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A BANACH ALGEBRA WITH ITS APPLICATIONS OVER PATHS OF BOUNDED VARIATION

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ABSTRACT. Let $C[0,T]$ denote the space of continuous real-valued functions on $[0,T]$. In this paper we introduce two Banach algebras: one of them is defined on $C[0,T]$ and the other is a space of equivalence classes of measures over paths of bounded variation on $[0,T]$. We establish an isometric isomorphism between them and evaluate analytic Feynman integrals of the functions in the Banach algebras, which play significant roles in the Feynman integration theories and quantum mechanics.

1. INTRODUCTION

Let $C_0[0,T]$ denote classical Wiener space; that is, the space of continuous real-valued functions x on the interval $[0,T]$ with $x(0)=0$. Cameron and Storvick [2] introduced a Banach algebra $\mathcal{S}'$ of functions on $C_0[0,T]$, a space of generalized Fourier–Stieltjes transforms of the $\mathbb{C}$-valued, and finite Borel measures over the functions of bounded variation on $[0,T]$. They showed that $\mathcal{S}'$ is isometrically embedded in the Banach algebra $\mathcal{S}$, a space of generalized Fourier–Stieltjes transforms of the complex Borel measures on $L^2[0,T]$.

On the other hand, let $C[0,T]$ denote an analogue of a generalized Wiener space, the space of continuous real-valued functions on the interval $[0,T]$. On the space $C[0,T]$, Ryu [9, 10] introduced a finite measure $w_{\alpha,\beta;\varphi}$ and investigated its properties, where $\alpha,\beta:[0,T]\to\mathbb{R}$ are continuous functions such that β is strictly increasing and φ is an arbitrary finite measure on the Borel class of $\mathbb{R}$.

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On this space $(C[0,T], w_{\alpha,\beta;\varphi})$, the author [3] introduced an Itô type integral $I_{\alpha,\beta}$ which generalizes the Paley–Wiener–Zygmund integrals on $C_0[0,T]$ and $C[0,T]$, and in [4, 5] he derived two Banach algebras $\mathcal{S}_{\alpha,\beta;\varphi}$ and $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$, by using $I_{\alpha,\beta}$, which generalize Cameron–Storvick's Banach algebra $\mathcal{S}$ with the mean function and the variance function determined by α and β, respectively.

In this paper, we introduce two Banach algebras $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ and $\bar{\mathcal{M}}(B[0,T])$; $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is defined on $C[0,T]$, and $\bar{\mathcal{M}}(B[0,T])$ is a space of equivalence classes of measures over the paths of bounded variation on $[0,T]$. We also establish an isomorphism between $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ and $\bar{\mathcal{M}}(B[0,T])$ and prove that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is embedded in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$. As an application, we derive analytic Feynman integrals of the functions in $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, which play significant roles in the Feynman integration theories and quantum mechanics. In particular, if $\alpha(t) = 0$, $\beta(t) = t$, for $t \in [0,T]$, and $\varphi = \delta_0$, which is the Dirac measure concentrated at 0, then $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi} = \mathcal{S}'$ and $\bar{\mathcal{S}}_{\alpha,\beta;\varphi} = \mathcal{S}$; so that the results of this paper generalize those in [2]. We also note that every path in $C[0,T]$ starts at an arbitrary point; so that $C[0,T]$ generalizes $C_0[0,T]$.

2. An analogue of a generalized Wiener space

In this section we introduce an analogue of a generalized Wiener space with preliminaries which will be used in the next sections.

