Adv. Oper. Theory 4 (2019), no. 1, 197–214
https://doi.org/10.15352/aot.1804-1359
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

NUMERICAL RADIUS INEQUALITIES FOR OPERATOR MATRICES

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Communicated by M. Martin

Abstract. Several numerical radius inequalities for operator matrices are proved by generalizing earlier inequalities. In particular, the following inequalities are obtained: if n is even,

$$2w(T) \leq \max\{\|A_1\|, \|A_2\|, \ldots, \|A_n\|\} + \frac{1}{2}\sum_{k=0}^{n-1} \| \, |A_{n-k}|^t |A_{k+1}^*|^{1-t}\|,$$

and if n is odd,

$$2w(T) \leq \max\{\|A_1\|, \|A_2\|, \ldots, \|A_n\|\} + w\left(\widetilde{A}_{(\frac{n+1}{2})t}\right) + \frac{1}{2}\sum_{k=0}^{n-1} \| \, |A_{n-k}|^t |A_{k+1}^*|^{1-t}\|,$$

for all $t \in [0,1]$, A_i's are bounded linear operators on the Hilbert space $\mathcal{H}$, and T is off diagonal matrix with entries $A_1, \cdots, A_n$.

1. Introduction

1.1. **Background and motivation.** Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle ., . \rangle$. The *numerical range* of $T \in \mathcal{L}(\mathcal{H})$, denoted by $W(T)$, is the subset of complex numbers given by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

Date: Received: Apr. 30, 2018; Accepted: Jun. 19, 2018.
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2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 47A63, 47A30.
Key words and phrases. Aluthge transform, spectral radius, numerical radius, operator matrix, polar decomposition.

The *numerical radius* of T, denoted by $w(T)$, is defined by $w(T) = \sup\{|z| : z \in W(T)\}$. Thus $W(T)$ is the image of unit circle in $\mathcal{H}$ under the quadratic form $f(x) = \langle Tx, x\rangle$ from $\mathcal{H}$ to $\mathbb{C}$, and $w(T)$ is the smallest radius of a circular disc centered at the origin which contains $W(T)$. It is well-known that $w(.)$ defines a norm on $\mathcal{H}$, which is equivalent to the usual operator norm $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$. For every $T \in \mathcal{L}(\mathcal{H})$, we have

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

Several numerical radius inequalities that provide alternative lower and upper bounds for $w(.)$ have attracted great research interest in recent years. The interested readers are referred to [3, 5] for the history and significance of numerical radius inequalities. The authors of [9] showed that if T is an operator in $\mathcal{L}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right). \tag{1.2}$$

Consequently, if $T^2 = 0$, then

$$w(T) = \frac{1}{2}\|T\|. \tag{1.3}$$

Let $T = U|T|$ be the polar decomposition of T. Here U is a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$. The *Aluthge transform* of the operator T, denoted by $\widetilde{T}$, is defined as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Okubo [11] introduced a more general notion called *t-Aluthge transform*. It is denoted by $\widetilde{T}_t$, and is defined as $\widetilde{T}_t = |T|^tU|T|^{1-t}$ for $0 \leq t \leq 1$. It coincides with the usual Aluthge transform for $t = \frac{1}{2}$. When $t = 1$, the operator $\widetilde{T}_1 = |T|U$ is called the *Duggal transform* of $T \in \mathcal{L}(\mathcal{H})$. Bakherad and Shebrawi [2] introduced *generalized Aluthge transform* of the operator T, denoted by $\widetilde{T}_{f,g}$. It is defined by $\widetilde{T}_{f,g} = f(|T|)Ug(|T|)$, where f, g are non-negative continuous functions such that $f(x)g(x) = x$ $(x \geq 0)$ and $T \in \mathcal{L}(\mathcal{H})$. Several refinements and generalizations of the inequality (1.2) are discussed in [1, 7, 14]. A nice refinement of the inequality (1.2) is recently obtained in [15]. It says that, for an operator T in $\mathcal{L}(\mathcal{H})$ and $0 \leq t \leq 1$,

$$w(T) \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T}_t)\right). \tag{1.4}$$

An important property of the numerical radius norm is its weak unitary invariance; that is, for $T \in \mathcal{L}(\mathcal{H})$,

$$w(U^*TU) = w(T).$$

for every unitary $U \in \mathcal{L}(\mathcal{H})$. Let $r(.)$ denote the spectral radius. It is well-known that, for every operator T in $\mathcal{L}(\mathcal{H})$, we have

$$r(T) \leq w(T) \leq \|T\|, \tag{1.5}$$

and the equality holds if T is normal. The above result basically focuses on the relation among the numerical radius, the spectral radius, and the norm of an operator. Hirzallah *et al.* [7] studied numerical radius inequality for certain 2×2 operator matrices. They showed that if $A, B \in \mathcal{L}(\mathcal{H})$, then

$$w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\| + \|B\|}{2}. \tag{1.6}$$

Shebrawi [13] then proved several numerical radius inequalities for 2×2 operator matrices, very recently. In particular, he obtained the following inequality: if A and B are operators in $\mathcal{L}(\mathcal{H})$, then

$$2w\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \max(\|A\|, \|B\|) + \frac{1}{2}\left(\left\||A|^t|B^*|^{1-t}\right\| + \left\||B|^t|A^*|^{1-t}\right\|\right) \tag{1.7}$$

for all $t \in [0, 1]$. The basic objective of this article is to find numerical radius inequalities for $n \times n$ operator matrices.

