quad \|Q_n^i\| \leq \kappa \tag{2.4}$$

for all $i \in \mathbb{Z}$ and $n \in [(i-1)q, iq]$.

We also recall a result from [9] (see their Lemma 3.1 and Claim 3.3).

Lemma 2.1. *Let $F, \overline{F} \colon X \to X$ be bounded linear operators such that*

$$\|F - \overline{F}\| \leq \delta < \sqrt{2} - 1.$$

Then F is invertible from $\operatorname{im}\overline{F}$ *onto* $\operatorname{im} F$ *and*

$$\|(F|\operatorname{im}\overline{F})^{-1}x\| \le \frac{1}{1-\delta}\|x\| \quad \text{for } x \in \operatorname{im}\overline{F}.$$

Now we establish some auxiliary results. We first show that if the projections P^i_{iq} and P^{i+1}_{iq} are sufficiently close uniformly on i, then the sequence $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy.

Lemma 2.2. *If there exists $\delta \in (0, \sqrt{2}-1)$ such that*

$$\|P^i_{iq} - P^{i+1}_{iq}\| \le \delta, \tag{2.5}$$

for all $i \in \mathbb{Z}$, then $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy on $\mathbb{Z}$ with constants $\overline{c}$ and $\overline{\kappa}$.

Proof. We define projections $\overline{P}_n$ and $\overline{Q}_n = \mathrm{Id} - \overline{P}_n$ for each $n \in \mathbb{Z}$ by

$$\overline{P}_n = P^i_n, \quad \overline{Q}_n = Q^i_n, \quad \text{for } i \in \mathbb{Z},\ n \in [(i-1)q, iq). \tag{2.6}$$

Note that

$$\overline{P}_{iq} = P^{i+1}_{iq}, \quad \overline{Q}_{iq} = Q^{i+1}_{iq} \quad \text{for } i \in \mathbb{Z}. \tag{2.7}$$

We also define linear operators $\overline{A}_n$ for each $n \in \mathbb{Z}$

$$\overline{A}_n = \begin{cases} A_n & \text{if } n \in [(i-1)q, iq-1), \\ P^{i+1}_{iq}A_{iq-1}P^i_{iq-1} + Q^{i+1}_{iq}A_{iq-1}Q^i_{iq-1} & \text{if } n = iq-1. \end{cases} \tag{2.8}$$

We first show that $(\overline{A}_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy with projections $\overline{P}_m$. Then we show that the two sequences $(A_m)_{m\in\mathbb{Z}}$ and $(\overline{A}_m)_{m\in\mathbb{Z}}$ are sufficiently close on m, which by the robustness property of the notion of a ρ-exponential dichotomy implies that the sequence $(A_m)_{m\in\mathbb{Z}}$ also has a ρ-exponential dichotomy.

Step 1. Invariance and invertibility along subspaces. We need to show that

$$\overline{P}_{n+1}\overline{A}_n = \overline{A}_n\overline{P}_n \tag{2.9}$$

for all $n \in \mathbb{Z}$. It follows readily from (2.6) and (2.8) that identity (2.9) holds for all $n \in \mathbb{Z}$ such that $n \in [(i-1)q, iq-1)$ for some $i \in \mathbb{Z}$. Now take $n = iq-1$. By (2.7) and (2.8) we obtain

$$\begin{aligned} \overline{P}_{n+1}\overline{A}_n = \overline{P}_{iq}\overline{A}_{iq-1} &= P^{i+1}_{iq}(P^{i+1}_{iq}A_{iq-1}P^i_{iq-1} + Q^{i+1}_{iq}A_{iq-1}Q^i_{iq-1}) \\ &= P^{i+1}_{iq}A_{iq-1}P^i_{iq-1} \\ &= (P^{i+1}_{iq}A_{iq-1}P^i_{iq-1} + Q^{i+1}_{iq}A_{iq-1}Q^i_{iq-1})P^i_{iq-1} \\ &= \overline{A}_{iq-1}P^i_{iq-1} \end{aligned}$$

and so identity (2.9) also holds in this case.