Let m_L denote the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. Let $C[0,T]$ denote the space of continuous real-valued functions on the interval $[0,T]$. Let $\alpha, \beta : [0,T] \to \mathbb{R}$ be two continuous functions, where β is strictly increasing. Let φ be a positive finite measure on $\mathcal{B}(\mathbb{R})$. For $\vec{t_n} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le T$, let $J_{\vec{t_n}} : C[0,T] \to \mathbb{R}^{n+1}$ be the function defined by

$$J_{\vec{t_n}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)).$$

For $\prod_{j=0}^n B_j$ in $\mathcal{B}(\mathbb{R}^{n+1})$, the subset $J_{\vec{t_n}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval I, and let $\mathcal{I}$ be the set of all such intervals I. Define a premeasure $m_{\alpha,\beta;\varphi}$ on $\mathcal{I}$ by

$$m_{\alpha,\beta;\varphi}(I) = \left[\frac{1}{\prod_{j=1}^n 2\pi[\beta(t_j) - \beta(t_{j-1})]}\right]^{\frac{1}{2}} \times \int_{B_0} \int_{\prod_{j=1}^n B_j} \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{[u_j - \alpha(t_j) - u_{j-1} + \alpha(t_{j-1})]^2}{\beta(t_j) - \beta(t_{j-1})}\right\} dm_L^n(u_1, \ldots, u_n)\, d\varphi(u_0).$$

The Borel σ-algebra $\mathcal{B}(C[0,T])$ of $C[0,T]$ with the supremum norm, coincides with the smallest σ-algebra generated by $\mathcal{I}$, and there exists a unique, positive, and finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for all $I \in \mathcal{I}$. This measure $w_{\alpha,\beta;\varphi}$ is called an analogue of a generalized Wiener measure on $(C[0,T], \mathcal{B}(C[0,T]))$ according to φ [9, 10].

For further work, we give additional conditions for α and β. Let α and β be absolutely continuous real-valued functions on $[0,T]$ such that β is strictly increasing and $|\alpha|'(t) + \beta'(t) > 0$ for $t \in [0,T]$. We note that both $|\alpha|'(t)$ and $\beta'(t)$ exist for m_L-almost everywhere t; since α is of bounded variation, so that $|\alpha|$ is increasing on $[0,T]$. We observe that the functions α and β induce a

Lebesgue–Stieltjes measure $\nu_{\alpha,\beta}$ on $[0,T]$ by

$$\nu_{\alpha,\beta}(E)=\int_E d(|\alpha|+\beta)(t)$$

for a Lebesgue measurable subset E of $[0,T]$. Define $L^2_{\alpha,\beta}[0,T]$ to be the space of functions on $[0,T]$ that are square integrable with respect to the measure $\nu_{\alpha,\beta}$; that is,

$$L^2_{\alpha,\beta}[0,T]=\left\{f:[0,T]\to\mathbb{R}:\int_0^T[f(t)]^2d\nu_{\alpha,\beta}(t)<\infty\right\}.$$

The space $L^2_{\alpha,\beta}[0,T]$ is a Hilbert space and has the obvious inner product [8]

$$\langle f,g\rangle_{\alpha,\beta}=\int_0^T f(t)g(t)d\nu_{\alpha,\beta}(t)\qquad\text{for } f,g\in L^2_{\alpha,\beta}[0,T].$$

We note that there exists a complete orthonormal set of functions in $L^2_{\alpha,\beta}[0,T]$; so that $L^2_{\alpha,\beta}[0,T]$ is separable [5].

Let $S[0,T]$ denote the collection of all step functions on $[0,T]$. For f in $L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}$ be a sequence of the step functions in $S[0,T]$ with $\lim_{n\to\infty}\|\phi_n-f\|_{\alpha,\beta}=0$. Define $I_{\alpha,\beta}(f)$ by the $L^2(C[0,T])$-limit

$$I_{\alpha,\beta}(f)(x)=\lim_{n\to\infty}\int_0^T\phi_n(t)dx(t),$$

for all $x\in C[0,T]$ for which this limit exists, where $\int_0^T\phi_n(t)dx(t)$ denotes the Riemann–Stieltjes integral of ϕ_n with respect to x. We note that $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$-almost everywhere $x\in C[0,T]$ and it is independent of choice of the sequence $\{\phi_n\}$ in $S[0,T]$ to define it [3]. We also note that $I_{\alpha,\beta}(f)$ is normally distributed with the mean $\int_0^T f(t)d\alpha(t)$ and the variance $\|f\|^2_{0,\beta}$ if φ is a probability measure [3].