1.2. **Outline.** The organization of this paper is as follows. In the next subsection, we present some preliminary results which are helpful to prove our main results. Section 2 is devoted to the main results. Here we establish certain general numerical radius inequalities for $n \times n$ operator matrices by applying t-Aluthge transform and generalized Aluthge transform of operators. We also obtain upper bounds for the numerical radius of $n \times n$ operator matrices.

1.3. **Primary results.** Here we collect all those results which will be used to prove the main results in the next section. Yamazaki [15] first proved that $w(A) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|$ for $A \in \mathcal{L}(\mathcal{H})$. It was recently restated by Shebrawi [13] in the following way.

Lemma 1.1. [13, Lemma 2.2]
Let $A \in \mathcal{L}(\mathcal{H})$. Then

$$w(A) = \max_{\theta \in \mathbb{R}} \|Re(e^{i\theta}A)\|.$$

The next theorem extends the inequality given in (1.4).

Theorem 1.2. [2, Theorem 2.3]
Let $A \in \mathcal{L}(\mathcal{H})$, and let f, g be non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$. Then

$$h(w(A)) \leq \frac{1}{4}\|h(g^2(|A|)) + h(f^2(|A|))\| + \frac{1}{2}h\left(w(\widetilde{A}_{f,g})\right), \tag{1.8}$$

for all non-negative nondecreasing convex function h on $[0, \infty)$.

Another generalization of the inequality (1.4) is recalled below.

Theorem 1.3. [2, Theorem 2.10]
Let $A \in \mathcal{L}(\mathcal{H})$, and let f, g, and h be non-negative nondecreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$. Then

$$h(w(A)) \leq \frac{1}{2}\Big(h(w(\widetilde{A}_{f,g})) + \|h(|A|)\|\Big).$$

The final result of this section is an extension of the inequality (1.6).

Theorem 1.4. [10, Theorem 2.2]

Let $A_i \in \mathcal{L}(\mathcal{H}),\ i = 1, 2, \ldots, n,$ *and let* $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$. *If* n *is even, then*

$$w(T) \le \frac{1}{2} \sum_{i=1}^{n} \|A_i\|.$$

On the other hand, if n *is an odd number,*

$$w(T) \le w\Big(A_{\frac{n+1}{2}}\Big) + \frac{1}{2} \sum_{\substack{i=1 \\ i \ne \frac{n+1}{2}}}^{n} \|A_i\|.$$

Next result deals with inequalities on block-norm matrix $\hat{A}$. For $\mathbf{H} = \oplus_{i=1}^{n} \mathcal{H}_i$ and $A \in \mathcal{L}(\mathcal{H})$, the operator A can be represented as an $n \times n$ operator matrix; that is, $A = (A_{ij})_{n\times n}$ with $A_i \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. The block-norm matrix $\hat{A}$ associated with an operator matrix $A = (A_{ij})_{n\times n}$ is defined by $\hat{A} = (\|A_{ij}\|)_{n\times n}$, which is an $n \times n$ non-negative matrix. The following lemma presents some basic inequalities on numerical radii, norms as well as spectral radii of an operator matrix A and its block-norm matrix $\hat{A}$.

Lemma 1.5. [8, Theorem 1.1]
Let $A = (A_{ij})_{n\times n}$ *be an operator matrix, and let* $\hat{A} = (\|A_{ij}\|)_{n\times n}$ *be its block-norm matrix. Then*
(i) $w(A) \le w(\hat{A})$, *(ii)* $\|A\| \le \|\hat{A}\|$, *(iii)* $r(A) \le r(\hat{A})$.

2. Main results

In this section, we prove some new numerical radius inequalities concerning n operators. The very first result generalizes the numerical radius inequality (1.7).

Theorem 2.1. *Let* $A_i \in \mathcal{L}(\mathcal{H}),\ i = 1, 2, \ldots, n,$ *and let* $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$.

If n *is even, then*

$$2w(T) \le \max\{\|A_1\|, \|A_2\|, \ldots, \|A_n\|\} + \frac{1}{2} \sum_{k=0}^{n-1} \|\ |A_{n-k}|^t |A_{k+1}^*|^{1-t} \|,$$

and if n *is odd, then*

$$2w(T) \le \max\{\|A_1\|, \|A_2\|, \ldots, \|A_n\|\} + w\Big(\widetilde{A}_{(\frac{n+1}{2})t}\Big) + \frac{1}{2} \sum_{k=0}^{n-1} \|\ |A_{n-k}|^t |A_{k+1}^*|^{1-t} \|,$$

for all $t \in [0, 1]$.

