Adv. Oper. Theory 3 (2018), no. 4, 807–815
https://doi.org/10.15352/aot.1803-1340
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

ON TENSORS OF FACTORIZABLE QUANTUM CHANNELS WITH THE COMPLETELY DEPOLARIZING CHANNEL

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Communicated by M. Tomforde

Abstract. In this paper, we obtain results for factorizability of quantum channels. Firstly, we prove that if a tensor $T \otimes S_k$ of a quantum channel T on $M_n(\mathbb{C})$ with the completely depolarizing channel S_k is written as a convex combination of automorphisms on the matrix algebra $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ with rational coefficients, then the quantum channel T has an exact factorization through some matrix algebra with the normalized trace. Next, we prove that if a quantum channel has an exact factorization through a finite dimensional von Neumann algebra with a convex combination of normal faithful tracial states with rational coefficients, then it also has an exact factorization through some matrix algebra with the normalized trace.

1. Introduction

In [1], Anantharaman-Delaroche introduced the class of factorizable Markov maps to deal with the noncommutative analogue of Rota's theorem which is one of the most important convergence theorems from classical probability theory. After the works of [1], Haagerup and Musat proved in [3, Theorem 6.1] that every nonfactorizable quantum channel on $M_n(\mathbb{C})$ $(n \geq 3)$ fails the asymptotic quantum Birkhoff conjecture which was raised by Smolin, Verstraete, and Winter (see [7]) as one of the most important problems in quantum information theory. In [3, 4], they also approached the Connes embedding problem by using factorizable quantum channels, in particular, tensors of factorizable quantum channels with the

Date: Received: Mar. 29, 2018; Accepted: May 24, 2018.
2010 *Mathematics Subject Classification.* Primary 46L07; Secondary 15A60, 47C15, 47L07.
Key words and phrases. Markov map, factorizable quantum channel, completely depolarizing channel.

completely depolarizing channel. In this paper, we focus on the relation between the factorizability of quantum channels and the property of tensors of factorizable quantum channels with the completely depolarizing channel. Haagerup and Musat proved in [4, Proposition 3.4] that a quantum channel T on $M_n(\mathbb{C})$ satisfies that $T \otimes S_k \in$conv(Aut($M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$)), (where S_k is the completely depolarizing channel on $M_k(\mathbb{C})$), if and only if T has an exact factoirzation through a tracial W^*-probability space $(M_n(\mathbb{C}) \otimes (M_k(\mathbb{C}) \otimes L^\infty([0,1],m)), \tau_n \otimes (\tau_k \otimes \tau_{L^\infty}))$, where m denotes the Lebesgue measure on $[0,1]$ and $\tau_{L^\infty}(f) := \int_0^1 f(t)dm(t)$ for all $f \in L^\infty([0,1],m)$. (Note that k is the same positive integer in the equivalent property). From the reason, it is very important to understand tensors $T \otimes S_k$ of a quantum channel T with the completely depolarizing channel S_k to know what T has an exact factorization through some W^*-probability space. Moreover they proved in [4, Corollary 3.5] that if a quantum channel T on $M_n(\mathbb{C})$ has an exact factorization through a tracial W^*-probability space $(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$, then $T \otimes S_k \in$ conv(Aut($M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$)). We raise the natural problem of whether the converse claim of the statement is true or not (see below).

Problem 1.1. Let n be a positive integer, and let T be a quantum channel on $M_n(\mathbb{C})$. Is it true that the following properties are equivalent?
(1) T has an exact factorization through $(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for some positive integer k.
(2) $T \otimes S_l \in$conv(Aut($M_n(\mathbb{C}) \otimes M_l(\mathbb{C})$)) for some positive integer l.

By [4, Corollary 3.5], the implication (1) $\Rightarrow$ (2) is true, and we should choose a positive integer k of the assumption (1) as a positive integer l of the condition (2), but the implication (2) $\Rightarrow$ (1) and the relation between positive integers k (of (1)) and l (of (2)) are unknown.