Now we show that for each $n \in \mathbb{Z}$ the linear operator

$$\overline{A}_n\overline{Q}_n \colon \operatorname{im}\overline{Q}_n \to \operatorname{im}\overline{Q}_{n+1} \tag{2.10}$$

is onto and injective. Since

$$\overline{A}_n\overline{Q}_n = A_nQ^i_n \quad \text{and} \quad \overline{Q}_{n+1} = Q^i_{n+1},$$

for all $n \in [(i-1)q, iq-1)$, it follows that the operator in (2.10) is onto and injective for these values of n. Now take $n = iq-1$. By (2.8) we have

$$\overline{A}_{iq-1}\overline{Q}_{iq-1} = Q_{iq}^{i+1}A_{iq-1}Q_{iq-1}^{i} = Q_{iq}^{i+1}\overline{A}_{iq-1}.$$

Since $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy, the linear operator A_{iq-1} is invertible from $\operatorname{im}\overline{Q}_{iq-1} = \operatorname{im} Q_{iq-1}^{i}$ onto $\operatorname{im} Q_{iq}^{i}$ and since

$$\|Q_{iq}^{i} - Q_{iq}^{i+1}\| \le \delta$$

(see (2.5)), it follows from Lemma 2.1 that the map

$$Q_{iq}^{i+1}\colon\ \operatorname{im} Q_{iq}^{i} \to \operatorname{im} Q_{iq}^{i+1} \tag{2.11}$$

is onto and injective. This implies that the linear operator

$$\overline{A}_{iq-1}\overline{Q}_{iq-1}\colon\ \operatorname{im}\overline{Q}_{iq-1} = \operatorname{im} Q_{iq-1} \to \operatorname{im}\overline{Q}_{iq-1}^{i+1} = \operatorname{im} Q_{iq-1}^{i+1}$$

is onto and injective. Moreover,

$$\big(\overline{A}_{iq-1}\overline{Q}_{iq-1}\big)^{-1} = \big(Q_{iq}^{i+1}A_{iq-1}Q_{iq-1}^{i}\big)^{-1} = \big(A_{iq-1}Q_{iq-1}^{i}\big)^{-1}\big(Q_{iq}^{i+1}\big)^{-1},$$

for $i \in \mathbb{Z}$, where $\big(Q_{iq}^{i+1}\big)^{-1}$ denotes the inverse of the map in (2.11). This shows that the linear operator in (2.10) is onto and injective for all $n \in \mathbb{Z}$. Finally, it follows from (2.3) and (2.5) together with Lemma 2.1 that

$$\|\big(\overline{A}_{iq-1}\overline{Q}_{iq-1}\big)^{-1}x\| \le \frac{\kappa e^{c(\rho(iq-1)-\rho(iq))}}{1-\delta}\|x\| \tag{2.12}$$

for $x \in \operatorname{im}\overline{Q}_{iq-1}^{i+1}$. This inequality will be used later one in the proof of the lemma.

Step 2. Existence of contraction and expansion. Finally, we show that the sequence $(\overline{A}_m)_{m\in\mathbb{Z}}$ exhibits contraction and expansion.

Before proceeding, we show that, for each $i \in \mathbb{Z}$, the linear operators A_{iq-1} and $\overline{A}_{iq-1}$ are sufficiently close. Note first that

$$\begin{aligned}
&\|A_{iq-1} - \overline{A}_{iq-1}\| \\
&= \big\|P_{iq}^{i}A_{iq-1} + Q_{iq}^{i}A_{iq-1} - P_{iq-1}^{i+1}\overline{A}_{iq-1} - Q_{iq}^{iq+1}\overline{A}_{iq-1}\big\| \\
&= \big\|P_{iq}^{i}A_{iq-1}P_{iq-1}^{i} - P_{iq}^{i+1}A_{iq-1}P_{iq-1}^{i} + Q_{iq}^{i}A_{iq-1}Q_{iq-1}^{i} - Q_{iq}^{i+1}A_{iq-1}Q_{iq-1}^{i}\big\| \\
&\le \big\|(P_{iq}^{i} - P_{iq}^{i+1})A_{iq-1}P_{iq-1}^{i}\big\| + \big\|(Q_{iq}^{i} - Q_{iq}^{i+1})A_{iq-1}Q_{iq-1}^{i}\big\|.
\end{aligned}$$

It follows from (1.2) and (2.4) that

$$\|\overline{A}_{iq-1} - A_{iq-1}\| \le 2K\kappa\delta e^{-c(\rho(iq)-\rho(iq-1))}, \tag{2.13}$$

and so

$$\begin{aligned}
\|\overline{A}_{iq-1}\| &\le \|\overline{A}_{iq-1} - A_{iq-1}\| + \|A_{iq-1}\| \\
&\le K(2\kappa\delta + 1)e^{-c(\rho(iq)-\rho(iq-1))}.
\end{aligned} \tag{2.14}$$