Let $\mathcal{M}_{\alpha,\beta}$ be the class of complex measures of finite variation on $L^2_{\alpha,\beta}[0,T]$ with the Borel σ-algebra $\mathcal{B}(L^2_{\alpha,\beta}[0,T])$ of $L^2_{\alpha,\beta}[0,T]$ as its class of measurable sets. If $\mu\in\mathcal{M}_{\alpha,\beta}$, then we set $\|\mu\|=\mathrm{var}\mu$, the total variation of μ over $L^2_{\alpha,\beta}[0,T]$. Then $\mathcal{M}_{\alpha,\beta}$ is a Banach algebra under convolution, with the total variation norm, since $L^2_{\alpha,\beta}[0,T]$ is a separable infinite dimensional Hilbert space [7]. Let $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ be the space of functions of the form

$$F(x)=\int_{L^2_{\alpha,\beta}[0,T]}\exp\{iI_{\alpha,\beta}(f)(x)\}d\mu(f),\tag{2.1}$$

for all $x\in C[0,T]$ for which the integral exists, where $\mu\in\mathcal{M}_{\alpha,\beta}$. Here we take

$$\|F\|=\inf\{\|\mu\|\},$$

where the infimum is taken over all μ's so that F and μ are related by (2.1). We note that F is well-defined for $w_{\alpha,\beta;\varphi}$-almost everywhere $x\in C[0,T]$ and it is an integrable function of x on $C[0,T]$. Moreover, it is not difficult to show that $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ is a Banach algebra with unit over $\mathbb{C}$ [5].

Let $F : C[0,T] \to \mathbb{C}$ be a measurable function and suppose that the integral

$$J_F(\lambda) \equiv \int_{C[0,T]} F(\lambda^{-\frac{1}{2}} x) dw_{\alpha,\beta;\varphi}(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in

$$\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$$

such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be a generalized analytic Wiener $w_{\alpha,\beta;\varphi}$-integral of F over $C[0,T]$ with the parameter λ, and it is denoted by

$$\int_{C[0,T]}^{anw_\lambda} F(x) dw_{\alpha,\beta;\varphi}(x) = J_F^*(\lambda)$$

for $\lambda \in \mathbb{C}_+$. Let q be a nonzero real number. If $\int_{C[0,T]}^{anw_\lambda} F(x) dw_{\alpha,\beta;\varphi}(x)$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then we call it a generalized analytic Feynman $w_{\alpha,\beta;\varphi}$-integral of F over $C[0,T]$ with the parameter q, and it is denoted by

$$\int_{C[0,T]}^{anf_q} F(x) dw_{\alpha,\beta;\varphi}(x) = \lim_{\lambda \to -iq} \int_{C[0,T]}^{anw_\lambda} F(x) dw_{\alpha,\beta;\varphi}(x).$$

3. A Banach algebra of classes of measures

In this section we introduce a Banach algebra of equivalence classes of measures over the paths of bounded variation on $[0,T]$.

Let $B[0,T]$ be the space of real right-continuous functions of bounded variation on $[0,T]$ that vanish at T. Let $\mathcal{A}'$ be the σ-algebra of subsets of $B[0,T]$ generated by the class of sets of the form

$$\{v \in B[0,T] : \langle v, f\rangle_{\alpha,\beta} < \lambda\},$$

where f and λ range over all elements of $L^2_{\alpha,\beta}[0,T]$ and all real numbers, respectively. Let $\mathcal{M}(B[0,T])$ be the class of complex measures of finite variation defined on subsets of $B[0,T]$ with $\mathcal{A}'$ as their class of measurable sets. If $\mu \in \mathcal{M}(B[0,T])$, we set $\|\mu\| = \operatorname{var}\mu$ over $B[0,T]$. Note that $\mathcal{M}(B[0,T])$ is a Banach algebra under convolution, with the total variation norm [2]. For $v \in B[0,T]$, let

$$J(x,v) = \exp\left\{ i \int_0^T v(t) dx(t) \right\} \qquad \text{for } x \in C[0,T]. \tag{3.1}$$