We obtain that Problem 1.1 is true in the special case of quantum channels. But we conclude that Problem 1.1 is not true in the general case.

Theorem 1.2. *Let T be a quantum channel on $M_n(\mathbb{C})$. If there exists a positive integer k such that $T \otimes S_k = \sum_{i=1}^{d(k)} \alpha_i ad(u_i) \in$conv(Aut($M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$)) for some positive integer $d(k)$, unitary matrices $u_1, \ldots, u_{d(k)} \in \mathcal{U}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))$, and positive rational numbers $\alpha_1, \ldots, \alpha_{d(k)}$ with $\sum_{i=1}^{d(k)} \alpha_i = 1$, then there exists a positive integer N such that T has an exact factorization through $(M_n(\mathbb{C}) \otimes M_N(\mathbb{C}), \tau_n \otimes \tau_L)$. In particular, if we can write $\alpha_i = \frac{L_i}{L}$, where $L_1, \ldots, L_{d(k)}, L$ are positive integers and $L_1 + \cdots + L_{d(k)} = L$, then we should choose $N = kL$.*

Theorem 1.3. *In general, Problem 1.1 has a negative answer.*

Moreover we also raise the following problem for the quantum channels which have an exact factorization through a finite dimensional W^*-probability space (see below).

Problem 1.4. Let T be a quantum channel on $M_n(\mathbb{C})$. Is it true that if there exists a finite dimensional W^*-probability space $(\mathcal{N}, \phi)$ (i.e. a pair of a finite dimensional von Neumann algebra $\mathcal{N}$ and a normal faithful state ϕ on $\mathcal{N}$) such that T has an exact factorization through $(M_n(\mathbb{C}) \otimes \mathcal{N}, \tau_n \otimes \phi)$, then there exists a positive integer k such that T also has an exact factorization through $(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$?

Note that every finite dimensional von Neumann algebra is $*$-isomorphic to a direct sum of some matrix algebras. We obtain that Problem 1.4 is true in the special case of normal faithful states on finite dimensional von Neumann algebras. But we also conclude that Problem 1.4 is not true in the general case.

Theorem 1.5. *Let T be a quantum channel on $M_n(\mathbb{C})$. If for $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Q}_+^d$ with $\alpha_1 + \cdots + \alpha_d = 1$ there exist positive integers $k_1, \dots, k_d$ such that T has an exact factorization through $(M_n(\mathbb{C}) \otimes (M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_d}(\mathbb{C})), \tau_n \otimes \tau_\alpha)$, where τ_α is a normal faithful tracial state on $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_d}(\mathbb{C})$ defined by*

$$\tau_\alpha(x_1, \dots, x_d) := \alpha_1 \tau_{k_1}(x_1) + \cdots + \alpha_d \tau_{k_d}(x_d)$$

for all $(x_1, \dots, x_d) \in M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_d}(\mathbb{C})$, then there exists a positive integer k such that T has an exact factorization through $(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$.

Theorem 1.6. *In general, Problem 1.4 has a negative answer.*

In section 2, we set notation and definitions in this paper. In section 3, we recall and discuss for the concepts and basic properties of quantum channels, factorizable quantum channels, and completely depolarizing channels. In sections 4 and 5, we prove that Theorems 1.2, 1.3 and Theorems 1.5, 1.6 hold, respectively.