To obtain the exponential estimates we first consider integers on a single interval $[(i-1)q, iq)$. Namely, take $i \in \mathbb{Z}$ and $m, n \in [(i-1)q, iq)$ with $m \ge n$. Then

$$\overline{\mathcal{A}}(m,n)\overline{P}_n = \mathcal{A}(m,n)P_n^{i} \quad \text{and} \quad \overline{\mathcal{A}}(n,m)\overline{Q}_m = \mathcal{A}(n,m)Q_m^{i}.$$

It follows from (2.2) and (2.3) that

$$\|\overline{\mathcal{A}}(m,n)\overline{P}_n\| \le \kappa e^{-c(\rho(m)-\rho(n))}$$

and

$$\|\overline{\mathcal{A}}(n,m)\overline{Q}_m\| \le \kappa e^{-c(\rho(m)-\rho(n))}.$$

Now take $i \in \mathbb{Z}$, $n \in [(i-1)q, iq]$ and $m = iq$. Then

$$\overline{\mathcal{A}}(iq,n)\overline{P}_n = \overline{P}_{iq} = P_{iq}^{i+1}, \quad \overline{\mathcal{A}}(n,iq)\overline{Q}_n = \overline{Q}_{iq} = Q_{iq}^{i+1},$$

for $n = iq$, and

$$\overline{\mathcal{A}}(iq,n)\overline{P}_n = \overline{A}_{iq-1}\mathcal{A}(iq-1,n)P_n^i,$$

$$\overline{\mathcal{A}}(n,iq)\overline{Q}_{iq} = \mathcal{A}(n,iq-1)Q_{iq-1}^i\big(\overline{A}_{iq-1}\overline{Q}_{iq-1}\big)^{-1}$$

for $n < iq$. Hence, it follows from the first case together with (2.14) that

$$\begin{aligned}\|\overline{\mathcal{A}}(m,n)\overline{P}_n\| &\le K(2\kappa\delta+1)\kappa e^{-c(\rho(iq)-\rho(iq-1))}e^{-c(\rho(iq-1)-\rho(n))}\\ &= K(2\kappa\delta+1)\kappa e^{-c(\rho(iq)-\rho(n))}\end{aligned}$$

and it follows from the first case and (2.12) that

$$\|\overline{\mathcal{A}}(n,m)\overline{Q}_m\| \le \frac{\kappa^2}{1-\delta}e^{-c(\rho(iq)-\rho(n))}.$$

Combining the inequalities, we find that, for all $i \in \mathbb{Z}$ and $m, n \in [(i-1)q, iq]$ with $m \ge n$, we have

$$\|\overline{\mathcal{A}}(m,n)\overline{P}_n\| \le \overline{\kappa} e^{-c(\rho(m)-\rho(n))} \tag{2.15}$$

and

$$\|\overline{\mathcal{A}}(n,m)\overline{Q}_m\| \le \overline{\kappa} e^{-c(\rho(m)-\rho(n))}, \tag{2.16}$$

where

$$\overline{\kappa} = \max\big\{1, \kappa, K(2\kappa\delta+1), \kappa^2/(1-\delta)\big\}. \tag{2.17}$$

One can take any fixed $\delta < \sqrt{2}-1$, which ensures that $\overline{\kappa} = \overline{\kappa}(\kappa, K)$.

It remains to consider arbitrary integers $m \ge n$ (that are not necessarily in the same interval $[(i-1)q, iq]$). Take $m, n \in \mathbb{Z}$ with $m \ge n$. Then there exist integers $j \le i$ such that

$$n \in [(j-1)q, jq] \quad \text{and} \quad m \in [(i-1)q, iq].$$

When $j = i$ we have the bounds in (2.15) and (2.16). Now assume that $j \le i-1$. Then

$$\begin{aligned}\|\overline{\mathcal{A}}(m,n)\overline{P}_n\| &= \|\overline{\mathcal{A}}(m,(i-1)q)\overline{P}_{(i-1)q}\overline{\mathcal{A}}((i-1)q,jq)\overline{P}_{jq}\overline{\mathcal{A}}(jq,n)\overline{P}_n\|\\ &\le \overline{\kappa}e^{-c(\rho(m)-\rho((i-1)q))}\overline{\kappa}^{i-1-j}e^{-c(\rho(jq)-\rho((i-1)q))}\overline{\kappa}e^{-c(\rho(jq)-\rho(n))}\\ &\le \overline{\kappa}^2\overline{\kappa}^{i-1-j}e^{-c(\rho(m)-\rho(n))}\\ &\le \overline{\kappa}^2(e^{-c}\overline{\kappa}^{1/q})^{\rho(m)-\rho(n)}\\ &\le \overline{\kappa}^2 e^{-(c-\frac{1}{q}\log\overline{\kappa})(\rho(m)-\rho(n))}\end{aligned}$$