Define a relation $\sim$ on $\mathcal{M}(B[0,T])$ by $\mu_1 \sim \mu_2$, for $\mu_1, \mu_2 \in \mathcal{M}(B[0,T])$, if

$$\int_{B[0,T]} J(x,v) d\mu_1(v) = \int_{B[0,T]} J(x,v) d\mu_2(v)$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$. It is obvious that $\sim$ is an equivalence relation on $\mathcal{M}(B[0,T])$. Let $\bar{\mathcal{M}}(B[0,T])$ be the set of equivalence classes under $\sim$. For $[\mu_1], [\mu_2] \in \bar{\mathcal{M}}(B[0,T])$ and $c \in \mathbb{C}$, define $[\mu_1] + [\mu_2] = [\mu_1 + \mu_2]$, $c[\mu_1] = [c\mu_1]$, and $[\mu_1][\mu_2] = [\mu_1 * \mu_2]$. In the following lemma, we prove that these operations are well-defined and that $\bar{\mathcal{M}}(B[0,T])$ is an algebra.

Lemma 3.1. *$\bar{\mathcal{M}}(B[0,T])$ is an algebra with unit over $\mathbb{C}$.*

Proof. It is obvious that the addition and scalar multiplication are well-defined. Let $\sigma_1 \in [\mu_1]$ and $\sigma_2 \in [\mu_2]$, where $[\mu_1], [\mu_2] \in \bar{\mathcal{M}}(B[0,T])$ for $\mu_1, \mu_2 \in \mathcal{M}(B[0,T])$. Then the multiplication is well-defined, since, for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$,

$$\begin{aligned}\int_{B[0,T]} J(x,v)d(\mu_1 * \mu_2)(v) &= \int_{B[0,T]}\int_{B[0,T]} J(x,u+v)d\mu_1(u)d\mu_2(v)\\ &= \left[\int_{B[0,T]} J(x,u)d\mu_1(u)\right]\left[\int_{B[0,T]} J(x,v)d\mu_2(v)\right]\\ &= \left[\int_{B[0,T]} J(x,u)d\sigma_1(u)\right]\left[\int_{B[0,T]} J(x,v)d\sigma_2(v)\right]\\ &= \int_{B[0,T]}\int_{B[0,T]} J(x,u+v)d\sigma_1(u)d\sigma_2(v)\\ &= \int_{B[0,T]} J(x,v)d(\sigma_1 * \sigma_2)(v).\end{aligned}$$

We also have, for $D \in \mathcal{A}'$ and $\mu \in \mathcal{M}(B[0,T])$,

$$(\delta_0 * \mu)(D) = \int_{B[0,T]}\int_{B[0,T]} \chi_D(u+v)d\delta_0(u)d\mu(v) = \int_{B[0,T]} \chi_D(v)d\mu(v) = \mu(D),$$

where δ_0 is the Dirac measure concentrated at the zero function in $B[0,T]$. So that $[\delta_0][\mu] = [\delta_0 * \mu] = [\mu]$; that is, $[\delta_0]$ is the unit of $\bar{\mathcal{M}}(B[0,T])$. Now it is easy to prove that $\bar{\mathcal{M}}(B[0,T])$ is an algebra with unit $[\delta_0]$ over $\mathbb{C}$. □

Lemma 3.2. *Define* $\|[\mu]\| = \inf\{\|\mu_1\| : \mu_1 \in [\mu]\}$ *for* $[\mu] \in \bar{\mathcal{M}}(B[0,T])$. *Then* $(\bar{\mathcal{M}}(B[0,T]), \|\cdot\|)$ *is a normed algebra with unit over* $\mathbb{C}$.