2. Notation and definitions

In this paper, we use the following notation:

- $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{Q}_+$ is the set of all positive rational numbers.
- $M_n(\mathbb{C})$ is the set of all $n \times n$ matrices with complex entries.
- $\mathcal{U}(n)$ is the set of all $n \times n$ unitary matrices with complex entries.
- τ_n is the normalized trace on $M_n(\mathbb{C})$; that is, $\tau_n((x_{ij})_{1 \le i,j \le n}) := \frac{x_{11} + \cdots + x_{nn}}{n}$.
- id_n is the identity map on $M_n(\mathbb{C})$.
- Define $\mathrm{ad}(u)(x) := u^* x u$ for all $x \in M_n(\mathbb{C})$, $u \in \mathcal{U}(n)$.
- $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ is the convex hull of the set $\mathrm{Aut}(M_n(\mathbb{C})) := \{\mathrm{ad}(u) : u \in \mathcal{U}(n)\}$.
- $1_{\mathcal{M}}$ is the unit in von Neumann algebra $\mathcal{M}$; in particular, $1_n := 1_{M_n(\mathbb{C})}$.

A pair $(\mathcal{M}, \phi)$ is called a *W^*-probability space* if $\mathcal{M}$ is a von Neumann algebra and ϕ is a normal faithful state on $\mathcal{M}$. In particular, we call $(\mathcal{M}, \phi)$ a *tracial* W^*-probability space when ϕ is tracial; that is, $\phi(xy) = \phi(yx)$ for all $x, y \in \mathcal{M}$.

3. Basic properties of factorizable quantum channels

In [1], Anantharaman-Delaroche considered factorizable Markov maps to prove a noncommutative analogue of Rota's theorem. We first recall the definition of Markov maps on a W^*-probability space. The concept is a noncommutative analogue of the measure-preserving Markov operator on a probability space.

Definition 3.1. Let $(\mathcal{M}, \phi)$ and $(\mathcal{N}, \psi)$ be W^*-probability spaces. A linear map $T : \mathcal{M} \to \mathcal{N}$ is called a *(ϕ, ψ)-Markov map* if
(1) T is completely positive,

(2) T is unital,
(3) T is (ϕ, ψ)-preserving; that is, $\psi \circ T = \phi$,
(4) $T \circ \sigma_t^{\phi} = \sigma_t^{\psi} \circ T$, where $\{\sigma_t^{\phi}\}_{t\in\mathbb{R}}$ denotes the automorphism group of ϕ.
In particular, we call it *ϕ-Markov map* when $(\mathcal{M}, \phi) = (\mathcal{N}, \psi)$.

If $(\mathcal{M}, \phi) = (\mathcal{N}, \psi) = (M_n(\mathbb{C}), \tau_n)$ in Definition 3.1, the fourth condition is removed since the operator σ_t^{τ} is trivial for any normal faithful tracial states τ on von Neumann algebras and $t \in \mathbb{R}$; so that a τ_n-Markov map means a unital completely positive trace-preserving map (*quantum channel*) on $M_n(\mathbb{C})$. Denote by

$$\mathcal{Q}(n) := \{T : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : T \text{ is a quantum channel}\}.$$

In [1, Definition 6.2], Anantharaman-Delaroche defined the class of factorizable Markov maps in the following sense.

Definition 3.2. A (ϕ, ψ)-Markov map $T : \mathcal{M} \to \mathcal{N}$ is called *factorizable* if there exist a W^*-probability space $(\mathcal{L}, \chi)$ and $*$-monomorphisms $\alpha : \mathcal{M} \to \mathcal{L}$ and $\beta : \mathcal{N} \to \mathcal{L}$ such that α is (ϕ, χ)-Markov, β is (ψ, χ)-Markov and $T = \beta^* \circ \alpha$, where $\beta^* : \mathcal{L} \to \mathcal{M}$ is the adjoint of β (see [3, Remark 1.2]).

The set of all factorizable (ϕ, ψ)-Markov maps is closed under composition, the adjoint operation, taking convex combinations, and w^*-limits (See [6, Proposition 2]). Haagerup and Musat proved in [3, Theorem 2.2] the following statement for the class of factorizable quantum channels.