since $(i-1)q \le m \le \rho(m)$ and $-jq \le -n \le -\rho(n)$ (because ρ is increasing) and $\overline{\kappa} \ge 1$ (see (2.17)). One can show in a similar manner that

$$\|\overline{\mathcal{A}}(n,m)\overline{Q}_m\| \le \overline{\kappa}^2 (e^{-c}\overline{\kappa}^{1/q})^{\rho(m)-\rho(n)} \le \overline{\kappa}^2 e^{-(c-\frac{1}{q}\log \overline{\kappa})(\rho(m)-\rho(n))}.$$

Step 3. Conclusion. Taking $q > \frac{1}{c}\log \overline{\kappa}$ sufficiently large, so that

$$c - \frac{1}{q}\log \overline{\kappa} > \overline{c},$$

we find that $(\overline{A}_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy with constants $\overline{\kappa}^2$ and $\overline{c}$. On the other hand, it follows readily from (2.8) and (2.13) that

$$\begin{aligned}\sup_{n\in\mathbb{Z}}\left(\|\overline{A}_n - A_n\| e^{c(\rho(n+1)-\rho(n))}\right) &= \sup_{i\in\mathbb{Z}}\left(\|\overline{A}_{iq-1} - A_{iq-1}\| e^{c(\rho(iq)-\rho(iq-1))}\right)\\ &\le 2K\kappa\delta.\end{aligned}$$

Hence, by the robustness property of the notion of a ρ-exponential dichotomy (see [2, 3] for details) that for δ sufficiently small the sequence $(A_m)_{m\in\mathbb{Z}}$ also has a ρ-exponential dichotomy with constant arbitrarily close to $\overline{\kappa}^2$ and $\overline{c}$. This concludes the proof of the lemma. $\square$

Now take $a, b, r \in \mathbb{Z}$ such that $b - a \ge 2r > 0$.

Lemma 2.3. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ has two ρ-exponential dichotomies on $[a,b]$ with constants c, κ and projections, respectively, $(P_m)_{m\in\mathbb{Z}}$ and $(\overline{P}_m)_{m\in\mathbb{Z}}$. Then*

$$\sup_{n\in[a+r,b-r]} \|P_n - \overline{P}_n\| \le 2\kappa^3 e^{-2cr}.$$

Proof. For each $m > n$, let $\mathcal{A}^u(n,m)$ be the inverse of the operator

$$\mathcal{A}(m,n)|\operatorname{im} Q_n\colon \operatorname{im} Q_n \to \operatorname{im} Q_n.$$

Now take $n \in [a+r, b-r]$ and $x \in \operatorname{im} Q_n$. We have

$$x = Q_n x = \mathcal{A}(n, n-r)\mathcal{A}^u(n-r,n)x,$$

and so

$$\overline{P}_n x = \mathcal{A}(n,n-r)\overline{P}_{n-r}\mathcal{A}^u(n-r,n)x.$$

Therefore,

$$\|\overline{P}_n x\| \le \kappa^2 e^{-2c(\rho(n)-\rho(n-r))}\|x\| \quad \text{for } x \in \operatorname{im} Q_n.$$

One can show in a similar manner that

$$\|P_n x\| \le \kappa^2 e^{-2c(\rho(n)-\rho(n-r))}\|x\| \quad \text{for } x \in \operatorname{im} \overline{Q}_n. \tag{2.18}$$

Now take $x \in \operatorname{im} \overline{P}_n$. Then

$$Q_n x = \mathcal{A}^u(n,n+r)\mathcal{A}(n+r,n)x = \mathcal{A}^u(n,n+r)\mathcal{A}(n+r,n)\overline{P}_n x,$$

and so

$$\|Q_n x\| \le \kappa^2 e^{-2c(\rho(n)-\rho(n-r))}\|x\| \quad \text{for } x \in \operatorname{im} \overline{P}_n. \tag{2.19}$$

Again, one can show in a similar manner that

$$\|\overline{Q}_n x\| \le \kappa^2 e^{-2c(\rho(n)-\rho(n-r))}\|x\| \quad \text{for } x \in \operatorname{im} Q_n.$$