Proof. By Lemma 3.1, it remains to prove that $\|\cdot\|$ is a norm on the algebra $\bar{\mathcal{M}}(B[0,T])$ with unit $[\delta_0]$. It is clear that $\|[0]\| = 0$. Suppose that $\|[\mu]\| = 0$ for $[\mu] \in \bar{\mathcal{M}}(B[0,T])$. Then, for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$, we have

$$\left|\int_{B[0,T]} J(x,f)d\mu(f)\right| = \left|\int_{B[0,T]} J(x,f)d\mu_1(f)\right| \le \|\mu_1\|$$

for all $\mu_1 \in [\mu]$; so that we have

$$\left|\int_{B[0,T]} J(x,f)d\mu(f)\right| \le \inf\{\|\mu_1\| : \mu_1 \in [\mu]\} = \|[\mu]\| = 0,$$

which implies that

$$\int_{B[0,T]} J(x,f)d\mu(f) = 0.$$

Now we have $[\mu] = [0]$. Let $c \in \mathbb{C}$ and $[\mu] \in \bar{\mathcal{M}}(B[0,T])$. If $c = 0$, then $\|c[\mu]\| = |c|\|[\mu]\|$. Suppose that $c \neq 0$. Then

$$\begin{aligned}\|c[\mu]\| &= \inf\{\|\sigma\| : \sigma \in [c\mu]\}\\ &= \inf\left\{|c|\left\|\frac{1}{c}\sigma\right\| : \frac{1}{c}\sigma \in [\mu]\right\} = |c|\inf\{\|\tau\| : \tau \in [\mu]\} = |c|\|[\mu]\|.\end{aligned}$$

Moreover let $[\mu_1], [\mu_2] \in \bar{\mathcal{M}}(B[0,T])$, and let $\epsilon > 0$ be arbitrary. Take $\sigma_1 \in [\mu_1]$ and $\sigma_2 \in [\mu_2]$ such that

$$\|\sigma_1\| < \|[\mu_1]\| + \frac{\epsilon}{2} \text{ and } \|\sigma_2\| < \|[\mu_2]\| + \frac{\epsilon}{2}.$$

Then

$$\|[\mu_1] + [\mu_2]\| = \|[\sigma_1 + \sigma_2]\| \leq \|\sigma_1 + \sigma_2\| \leq \|\sigma_1\| + \|\sigma_2\| < \|[\mu_1]\| + \|[\mu_2]\| + \epsilon$$

and

$$\|[\mu_1][\mu_2]\| = \|[\sigma_1 * \sigma_2]\| \leq \|\sigma_1 * \sigma_2\| \leq \|\sigma_1\|\|\sigma_2\| < \left(\|[\mu_1]\| + \frac{\epsilon}{2}\right)\left(\|[\mu_2]\| + \frac{\epsilon}{2}\right).$$

Since ϵ is arbitrary, we have

$$\|[\mu_1] + [\mu_2]\| \leq \|[\mu_1]\| + \|[\mu_2]\| \text{ and } \|[\mu_1][\mu_2]\| \leq \|[\mu_1]\|\|[\mu_2]\|.$$

We also have $\|[\delta_0]\| \leq \|\delta_0\| = 1$ and $\|[\delta_0]\| = \|[\delta_0][\delta_0]\| \leq \|[\delta_0]\|^2$ which imply $\|[\delta_0]\| = 1$ because $[0] \neq [\delta_0]$. Now $\|\cdot\|$ is a norm on $\bar{\mathcal{M}}(B[0,T])$ which completes the proof. □

Theorem 3.3. *$\bar{\mathcal{M}}(B[0,T])$ is a Banach algebra with unit.*

Proof. It only remains to be shown that $\bar{\mathcal{M}}(B[0,T])$ is complete under the norm given by Lemma 3.2. Let $\{[\mu_n]\}_{n=1}^{\infty}$ be a Cauchy sequence of elements in $\bar{\mathcal{M}}(B[0,T])$ and take a subsequence $\{[\mu_{n_k}]\}_{k=1}^{\infty}$ of $\{[\mu_n]\}_{n=1}^{\infty}$ satisfying

$$\|[\mu_{n_k}] - [\mu_{n_{k-1}}]\| < \frac{1}{2^k} \qquad \text{for } k = 2, 3, \ldots.$$