Proposition 3.3. *Consider $T \in \mathcal{Q}(n)$. Then the following properties are equivalent:*
(1) T is factorizable,
(2) There exist a tracial W^*-probability space $(\mathcal{M}, \phi)$ and a unitary u in $M_n(\mathbb{C}) \otimes \mathcal{M}$ such that

$$Tx = (id_n \otimes \phi)(u^*(x \otimes 1_{\mathcal{M}})u), \qquad x \in M_n(\mathbb{C}).$$

In this case, we say that T *has an exact factorization through* $(M_n(\mathbb{C}) \otimes \mathcal{M}, \tau_n \otimes \phi)$. A factorization of quantum channels is not unique. We have two examples of a factorization of quantum channels. Firstly we show the following statement.

Lemma 3.4. *If $T \in \mathcal{Q}(n)$ has an exact factorization through a tracial W^*-probability space $(M_n(\mathbb{C}) \otimes \mathcal{M}, \tau_n \otimes \phi)$ and there exist a tracial W^*-probability space $(\mathcal{N}, \psi)$ and a (ϕ, ψ)-Markov *-homomorphism $S : (\mathcal{M}, \phi) \to (\mathcal{N}, \psi)$, then T also has an exact factorization through $(M_n(\mathbb{C}) \otimes \mathcal{N}, \tau_n \otimes \psi)$.*

Proof. Since T has an exact factorization through $(M_n(\mathbb{C}) \otimes \mathcal{M}, \tau_n \otimes \phi)$, there exists a unitary $u \in M_n(\mathbb{C}) \otimes \mathcal{M}$ such that

$$Tx = (id_n \otimes \phi)(u^*(x \otimes 1_{\mathcal{M}})u), \qquad x \in M_n(\mathbb{C}).$$

Since S is a *-homomorphism, $(id_n \otimes S)(u)$ is a unitary in $M_n(\mathbb{C}) \otimes \mathcal{N}$. Hence,

$$\begin{aligned}
&(id_n \otimes \psi)\big((id_n \otimes S)(u)^*(x \otimes 1_{\mathcal{N}})(id_n \otimes S)(u)\big) \\
&\qquad = (id_n \otimes \psi)(id_n \otimes S)(u^*(x \otimes 1_{\mathcal{M}})u) \\
&\qquad = (id_n \otimes \phi)(u^*(x \otimes 1_{\mathcal{M}})u) \\
&\qquad = Tx
\end{aligned}$$

for all $x \in M_n(\mathbb{C})$. Therefore T has an exact factorization through $(M_n(\mathbb{C}) \otimes \mathcal{N}, \tau_n \otimes \psi)$. $\square$

As the second example, we consider a linear map T defined by

$$Tx = \sum_{k=1}^{d} E_{kk} x E_{kk}, \qquad x \in M_d(\mathbb{C}),$$

where $\{E_{kl}\}_{1\leq k,l\leq d}$ is the set of standard matrix units in $M_d(\mathbb{C})$. By [2], it is clear that T is a quantum channel on $M_d(\mathbb{C})$. Then we have the following proposition.

Proposition 3.5. *Consider $d \geq 2$. Let T be the above quantum channel on $M_d(\mathbb{C})$. Then the following conditions hold.*
(1) T has an exact factorization through $(M_d(\mathbb{C}) \otimes \mathcal{L}\mathbb{F}_d, \tau_d \otimes \tau_{\mathcal{L}\mathbb{F}_d})$.
(2) T has an exact factorization through $(M_d(\mathbb{C}) \otimes M_d(\mathbb{C}), \tau_d \otimes \tau_d)$.

Proof. (1) Let $g_1, \ldots, g_d$ be generators of the free group $\mathbb{F}_d$ of degree d, and set $u := \sum_{k=1}^{d} E_{kk} \otimes \lambda_{g_k} \in \mathcal{U}(M_d(\mathbb{C}) \otimes \mathcal{L}\mathbb{F}_d)$, where λ_g is the left representation of $g \in \mathbb{F}_d$; that is,

$$\lambda_g(f)(h) := f(g^{-1}h), \qquad f \in l^2\mathbb{F}_d, \;\; g, h \in \mathbb{F}_d,$$