Moreover, we have

$$\begin{aligned}\|\overline{P}_n - P_n\| &= \|(\mathrm{Id} - P_n)\overline{P}_n - P_n(\mathrm{Id} - \overline{P}_n)\| \\ &= \|Q_n\overline{P}_n - P_n\overline{Q}_n\| \\ &\le \|Q_n\overline{P}_n\| + \|P_n\overline{Q}_n\|,\end{aligned}$$

and it follows from (2.18) and (2.19) that

$$\begin{aligned}\|\overline{P}_n - P_n\| &\le \kappa^2 e^{-2c(\rho(n)-\rho(n-r))}\big(\|\overline{P}_n\| + \|\overline{Q}_n\|\big) \\ &\le 2\kappa^3 e^{-2c(\rho(n)-\rho(n-r))} \le e^{-2cr},\end{aligned}$$

where we have used that $\rho(l) - \rho(j) \ge l - j$ whenever $l \ge n$. This completes the proof of the lemma. $\square$

We proceed with the proof of the theorem. Let $\ell = 6q > 3\tau$ be an integer multiple of 6 with q even (note that if necessary one can take ℓ larger in the statement of the theorem). Then, for each $n \in \mathbb{Z}$, the interval $[n-2q, n]$ contains at least one number α_j and

$$[n+q, n+2q] \subset [n, n+3q] \subset [\alpha_j, \alpha_j + 6q].$$

In particular, taking $n = iq$ with $i \in \mathbb{Z}$, we obtain

$$J_i := [(i+1)q, (i+2)q] \subset I_i := [iq, (i+3)q] \subset [\alpha_j, \alpha_j + 6q].$$

Since the sequence $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy on each interval $[\alpha_j, \alpha_j + 6q]$, it has also a ρ-exponential dichotomy on each interval J_i, which as we have seen can be extended to the whole interval I_i. Therefore, one can readily apply Lemma 2.2 on $[a, b] = J_i$ provided that the integer $r = q/2$ in Lemma 2.3 can be taken sufficiently large, so that

$$2\kappa^3 e^{-2cr} = 2\kappa^2 e^{-cq} < \delta < \sqrt{2} - 1$$

with δ at least as small as in Step 3 of the proof of Lemma 2.2. But this is always possible by taking q sufficiently large.

3. The case of continuous time

In this section we obtain a corresponding version of Theorem 1.1 for a dynamics with continuous time. Let $A\colon \mathbb{R} \to B(X)$ be a continuous function, where $B(X)$ is the set of all bounded linear operators acting on a Banach space X. Consider the linear equation

$$x' = A(t)x, \tag{3.1}$$

and let $T(t, s)$ be the linear evolution operator associated with equation (3.1). Given an increasing function $\rho\colon \mathbb{R} \to \mathbb{R}$ with $\rho(0) = 0$ and an interval $I \subset \mathbb{R}$, we say that equation (3.1) has a *ρ-exponential dichotomy on I* if:

(1) there exist projections $P(t)$, for $t \in I$, satisfying

$$P(t)T(t,s) = T(t,s)P(s) \quad \text{for } t \ge s$$

such that, writing $Q(t) = \mathrm{Id} - P(t)$, the map

$$T(t,s)|\operatorname{im} Q(s)\colon \operatorname{im} Q(s) \to \operatorname{im} Q(t) \tag{3.2}$$

is onto and injective;

(2) there exist $c, \kappa > 0$ such that, for all $t, s \in I$ with $t \geq s$, we have

$$\|T(t,s)P(s)\| \leq \kappa e^{-c(\rho(t)-\rho(s))}$$

and

$$\|T(s,t)Q(t)\| \leq \kappa e^{-c(\rho(t)-\rho(s))},$$

where $T(s,t)$ denotes the inverse of the map in (3.2).

The vector spaces $\operatorname{im} P(s)$ and $\operatorname{im} Q(s)$ are then called, respectively, *stable* and *unstable* spaces at time s.

The following result is a version of Theorem 1.1 for continuous time.