Take $\sigma_1 \in [\mu_{n_1}]$ with

$$\|\sigma_1\| < \|[\mu_{n_1}]\| + 1.$$

For each $k = 2, 3, \ldots$, take $\sigma_k \in [\mu_{n_k} - \mu_{n_{k-1}}]$ with

$$\|\sigma_k\| < \|[\mu_{n_k}] - [\mu_{n_{k-1}}]\| + \frac{1}{2^k}.$$

Then we have

$$\sum_{k=1}^{\infty} \|\sigma_k\| < \|\mu_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} < \infty.$$

Let $\mu = \sum_{k=1}^{\infty} \sigma_k \in \mathcal{M}(B[0,T])$. Then we also have

$$\begin{aligned}
\|[\mu] - [\mu_{n_k}]\| &= \left\|[\mu] - \sum_{j=2}^{k}[\mu_{n_j} - \mu_{n_{j-1}}] - [\mu_{n_1}]\right\| = \left\|[\mu] - \sum_{j=1}^{k}[\sigma_j]\right\| \\
&\leq \left\|\sum_{j=k+1}^{\infty} \sigma_j\right\| \leq \sum_{j=k+1}^{\infty} \|\sigma_j\| \leq \sum_{j=k+1}^{\infty} \frac{1}{2^{j-1}} = \frac{1}{2^{k-1}},
\end{aligned}$$

which converges to 0 as $k \to \infty$. Since $\{[\mu_n]\}_{n=1}^{\infty}$ is a Cauchy sequence it follows that

$$\lim_{n \to \infty} \|[\mu] - [\mu_n]\| = 0;$$

so that $\bar{\mathcal{M}}(B[0,T])$ is complete as desired. □

4. A Banach algebra of transforms of measures

In this section, we introduce a Banach algebra of generalized Fourier–Stieltjes transforms of the $\mathbb{C}$-valued finite Borel measures over $B[0,T]$.

Let $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ be the space of functions of the form

$$F(x) = \int_{B[0,T]} J(x,f)d\mu(f), \tag{4.1}$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$, where $\mu \in \mathcal{M}(B[0,T])$ and where J is defined by (3.1). Here we take

$$\|F\|' = \inf\{\|\mu\|\},$$

where the infimum is taken over all μ's; so that F and μ are related by (4.1). By using the same method as the proof of Lemma 3.2, we can prove that $(\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}, \|\cdot\|')$ is a normed space over $\mathbb{C}$.

Lemma 4.1. *For each positive integer n, let $F_n \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ with*

$$\sum_{n=1}^{\infty} \|F_n\|' < \infty.$$

Then the sum defined by

$$F(x) \equiv \sum_{n=1}^{\infty} F_n(x), \qquad \text{for } w_{\alpha,\beta;\varphi}\text{-almost everywhere } x \in C[0,T], \tag{4.2}$$

converges absolutely and uniformly, and it is an element of $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$.

Proof. For each positive integer n, take $\mu_n \in \mathcal{M}(B[0,T])$ such that

$$\|\mu_n\| < \|F_n\|' + \frac{1}{2^n}$$

and F_n and μ_n are related by (4.1). Then, for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$,

$$\sum_{n=1}^{\infty} |F_n(x)| \le \sum_{n=1}^{\infty} \|\mu_n\| \le \sum_{n=1}^{\infty} \left(\|F_n\|' + \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \|F_n\|' + 1 < \infty.$$

Hence the absolute and uniform convergences of the right-hand side of (4.2) follow for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$. Define $\mu \in \mathcal{M}(B[0,T])$ by $\mu = \sum_{n=1}^{\infty} \mu_n$. For $f \in B[0,T]$ and $x \in C[0,T]$, let

$$J_k(x,f) = \exp\left\{ \frac{m}{2^k} \pi i \right\}$$

when, for $m = -2^k+1, -2^k+2, \ldots, 2^k$; $k = 1, 2, \ldots$,

$$\frac{m-1}{2^k}\pi < \mathrm{Arg} J(x,f) \le \frac{m}{2^k}\pi.$$

Then $J_k(x,\cdot)$ is a simple function with respect to each of the measures μ_1, μ_2, $\ldots$, and μ, and