and $\mathcal{L}\mathbb{F}_d$ is the free group von Neumann algebra. For all $x \in M_d(\mathbb{C})$,

$$\begin{aligned}(id_d \otimes \tau_{\mathcal{L}\mathbb{F}_d})(u^*(x \otimes 1_{\mathcal{L}\mathbb{F}_d})u) &= \sum_{k,l=1}^{d} \tau_{\mathcal{L}\mathbb{F}_d}(\lambda_{g_k}^* \lambda_{g_l}) E_{kk}^* x E_{ll} \\ &= \sum_{k,l=1}^{d} \delta_{kl} E_{kk} x E_{ll} = Tx,\end{aligned}$$

where

$$\tau_{\mathcal{L}\mathbb{F}_d}(\lambda) := < \lambda\delta_e, \delta_e >_{l^2\mathbb{F}_d}, \qquad \lambda \in \mathcal{L}\mathbb{F}_d,$$

and $e \in \mathbb{F}_d$ is the unit of $\mathbb{F}_d$. Therefore T has an exact factorization through $(M_d(\mathbb{C}) \otimes \mathcal{L}\mathbb{F}_d, \tau_d \otimes \tau_{\mathcal{L}\mathbb{F}_d})$.
(2) We can write T as $Tx = \frac{1}{d}\sum_{k=1}^{d} (u^*)^k x u^k$ with

$$u = \mathrm{diag}(e^{\frac{2\pi i}{d}}, e^{\frac{2\pi \cdot 2i}{d}}, \ldots, e^{\frac{2\pi \cdot di}{d}}) \in \mathcal{U}(M_d(\mathbb{C}))$$

and define the unitary element $v = \sum_{k=1}^{d} u^k \otimes E_{kk} \in \mathcal{U}(M_d(\mathbb{C}) \otimes M_d(\mathbb{C}))$. Then we have

$$(id_d \otimes \tau_d)(v^*(x \otimes 1_d)v) = \sum_{k,l=1}^{d} \tau_d(E_{kk}^* E_{ll})(u^*)^k x u^l = Tx$$

for all $x \in M_d(\mathbb{C})$. The proof is complete. $\square$

Set

$$\mathcal{F}(n) := \{T \in \mathcal{Q}(n) : T \text{ is factorizable}\}.$$

By the statements before Proposition 3.3, we have that $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))\subset \mathcal{F}(n)$ for all positive integers n. Haagerup and Musat found a quantum channel in $\mathcal{F}(n)\backslash\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ in [3, Example 3.3]. In particular, Kümmerer proved in [5] that conv(Aut($M_2(\mathbb{C})$))=$\mathcal{F}(2)$. Haagerup and Musat pointed out the important relations between the factorizable quantum channels and the Connes embedding problem in [3, 4].

Recall the completely depolarizing channels. Let $S_k : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map defined by

$$S_k(x) := \tau_k(x)1_k, \qquad x \in M_k(\mathbb{C}).$$

The map S_k is called the *completely depolarizing channel* on $M_n(\mathbb{C})$. Note that S_k is in conv(Aut($M_k(\mathbb{C})$)), and therefore it is a factorizable quantum channel on $M_n(\mathbb{C})$. By [3, Corollary 2.5], Haagerup and Musat found a quantum channel $T \in \mathcal{F}(n)\setminus$ conv(Aut($M_n(\mathbb{C})$)) ($n \geq 3$) such that it has an exact factorization through $(M_n(\mathbb{C}) \otimes M_{2^d}(\mathbb{C}), \tau_n \otimes \tau_{2^d})$ for some $d \geq 3$. By [4, Corollary 3.5], we have that $T \otimes S_{2^d} \in$ conv(Aut($M_n(\mathbb{C}) \otimes M_{2^d}(\mathbb{C})$)).