Proof. Let $T = Y_1 + Y_2 + Y_3 + \cdots + Y_n$, where

$$Y_1 = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, Y_2 = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ldots, Y_n = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}.$$

Theorem 1.4 yields $Y_i^2 = 0$ for all $i = 1, 2, \ldots, n$, if n is an even number and so

$$w(T) = w\left(\sum_{i=1}^{n} Y_i\right) \le \sum_{i=1}^{n} w(Y_i) = \frac{1}{2}\sum_{i=1}^{n} \|Y_i\| = \frac{1}{2}\sum_{i=1}^{n} \|A_i\|. \tag{2.1}$$

On the other hand, we have

$$w(T) = w\left(\sum_{i=1}^{n} Y_i\right) \le w\left(Y_{\frac{n+1}{2}}\right) + \sum_{i \neq \frac{n+1}{2}}^{n} w(Y_i) = w\left(A_{\frac{n+1}{2}}\right) + \frac{1}{2}\sum_{i \neq \frac{n+1}{2}}^{n} \|A_i\|, \tag{2.2}$$

if n is an odd number. Let $A_i = U_i|A_i|,\ i = 1, 2, \ldots, n$ be the polar decompositions of the operators A_i. Then $\|T\| = \max\{\|A_1\|, \|A_2\|, \ldots, \|A_n\|\}$ and

$$T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} |A_n| & 0 & \cdots & 0 \\ 0 & |A_{n-1}| & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |A_1| \end{bmatrix}$$

is the polar decomposition of T. The t-Aluthge transform of T, for $0 \le t \le 1$, is given by

$$\begin{aligned} \widetilde{T_t} =& |T|^t \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} |T|^{1-t} \\ =& \begin{bmatrix} |A_n|^t & 0 & \cdots & 0 \\ 0 & |A_{n-1}|^t & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |A_1|^t \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} |A_n|^{1-t} & 0 & \cdots & 0 \\ 0 & |A_{n-1}|^{1-t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |A_1|^{1-t} \end{bmatrix} \\ =& \begin{bmatrix} 0 & \cdots & 0 & |A_n|^t U_1 |A_1|^{1-t} \\ \vdots & & |A_{n-1}|^t U_2 |A_2|^{1-t} & 0 \\ 0 & \iddots & & \vdots \\ |A_1|^t U_n |A_n|^{1-t} & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

Applying the triangle inequality for numerical radius, we have

$$\begin{aligned}
w(\widetilde{T_t}) &= w\left(\begin{bmatrix} 0 & \cdots & 0 & |A_n|^t U_1 |A_1|^{1-t} \\ \vdots & & |A_{n-1}|^t U_2 |A_2|^{1-t} & 0 \\ 0 & \iddots & & \vdots \\ |A_1|^t U_n |A_n|^{1-t} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} 0 & \cdots & 0 & |A_n|^t U_1 |A_1|^{1-t} \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & |A_{n-1}|^t U_2 |A_2|^{1-t} & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right. \\
&\quad \left. + \cdots + \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ |A_1|^t U_n |A_n|^{1-t} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\le w\left(\begin{bmatrix} 0 & \cdots & 0 & |A_n|^t U_1 |A_1|^{1-t} \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + w\left(\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & |A_{n-1}|^t U_2 |A_2|^{1-t} & 0 \\ 0 & \iddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \cdots + w\left(\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & 0 & 0 \\ 0 & \iddots & & \vdots \\ |A_1|^t U_n |A_n|^{1-t} & 0 & \cdots & 0 \end{bmatrix}\right).
\end{aligned}$$

Applying inequalities (2.1) and (2.2), we get

$$w(\widetilde{T_t}) \le \frac{1}{2} \sum_{k=0}^{n-1} \| |A_{n-k}|^t U_{k+1} |A_{k+1}|^{1-t} \|,$$

when n is even, and we have

$$w(\widetilde{T_t}) \le w\left(|A_{\frac{n+1}{2}}|^t U_{\frac{n+1}{2}} |A_{\frac{n+1}{2}}|^{1-t} \right) + \frac{1}{2} \sum_{k=0}^{n-1} \| |A_{n-k}|^t U_{k+1} |A_{k+1}|^{1-t} \|,$$

when n is odd. Now,

$$|A_i^*|^2 = A_i A_i^* = U_i |A_i| (U_i |A_i|)^* = U_i |A_i| |A_i|^* U_i^* = U_i |A_i|^2 U_i^*,$$

and so $|A_i|^{1-t} = U_i^* |A_i^*|^{1-t} U_i$ for all $t \in [0,1]$. Thus,

$$\| |A_{n-k}|^t U_{k+1} |A_{k+1}|^{1-t} \| = \| |A_{n-k}|^t U_{k+1} U_{k+1}^* |A_{k+1}^*|^{1-t} U_{k+1} \| = \| |A_{n-k}|^t |A_{k+1}^*|^{1-t} \|.$$