Theorem 3.1. *Let $A\colon \mathbb{R} \to B(X)$ be a continuous function such that*

$$K' := \sup_{t,s\in\mathbb{R}, t\geq s} \left(\|T(t,s)\| e^{c(\rho(t)-\rho(s))}\right) < +\infty,$$

for some $c > 0$, and let $(\alpha_i)_{i\in\mathbb{Z}}$ be a τ-dense sequence. Given $c > \overline{c} > 0$, there exist $\ell = \ell(\tau, c, \overline{c}, \kappa, K')$ and $\overline{\kappa} = \overline{\kappa}(\kappa, K')$ such that if equation (3.1) has a ρ-exponential dichotomy on $[\alpha_i, \alpha_i + \ell]$ for each $i \in \mathbb{Z}$ with constants c and κ, then it has a ρ-exponential dichotomy on $\mathbb{R}$ with constants $\overline{c}$ and $\overline{\kappa}$.

Proof. Let

$$A_m = T(m+1, m) \quad \text{for } m \in \mathbb{Z}.$$

It follows from the assumptions in the theorem that $(A_m)_{m\in\mathbb{Z}}$ is a sequence of bounded linear operators satisfying (1.2) having a ρ-exponential dichotomy on $[\alpha_i, \alpha_i+\ell]$ for $i \in \mathbb{Z}$ with constants c and κ. By Theorem 1.1, given $c > \overline{c} > 0$, there exist $\ell = \ell(\tau, c, \overline{c}, \kappa, K')$ and $\overline{\kappa} = \overline{\kappa}(\kappa, K')$ such that $(A_m)_{m\in\mathbb{Z}}$ has a ρ-exponential dichotomy on $\mathbb{Z}$ with constants $\overline{c}$ and $\overline{\kappa}$. Finally, applying Theorem 1.3 in [12] (which allows one to pass from discrete time to continuous time), we conclude that equation (3.1) has a ρ-exponential dichotomy on the whole $\mathbb{R}$. ☐

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References

1. L. Barreira and C. Valls, *Growth rates and nonuniform hyperbolicity*, Discrete Contin. Dyn. Syst. **22** (2008), 509–528.
2. L. Barreira and C. Valls, *Smooth robustness of parameterized perturbations of exponential dichotomies*, J. Differential Equations **249** (2010), no. 8, 2021–2043.
3. L. Barreira and C. Valls, *Robust nonuniform dichotomies and parameter dependence*, J. Math. Anal. Appl. **373** (2011), 690–708.
4. S.-N. Chow and H. Leiva, *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, J. Differential Equations **120** (1995), 429–477.
5. W. Coppel, *Dichotomies and reducibility*, J. Differential Equations **3** (1967), 500–521.
6. W. Coppel, *Dichotomies in stability theory*, Lect. Notes in Math. 629, Springer, 1978.
7. Ju. Dalec'kiĭ and M. Kreĭn, *Stability of solutions of differential equations in Banach space*, Translations of Mathematical Monographs 43, Amer. Math. Soc., 1974.
8. A. Ducrot, P. Magal, and O. Seydi, *A finite-time condition for exponential trichotomy in infinite dynamical systems*, Canad. J. Math. **67** (2015), 1065–1090.

9. A. Ducrot, P. Magal, and O. Seydi, *Persistence of exponential trichotomy for linear operators: a Lyapunov–Perron approach*, J. Dynam. Differential Equations **28** (2016), 93–126.
10. J. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs 25, Amer. Math. Soc., 1988.
11. D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes in Math. 840, Springer, 1981.
12. D. Henry, *Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces*, Resenhas **1** (1994), 381–401.
13. J. Massera and J. Schäffer, *Linear differential equations and functional analysis. I*, Ann. of Math. (2) **67** (1958), 517–573.
14. J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics 21, Academic Press, 1966.
15. R. Naulin and M. Pinto, *Admissible perturbations of exponential dichotomy roughness*, Nonlinear Anal. **31** (1998), 559–571.
16. V. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.
17. K. Palmer, *Shadowing in Dynamical Systems. Theory and Applications*, Mathematics and its Applications 501, Kluwer Academic Publishers, 2000.
18. K. Palmer, *A finite-time condition for exponential dichotomy*, J. Difference. Equ. Appl. **17** (2011), 221–234.
19. O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.
20. Ya. Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **10** (1976), 1261–1305.
21. Ya. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55–114.
22. Ya. Pesin, *Geodesic flows on closed Riemannian manifolds without focal points*, Math. USSR-Izv. **11** (1977), 1195–1228.
23. V. Pliss and G. Sell, *Robustness of exponential dichotomies in infinite-dimensional dynamical systems*, J. Dynam. Differential Equations **11** (1999), 471–513.
24. L. Popescu, *Exponential dichotomy roughness on Banach spaces*, J. Math. Anal. Appl. **314** (2006), 436–454.
25. G. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences 143, Springer, 2002.

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