4. Proof of Theorems 1.2 and 1.3

We prove that Theorems 1.2 and 1.3 hold in this section. We first introduce the following set.

$$\mathcal{J}_n := \left\{ T \in \mathcal{F}(n) \,\middle|\, \begin{array}{l} \exists k \in \mathbb{N} \text{ s.t. } T \text{ has an exact factorization through} \\ (M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k) \end{array} \right\}.$$

To prove Theorem 1.2, we give the following property for $\mathcal{J}_n$.

Lemma 4.1. *$\mathcal{J}_n$ is closed under convex combinations with rational coefficients.*

Proof. Suppose that $n \in \mathbb{N}$. We claim that if $T_1, \dots, T_d \in \mathcal{J}_n$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Q}_+$ with $\alpha_1 + \cdots + \alpha_d = 1$, then $\sum_{i=1}^d \alpha_i T_i \in \mathcal{J}_n$. By Lemma 3.4 and the definition of $\mathcal{J}_n$, there exists $k \in \mathbb{N}$ such that T_i has an exact factorization through $(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for all $i = 1, \dots, d$. Therefore for each $i = 1, \dots, d$, we can find a unitary matrix $u_i \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ such that

$$T_i(x) = (id_n \otimes \tau_k)(u_i^*(x \otimes 1_k)u_i), \qquad x \in M_n(\mathbb{C}).$$

Since $\alpha_1, \dots, \alpha_d$ are positive rational numbers such that $\alpha_1 + \cdots + \alpha_d = 1$, we can write $\alpha_i = \frac{L_i}{L}$, for $i = 1, \dots, d$, where $L_1, \dots, L_d, L > 0$ and $L_1 + \cdots + L_d = L$. Then we define

$$U := \mathrm{diag}(\overbrace{u_1, \dots, u_1}^{L_1}, \dots, \overbrace{u_d, \dots, u_d}^{L_d}) \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes M_L(\mathbb{C}).$$

Clearly the block matrix U is a unitary in $M_n(\mathbb{C})\otimes M_k(\mathbb{C})\otimes M_L(\mathbb{C})$, and we have that

$$\begin{aligned}
&(id_n\otimes\tau_k\otimes\tau_L)(U^*(x\otimes 1_k\otimes 1_L)U)\\
&\quad=(id_n\otimes\tau_k\otimes\tau_L)\begin{pmatrix}\mathrm{diag}(\overbrace{u_1^*(x\otimes 1_k)u_1,\dots,u_1^*(x\otimes 1_k)u_1}^{L_1}\\ \qquad\qquad,\dots,\overbrace{u_d^*(x\otimes 1_k)u_d,\dots,u_d^*(x\otimes 1_k)u_d}^{L_d})\end{pmatrix}\\
&\quad=(id_n\otimes\tau_k\otimes\tau_L)\begin{pmatrix}\sum_{i=1}^{L_1}u_1^*(x\otimes 1_k)u_1\otimes E_{ii}\\ \qquad+\cdots+\sum_{i=1}^{L_d}u_d^*(x\otimes 1_k)u_d\otimes E_{L-L_d+i,L-L_d+i}\end{pmatrix}\\
&\quad=\sum_{i=1}^{L_1}\tau_L(E_{ii})(id_n\otimes\tau_k)(u_1^*(x\otimes 1_k)u_1)\\
&\qquad+\cdots+\sum_{i=1}^{L_d}\tau_L(E_{L-L_d+i,L-L_d+i})(id_n\otimes\tau_k)(u_d^*(x\otimes 1_k)u_d)\\
&\quad=\sum_{i=1}^{d}\frac{L_i}{L}(id_n\otimes\tau_k)(u_i^*(x\otimes 1_k)u_i)\\
&\quad=\sum_{i=1}^{d}\alpha_iT_i(x),
\end{aligned}$$

for all $x\in M_n(\mathbb{C})$, where $\{E_{ij}\}_{1\le i,j\le L}$ is the set of standard matrix units in $M_L(\mathbb{C})$.Therefore $\sum_{i=1}^d\alpha_iT_i$ has an exact factorization through $(M_n(\mathbb{C})\otimes(M_k(\mathbb{C})\otimes M_L(\mathbb{C})),\tau_n\otimes(\tau_k\otimes\tau_L))$. Thus $\sum_{i=1}^d\alpha_iT_i\in\mathcal{J}_n$. □

Finally, it is easy to prove that Theorem 1.2 holds.