Therefore, we have

$$\begin{aligned} w(\widetilde{T_t}) &\le \frac{1}{2}\sum_{k=0}^{n-1} \||A_{n-k}|^t U_{k+1}|A_{k+1}|^{1-t}\| \\ &= \frac{1}{2}\sum_{k=0}^{n-1} \||A_{n-k}|^t |A_{k+1}^*|^{1-t}\| \end{aligned}$$

for n is even, and

$$\begin{aligned} w(\widetilde{T_t}) &\le w\left(|A_{\frac{n+1}{2}}|^t U_{\frac{n+1}{2}}|A_{\frac{n+1}{2}}|^{1-t}\right) + \frac{1}{2}\sum_{k=0}^{n-1} \||A_{n-k}|^t U_{k+1}|A_{k+1}|^{1-t}\| \\ &= w\left(\widetilde{A}_{(\frac{n+1}{2})t}\right) + \frac{1}{2}\sum_{k=0}^{n-1} \|\, |A_{n-k}|^t |A_{k+1}^*|^{1-t}\|, \end{aligned}$$

when n is odd. Since $w(T) \le \frac{1}{2}\left(\|T\| + w(\widetilde{T_t})\right)$, so

$$2w\left(\begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}\right) = 2w(T) \le 2 \times \frac{1}{2}\left(\|T\| + w(\widetilde{T_t})\right) = \|T\| + w(\widetilde{T_t}).$$

If n is even, then

$$\begin{aligned} 2w\left(\begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}\right) &\le \max\{\|A_1\|, \|A_2\|, \dots, \|A_n\|\} \\ &\quad + \frac{1}{2}\sum_{k=0}^{n-1} \|\, |A_{n-k}|^t |A_{k+1}^*|^{1-t}\|, \end{aligned}$$

and if n is odd, then

$$\begin{aligned} 2w\left(\begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}\right) &\le \max\{\|A_1\|, \|A_2\|, \dots, \|A_n\|\} + w\left(\widetilde{A}_{(\frac{n+1}{2})t}\right) \\ &\quad + \frac{1}{2}\sum_{k=0}^{n-1} \|\, |A_{n-k}|^t |A_{k+1}^*|^{1-t}\|. \end{aligned}$$

□

The next result provides the numerical radius inequality for off-diagonal $n \times n$ operator matrix.

Theorem 2.2. *Let* $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$, *where* $A_i \in \mathcal{L}(\mathcal{H})$, $i = 1, 2, \ldots, n$. *Suppose that* f *and* g *are two non-negative continuous functions on* $[0, \infty)$ *such that* $f(x)g(x) = x$ $(x \geq 0)$ *and* $r \geq 1$. *If* n *is even, then*

$$w^r(T) \leq \frac{1}{4} \max_{1 \leq i \leq n} \left\{ \| g^{2r}(|A_i|) + f^{2r}(|A_i|) \| \right\} + \frac{n^{r-1}}{2^{r+1}} \sum_{i=1}^{n} \| f(|A_{n-i+1}|) g(|A_i^*|) \|^r,$$

and if n *is odd, then*

$$\begin{aligned} w^r(T) \leq & \frac{1}{4} \max_{1 \leq i \leq n} \left\{ \| g^{2r}(|A_i|) + f^{2r}(|A_i|) \| \right\} \\ & + \frac{1}{2} \left[w\left(\widetilde{A}_{(\frac{n+1}{2})f,g} \right) + \frac{1}{2} \sum_{\substack{i=1 \\ i \neq \frac{n+1}{2}}}^{n} \| f(|A_{n-i+1}|) g(|A_i^*|) \| \right]^r . \end{aligned}$$

Proof. Let $A_i = U_i |A_i|$, $i = 1, 2, \ldots, n$ be the polar decomposition of the operators A_i, $i = 1, 2, \ldots, n$, and let $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$. It follows from the polar decomposition of

$$T = \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} |A_n| & 0 & \cdots & 0 \\ 0 & |A_{n-1}| & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |A_1| \end{bmatrix} \quad \textit{that}$$

$$\begin{aligned} \widetilde{T}_{f,g} =& f(|T|) \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} g(|T|) \\ =& \begin{bmatrix} f(|A_n|) & 0 & \cdots & 0 \\ 0 & f(|A_{n-1}|) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & f(|A_1|) \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & U_1 \\ \vdots & & U_2 & 0 \\ 0 & \iddots & & \vdots \\ U_n & 0 & \cdots & 0 \end{bmatrix} \\ & \begin{bmatrix} g(|A_n|) & 0 & \cdots & 0 \\ 0 & g(|A_{n-1}|) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g(|A_1|) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & f(|A_n|)U_1 g(|A_1|) \\ \vdots & & f(|A_{n-1}|)U_2 g(|A_2|) & 0 \\ 0 & \iddots & & \vdots \\ f(|A_1|)U_n g(|A_n|) & 0 & \cdots & 0 \end{bmatrix}.$$