Proof of Theorem 1.2. Suppose that $n\in\mathbb{N}$ and that T is a quantum channel on $M_n(\mathbb{C})$. Assume that there exists a positive integer $k>0$ such that $T\otimes S_k(z)=\sum_{i=1}^{d(k)}\alpha_iu_i^*zu_i$, $(z\in M_n(\mathbb{C})\otimes M_k(\mathbb{C}))$, for some positive integer $d(k)>0$, unitaries $u_1,\dots,u_{d(k)}\in M_n(\mathbb{C})\otimes M_k(\mathbb{C})$, and positive rational numbers $\alpha_1,\dots,\alpha_{d(k)}>0$ with $\sum_{i=1}^{d(k)}\alpha_i=1$. Therefore

$$\begin{aligned}
Tx&=(id_n\otimes\tau_k)(T\otimes S_k)(x\otimes 1_k)\\
&=\sum_{i=1}^{d(k)}\alpha_i(id_n\otimes\tau_k)(u_i^*(x\otimes 1_k)u_i),\ \ x\in M_n(\mathbb{C}).
\end{aligned}$$

Since $T_i(\cdot):=(id_n\otimes\tau_k)(u_i^*(\cdot\otimes 1_k)u_i)\in\mathcal{J}_n$ for all $i=1,\dots,d(k)$, we have that $T\in\mathcal{J}_n$ by Lemma 4.1. This means that T has an exact factorization through $(M_n(\mathbb{C})\otimes M_N(\mathbb{C}),\tau_n\otimes\tau_N)$ for some positive integer N. In particular, if we write $\alpha_i=\frac{L_i}{L}$ where $L_1,\dots,L_{d(k)},L>0$ and $L_1+\cdots+L_{d(k)}=L$, then T has an exact factorization through $(M_n(\mathbb{C})\otimes M_{kL}(\mathbb{C}),\tau_n\otimes\tau_{kL})$ by the proof of Lemma 4.1. Thus the proof is complete. □

We prove that Theorem 1.3 holds.

Proof of Theorem 1.3. We define the quantum channel T_λ on $M_3(\mathbb{C})$ given by $T_\lambda(x) := \lambda x + (1-\lambda)u^*xu$ for all $x \in M_3(\mathbb{C})$, where $u := \mathrm{diag}(1, i, -1) \in \mathcal{U}(M_3(\mathbb{C}))$ and $\lambda \in [0,1] \setminus \mathbb{Q}$. It is clear that $T_\lambda \otimes S_l \in \mathrm{conv}(\mathrm{Aut}(M_3(\mathbb{C}) \otimes M_l(\mathbb{C})))$ for all positive integers l. Assume that T_λ has an exact factorization through $(M_3(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for some positive integer k. Since T_λ is the Schur multiplier associated to the following matrix

$$\begin{pmatrix} 1 & \lambda + (1-\lambda)i & 2\lambda - 1 \\ \lambda - (1-\lambda)i & 1 & \lambda + (1-\lambda)i \\ 2\lambda - 1 & \lambda - (1-\lambda)i & 1 \end{pmatrix},$$

we can find three unitary matrices $u_0, u_1, u_2 \in M_k(\mathbb{C})$ such that $\tau_k(u_p^* u_q) = \lambda + (1-\lambda)i^{q-p}$ for all $p, q = 0, 1, 2$ by [3, Proposition 2.8]. Without loss of generality we may assume that $u_0 = 1_k$ by replacing u_p by $u_0^* u_p$ for all $p = 0, 1, 2$. Then we have the linear relation $u_2 = -i1_k + (1+i)u_1$. From the linear relation, the matrices u_1 and u_2 commute, and therefore we can choose a unitary matrix $V \in M_k(\mathbb{C})$ such that V^*u_1V and V^*u_2V are diagonal. Consider that u_1 has eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_k}$. Then u_2 has eigenvalues $-i + (1+i)e^{i\theta_j}$ $(j = 1, \ldots, k)$ by using the above relation and the unitary matrix V. Since u_2 is unitary, we have that

$$1 = |-i + (1+i)e^{i\theta_j}|^2 = 3 - 2\sqrt{2}\sin\left(\theta_j + \frac{\pi}{4}\right), \qquad j = 1, \ldots, k.$$

This implies that u_2 has eigenvalues 1 (l times) or -1 ($k-l$ times) for some $0 \le l \le k$. Therefore we have that $\tau_k(u_2) = \frac{2l-k}{k}$. But this implies that $\lambda \in \mathbb{Q}$. Hence T_λ does not have an exact factorization through $(M_3(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for any positive integers k. □

5. Proof of Theorems 1.5 and 1.6

We prove that Theorem 1.5 holds. The proof is similar to Theorem 1.2.

Proof of Theorem 1.5. Since T has an exact factorization through $(M_n(\mathbb{C}) \otimes (M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_d}(\mathbb{C})), \tau_n \otimes \tau_\alpha)$, there exist unitary matrices $u_i \in M_n(\mathbb{C}) \otimes M_{k_i}(\mathbb{C})$, for each $i = 1, \ldots, d$, such that

$$Tx = \sum_{i=1}^{d} \alpha_i (id_n \otimes \tau_{k_i})(u_i^*(x \otimes 1_{k_i})u_i), \qquad x \in M_n(\mathbb{C}).$$

Note that $T_i(\cdot) := (id_n \otimes \tau_{k_i})(u_i^*(\cdot \otimes 1_{k_i})u_i) \in \mathcal{J}_n$ for all $i = 1, \ldots, d$. By Lemma 4.1, we have that $T = \sum_{i=1}^{d} \alpha_i T_i \in \mathcal{J}_n$; that is, there exists a positive integer k such that T has an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$. □

We prove that Theorem 1.6 holds.

Proof of Theorem 1.6. Consider $\lambda \in [0,1] \setminus \mathbb{Q}$ and the quantum channel T_λ on $M_3(\mathbb{C})$ in the proof of Theorem 1.3. Since we have that

$$T_\lambda(x) = \lambda(id_3 \otimes \tau_3)(x \otimes 1_3) + (1-\lambda)(id_3 \otimes \tau_3)((u \otimes 1_3)^*(x \otimes 1_3)(u \otimes 1_3)),$$

for all $x \in M_3(\mathbb{C})$, the quantum channel T_λ has an exact factorization through $(M_3(\mathbb{C}) \otimes (M_3(\mathbb{C}) \oplus M_3(\mathbb{C})), \tau_3 \otimes \tau_\lambda)$, where τ_λ is a tracial state on $M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$

given by $\tau_\lambda(x, y) := \lambda\tau_3(x) + (1 - \lambda)\tau_3(y)$ for all $x, y \in M_3(\mathbb{C})$. But T_λ does not have an exact factorization through $(M_3(\mathbb{C}) \otimes M_k(\mathbb{C}), \tau_n \otimes \tau_k)$ for any positive integers k by Theorem 1.3. □

Acknowledgments. The author expresses sincere thanks to the referee who gave useful comments to improve the manuscript. In particular, thanks to referee's important advices, the author could advance researches related to Proposition 3.5 and Problems 1.1 and 1.4. The author would like to express his heartly thanks to Professor Benoît Collins for pointing out some inaccuracies. The author had a chance to visit Professor Magdalena Musat and Professor Mikael Rørdam (in Copenhagen university) during November-December, 2016 with the support of the Top Global University project for Kyoto University.

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