Then

$$w(\widetilde{T}_{f,g}) = w\left(\begin{bmatrix} 0 & \cdots & 0 & f(|A_n|)U_1 g(|A_1|) \\ \vdots & & f(|A_{n-1}|)U_2 g(|A_2|) & 0 \\ 0 & \iddots & & \vdots \\ f(|A_1|)U_n g(|A_n|) & 0 & \cdots & 0 \end{bmatrix}\right).$$

It follows from inequalities (2.1) and (2.2); that is, if n is an even number, then

$$w(\widetilde{T}_{f,g}) \leq \frac{1}{2}\sum_{i=1}^{n} \|f(|A_{n-i+1}|)U_i g(|A_i|)\|,$$

and if n is an odd number, then

$$w(\widetilde{T}_{f,g}) \leq w\Big(f(|A_{\frac{n+1}{2}}|)U_{\frac{n+1}{2}} g(|A_{\frac{n+1}{2}}|)\Big) + \frac{1}{2}\sum_{\substack{i=1 \\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)U_i g(|A_i|)\|.$$

Using $|A_i^*|^2 = A_iA_i^* = U_i|A_i|^2U_i^*$, where $i = 1, 2, \ldots, n$, we have $g(|A_i|) = U_i^* g(|A_i^*|)U_i$ for every non-negative continuous function g on $[0, \infty)$. Therefore we have

$$\begin{aligned} w(\widetilde{T}_{f,g}) \leq& \frac{1}{2}\sum_{i=1}^{n} \|f(|A_{n-i+1}|)U_iU_i^* g(|A_i^*|)U_i\| \\ =& \frac{1}{2}\sum_{i=1}^{n} \|f(|A_{n-i+1}|)g(|A_i^*|)\|, \end{aligned} \tag{2.3}$$

for even n, and

$$\begin{aligned} w(\widetilde{T}_{f,g}) \leq& w\Big(\widetilde{A}_{(\frac{n+1}{2})f,g}\Big) + \frac{1}{2}\sum_{\substack{i=1 \\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)U_iU_i^* g(|A_i^*|)U_i\| \\ =& w\Big(\widetilde{A}_{(\frac{n+1}{2})f,g}\Big) + \frac{1}{2}\sum_{\substack{i=1, \\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)g(|A_i^*|)\| \end{aligned} \tag{2.4}$$

for odd n. Applying Theorem 1.2, we get

$$w^r(T) \leq \frac{1}{4}\|g^{2r}(|T|) + f^{2r}(|T|)\| + \frac{1}{2}\left(w^r(\widetilde{T}_{f,g})\right).$$

If n is an even number, we then have

$$\begin{aligned} w^r(T) &\leq \frac{1}{4}\max_{1\leq i\leq n}\{\|g^{2r}(|A_i|) + f^{2r}(|A_i|)\|\} + \frac{1}{2}\left[\frac{1}{2}\sum_{i=1}^{n}\|f(|A_{n-i+1}|)g(|A_i^*|)\|\right]^r \\ &\leq \frac{1}{4}\max_{1\leq i\leq n}\{\|g^{2r}(|A_i|) + f^{2r}(|A_i|)\|\} + \frac{n^{r-1}}{2^{r+1}}\sum_{i=1}^{n}\|f(|A_{n-i+1}|)g(|A_i^*|)\|^r. \end{aligned}$$

While for odd n, we obtain

$$\begin{aligned} w^r(T) \leq & \frac{1}{4} \max_{1\leq i\leq n}\{\|g^{2r}(|A_i|)+f^{2r}(|A_i|)\|\} \\ & + \frac{1}{2}\left[w\left(\widetilde{A}_{\left(\frac{n+1}{2}\right)f,g}\right) + \frac{1}{2}\sum_{\substack{i=1 \\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)g(|A_i^*|)\| \right]^r, \end{aligned}$$

by using the convexity of $h(x)=x^r$. $\square$

Numerical radius inequality for off-diagonal 2×2 operator matrix is obtained below as a corollary.

Corollary 2.3. [2, Theorem 2.6]
Let $A, B \in \mathcal{L}(\mathcal{H})$. Suppose that f and g are two non-negative continuous functions on $[0,\infty)$ such that $f(x)g(x)=x$ $(x\geq 0)$ and $r\geq 1$. Then

$$\begin{aligned} w^r\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq & \frac{1}{4}\max\left(\|g^{2r}(|A|)+f^{2r}(|A|)\|, \|g^{2r}(|B|)+f^{2r}(|B|)\|\right) \\ & + \frac{1}{4}(\|f(|B|)g(|A^*|)\|^r + \|f(|A|)g(|B^*|)\|^r). \end{aligned}$$

We next prove another numerical radius inequality which yields a new upper bound and is stated in the next theorem. From Theorem 1.3, we obtain

$$2w^r(T) \leq w^r(\widetilde{T}_{f,g}) + \|T\|^r.$$

Applying inequalities (2.3) and (2.4), when n is an even number, we have

$$2w^r(T) \leq \max_{1\leq i\leq n}\{\|A_i\|^r\} + \left[\frac{1}{2}\sum_{i=1}^{n}\|f(|A_{n-i+1}|)g(|A_i^*|)\|\right]^r,$$

and when n is an odd number, we have

$$2w^r(T) \leq \max_{1\leq i\leq n}\{\|A_i\|^r\} + \left[w\left(\widetilde{A}_{\left(\frac{n+1}{2}\right)f,g}\right) + \frac{1}{2}\sum_{\substack{i=1 \\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)g(|A_i^*|)\| \right]^r.$$

Thus, we have the following result.

Theorem 2.4. *Let $A_i \in \mathcal{L}(\mathcal{H})$, $i=1,2,\ldots,n$, and let* $T = \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ \vdots & & A_2 & 0 \\ 0 & \iddots & & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$.
Suppose that f and g are two non-negative nondecreasing continuous functions on $[0,\infty)$ such that $f(x)g(x)=x$ $(x\geq 0)$ and $r\geq 1$. Then, for an even number n, we get

$$2w^r(T) \leq \max_{1\leq i\leq n}\{\|A_i\|^r\} + \left[\frac{1}{2}\sum_{i=1}^{n}\|f(|A_{n-i+1}|)g(|A_i^*|)\|\right]^r,$$

and for an odd number n, we have

$$2w^r(T) \leq \max_{1\leq i\leq n}\{\|A_i\|^r\} + \left[w\left(\widetilde{A}_{(\frac{n+1}{2})f,g}\right) + \frac{1}{2}\sum_{\substack{i=1\\ i\neq \frac{n+1}{2}}}^{n} \|f(|A_{n-i+1}|)g(|A_i^*|)\| \right]^r.$$

Another estimate of numerical radius inequality for a special type of $n\times n$ operator matrix is presented below.

Theorem 2.5. *Let $A_i \in \mathcal{L}(\mathcal{H})$, $i = 1, 2, \ldots, n$, and let* $T = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

Then

$$w(T) \leq \frac{1}{2}\left(\|A_1\| + \|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|^{\frac{1}{2}}\right).$$

Proof. Let $T = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Then

$$\begin{aligned}
\|Re(e^{i\theta}T)\| &= r(Re(e^{i\theta}T)) \\
&= r\left(\frac{e^{i\theta}T + (e^{i\theta}T)^*}{2}\right) \\
&= \frac{r}{2}\left(e^{i\theta}\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{r}{2}\left(\begin{bmatrix} e^{i\theta}A_1 + e^{-i\theta}A_1^* & e^{i\theta}A_2 & \cdots & e^{i\theta}A_n \\ e^{-i\theta}A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ e^{-i\theta}A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{r}{2}\left(\begin{bmatrix} A_1^* & 0 & \cdots & e^{i\theta}I \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}\right).
\end{aligned}$$

Using the commutativity property of the spectral radius, we get

$$
\begin{aligned}
&\|Re(e^{i\theta}T)\| \\
&= \frac{r}{2}\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}\begin{bmatrix} A_1^* & 0 & \cdots & e^{i\theta}I \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \frac{r}{2}\left(\begin{bmatrix} e^{-i\theta}A_1^* & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^* & 0 & \cdots & e^{i\theta}A_1 \end{bmatrix}\right) \\
&\leq \frac{r}{2}\left(\begin{bmatrix} \|A_1^*\| & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\| & 0 & \cdots & \|A_1\| \end{bmatrix}\right) \\
&= \frac{r}{2}\left(\begin{bmatrix} \|A_1^*\| & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \|A_1\| \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & & \vdots \\ \|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\| & \cdots & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\left(\|A_1\| + \|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|^{\frac{1}{2}}\right).
\end{aligned}
$$

Applying Lemma 1.1, we have

$$
\begin{aligned}
w(T) &= \max_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\| \\
&\leq \frac{1}{2}\left(\|A_1\| + \|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|^{\frac{1}{2}}\right).
\end{aligned}
$$

□

The following result is a numerical radius inequality for a 2×2 operator matrix.

Corollary 2.6. [13, Theorem 3.1]
Let A_1 and A_2 be operators in $\mathcal{L}(\mathcal{H})$. Then

$$
w\left(\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(\|A_1\| + \|A_1A_1^* + A_2A_2^*\|^{\frac{1}{2}}\right).
$$

The next result deals with the same class of operators as mentioned in Theorem 2.5.

Lemma 2.7. *Let $A_i \in \mathcal{L}(\mathcal{H})$, $i = 1, 2, \ldots, n$, and let $T = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.*

Then

$$
w(T) \leq \frac{w(A_1)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|.
$$

Proof. Let $T = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Then

$$\begin{aligned}
\|Re(e^{i\theta}T)\| =& r(Re(e^{i\theta}T)) \\
=& r\left(\frac{e^{i\theta}T + (e^{i\theta}T)^*}{2}\right) \\
=& \frac{r}{2}\left(e^{i\theta}\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} + e^{-i\theta}\begin{bmatrix} A_1^* & 0 & \cdots & 0 \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
=& \frac{r}{2}\left(\begin{bmatrix} e^{i\theta}A_1 + e^{-i\theta}A_1^* & e^{i\theta}A_2 & \cdots & e^{i\theta}A_n \\ e^{-i\theta}A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ e^{-i\theta}A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
=& \frac{r}{2}\left(\begin{bmatrix} A_1^* & 0 & \cdots & e^{i\theta}I \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}\right).
\end{aligned}$$

Using the commutativity property of the spectral radius and $r(T) \leq w(T) \leq \|T\|$, we get

$$\begin{aligned}
\|Re(e^{i\theta}T)\| =& \frac{r}{2}\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}\begin{bmatrix} A_1^* & 0 & \cdots & e^{i\theta}I \\ A_2^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
=& \frac{r}{2}\left(\begin{bmatrix} e^{-i\theta}A_1^* & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^* & 0 & \cdots & e^{i\theta}A_1 \end{bmatrix}\right) \\
\leq& \frac{1}{2}w\left(\begin{bmatrix} e^{-i\theta}A_1^* & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^* & 0 & \cdots & e^{i\theta}A_1 \end{bmatrix}\right) \\
=& \frac{1}{2}w\left(\begin{bmatrix} e^{-i\theta}A_1^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{i\theta}A_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
=& \frac{w(A_1)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|.
\end{aligned}$$

Applying Lemma 1.1, we have

$$\begin{aligned} w(T) &= \max_{\theta\in\mathbb{R}} \|Re(e^{i\theta}T)\| \\ &\leq \frac{w(A_1)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|. \end{aligned}$$

□

Similarly, one can easily prove the following result for $A_i \in \mathcal{L}(\mathcal{H})$, $i = 1, 2, \ldots, n$.

Lemma 2.8. (i) *If* $T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1 & A_2 & \cdots & A_n \end{bmatrix}$, *then*

$$w(T) \leq \frac{w(A_n)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^*\|.$$

(ii) *If* $T = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix}$, *then*

$$w(T) \leq \frac{w(A_1)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1^*A_1 + A_2^*A_2 + \cdots + A_n^*A_n\|.$$

(iii) *If* $T = \begin{bmatrix} 0 & 0 & \cdots & A_1 \\ 0 & 0 & \cdots & A_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$, *then*

$$w(T) \leq \frac{w(A_n)}{2} + \frac{1}{4} + \frac{1}{4}\|A_1^*A_1 + A_2^*A_2 + \cdots + A_n^*A_n\|.$$

Using Lemma 2.7, we prove the next result for $n \times n$ arbitrary operator matrix.

Theorem 2.9. *Let* $A_{ij} \in \mathcal{L}(\mathcal{H})$ *where* $1 \leq i, j \leq n$, *and let* $T = [A_{ij}]$. *Then*

$$w(T) \leq \frac{1}{2}\sum_{i=1}^{n} w(A_{ii}) + \frac{n}{4} + \frac{1}{4}\sum_{i,j=1}^{n} \|A_{ij}\|^2.$$

Proof. Let

$$U_k = \left[\begin{array}{c|c} J_{k\times k} & 0_{k\times n-k} \\ \hline 0_{n-k\times k} & I_{n-k\times n-k} \end{array}\right],$$

where

$$J_{k\times k} = \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & & I & 0 \\ 0 & \iddots & & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}.$$

Then U_k is unitary and

$$
\begin{aligned}
&w\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \cdots + w\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w\left(U_2^* \begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} U_2\right) \\
&\quad + \cdots + w\left(U_n^* \begin{bmatrix} A_{nn} & A_{nn-1} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} U_n\right).
\end{aligned}
$$

Since $w(.)$ is weak unitary, then

$$
\begin{aligned}
w(T) \leq\ & w\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} A_{22} & A_{21} & \cdots & A_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&+ \cdots + w\left(\begin{bmatrix} A_{nn} & A_{nn-1} & \cdots & A_{n1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{w(A_{11})}{2} + \frac{1}{4} + \frac{1}{4}\|A_{11}A_{11}^* + A_{12}A_{12}^* + \cdots + A_{1n}A_{1n}^*\| \\
&\quad + \frac{w(A_{22})}{2} + \frac{1}{4} + \frac{1}{4}\|A_{22}A_{22}^* + A_{21}A_{21}^* + \cdots + A_{2n}A_{2n}^*\| \\
&\quad + \cdots + \frac{w(A_{nn})}{2} + \frac{1}{4} + \frac{1}{4}\|A_{nn}A_{nn}^* + A_{nn-1}A_{nn-1}^* + \cdots + A_{n1}A_{n1}^*\| \\
&= \frac{1}{2}\Big[w(A_{11}) + w(A_{22}) + \cdots + w(A_{nn})\Big] \\
&\quad + \frac{1}{4} + \frac{1}{4}\Big[\|A_{11}A_{11}^* + A_{12}A_{12}^* + \cdots + A_{1n}A_{1n}^*\|\Big] \\
&\quad + \frac{1}{4} + \frac{1}{4}\Big[\|A_{21}A_{21}^* + A_{22}A_{22}^* + \cdots + A_{2n}A_{2n}^*\|\Big] \\
&\quad + \cdots + \frac{1}{4} + \frac{1}{4}\Big[\|A_{n1}A_{n1}^* + A_{n2}A_{n2}^* + \cdots + A_{nn}A_{nn}^*\|\Big] \\
&\leq \frac{1}{2}\sum_{i=1}^{n} w(A_{ii}) + \frac{n}{4} + \frac{1}{4}\sum_{i,j=1}^{n} \|A_{ij}A_{ij}^*\| \\
&= \frac{1}{2}\sum_{i=1}^{n} w(A_{ii}) + \frac{n}{4} + \frac{1}{4}\sum_{i,j=1}^{n} \|A_{ij}\|^2.
\end{aligned}
$$

□

The above theorem is explained below for the case of right shift operators.

Example 2.10. Let $T = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix}$. Then

$$
\begin{aligned}
w(T) &\leq \frac{1}{2}\sum_{i=1}^{n} w(S_{ii}) + \frac{n}{4} + \frac{1}{4}\sum_{i,j=1}^{n} \|S_{ij}\|^2. \\
&= \frac{1}{2}n\ cos\frac{\pi}{n+1} + \frac{n}{4} + \frac{n}{4} \\
&= \frac{n}{2}\Big[1 + cos\frac{\pi}{n+1}\Big],
\end{aligned}
$$

where S_{ij}'s are right shift operators on $\mathbb{C}^n$ and for $1 \leq i, j \leq n$.

An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* if there exists an $M \geq 1$ such that $\|p(A)\| \leq$ M $\sup\{|p(z)| : |z| = 1\}$ for all polynomials p. An operator $A \in \mathcal{L}(\mathcal{H})$ is similar to contraction if there exists a bounded invertible operator L such that $\|LTL^{-1}\| \leq 1$. Halmos [6] raised a question whether every *polynomially bounded* operator is similar to a contraction. There is a number of results dealing with sufficient conditions for a *polynomially bounded* operator to be similar to a contraction. Foias and Williams [4] studied operators of the form T(X)=$\begin{pmatrix} S^* & X \\ 0 & S \end{pmatrix}$ acting on $H^2 \oplus H^2$, where S is the forward unilateral shift on the

Hardy space H^2 and $X \in \mathcal{L}(H^2)$. They conjectured that there exists a Hankel operator H_g with symbol g such that the operator $T(H_g)$ is *polynomially bounded* and is not similar to a contraction. Petrovic [12] then exhibited a relationship between the similarity of $T(H_g)$ and a contraction. Now, we present the numerical radius inequality for a *Foias–Williams operator*. Let $T(H_g) = \begin{pmatrix} S^* & H_g \\ 0 & S \end{pmatrix}$. Then, by triangle inequality for numerical radius, we obtain

$$\begin{aligned} w\begin{pmatrix} S^* & H_g \\ 0 & S \end{pmatrix} &\leq w\begin{pmatrix} S^* & H_g \\ 0 & 0 \end{pmatrix} + w\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \\ &\leq \frac{1}{2}w(S^*) + \frac{1}{4}\left\| I + S^*S + H_gH_g^* \right\| + w(S) \\ &\leq \frac{1}{2}w(S^*) + \frac{1}{4}\left\{1 + \|S^*\|\,\|S\| + \|H_g\|\,\left\|H_g^*\right\|\right\} + w(S) \\ &= \frac{1}{2} + \frac{\pi^2}{4} + w(S) + \frac{w(S^*)}{2}. \end{aligned}$$

We conclude this article with the following observation for a Foias–Williams operator T.

Remark 2.11. The following inequality holds:

$$w(T) \leq w(S) + \frac{w(S^*)}{2} + \frac{1}{2} + \frac{\pi^2}{4}$$

for a Foias–Williams operator T.

Acknowledgment. The authors thank the referee for his/her suggestions that led to a better presentation.

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