$$\lim_{k\to\infty} J_k(x,f) = J(x,f)$$

uniformly for all $f \in B[0,T]$. Since $J_k(x,\cdot)$ is a simple function and $\sum_{n=1}^{\infty}\|\mu_n\|$ converges, it follows that

$$\int_{B[0,T]} J_k(x,f)d\mu(f) = \sum_{n=1}^{\infty}\int_{B[0,T]} J_k(x,f)d\mu_n(f)$$

uniformly. Taking limits on the both sides of the above equality, we obtain

$$\begin{aligned}\int_{B[0,T]} J(x,f)d\mu(f) &= \lim_{k\to\infty}\int_{B[0,T]} J_k(x,f)d\mu(f)\\ &= \lim_{k\to\infty}\sum_{n=1}^{\infty}\int_{B[0,T]} J_k(x,f)d\mu_n(f)\\ &= \sum_{n=1}^{\infty}\lim_{k\to\infty}\int_{B[0,T]} J_k(x,f)d\mu_n(f)\\ &= \sum_{n=1}^{\infty} F_n(x) = F(x)\end{aligned}$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$; so that $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$. □

Theorem 4.2. *$\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is a Banach space.*

Proof. It suffices to be shown that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is complete under the norm $\|\cdot\|'$. Let $\{F_n\}_{n=1}^{\infty}$ be a Cauchy sequence of elements in $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, and take a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ of $\{F_n\}_{n=1}^{\infty}$ satisfying

$$\|F_{n_k} - F_{n_{k-1}}\|' < \frac{1}{2^k} \qquad \text{for } k = 2,3,\dots.$$

Let $G_1 = F_{n_1}$ and $G_k = F_{n_k} - F_{n_{k-1}}$ for $k = 2,3,\dots$. Then $\sum_{k=1}^{\infty}\|G_k\|' < \infty$ and each $G_k \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$. By Lemma 4.1, there exists a function $G \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ such that

$$G(x) = \sum_{k=1}^{\infty} G_k(x)$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of measures in $\mathcal{M}(B[0,T])$ such that G_k and μ_k are related by (4.1) with

$$\|\mu_k\| < \|G_k\|' + \frac{1}{2^k}.$$

By Lemma 4.1, G and $\sum_{k=1}^{\infty}\mu_k$ are related by (4.1). Now, for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$,

$$G(x) - F_{n_k}(x) = G(x) - \sum_{j=1}^{k} G_j(x) = \int_{B[0,T]} J(x,f)d\Big(\sum_{j=k+1}^{\infty}\mu_j\Big)(f);$$

so that

$$\|G - F_{n_k}\|' \le \Big\|\sum_{j=k+1}^{\infty}\mu_j\Big\| \le \sum_{j=k+1}^{\infty}\|\mu_j\| \le \sum_{j=k+1}^{\infty}\Big(\|G_j\|' + \frac{1}{2^j}\Big) \le \frac{1}{2^{k-1}},$$

which converges to 0 as $k \to \infty$. Since $\{F_n\}_{n=1}^{\infty}$ is a Cauchy sequence, it follows that

$$\lim_{n\to\infty} \|G - F_n\|' = 0,$$

which proves this theorem. □

Theorem 4.3. *The space* $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ *is a Banach algebra with unit. Moreover,* $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ *is isometrically isomorphic to* $\bar{\mathcal{M}}(B[0,T])$.

Proof. Let $F, G \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, and let $\epsilon > 0$ be arbitrary. Let F, G and μ, ν be related by (4.1), respectively, with

$$\|\mu\| < \|F\|' + \epsilon \text{ and } \|\nu\| < \|G\|' + \epsilon.$$

It is not difficult to show that FG and $\mu * \nu$ are related by (4.1). Moreover,

$$\|FG\|' \le \|\mu * \nu\| \le \|\mu\|\|\nu\| < (\|F\|' + \epsilon)(\|G\|' + \epsilon);$$

so that

$$\|FG\|' \le \|F\|'\|G\|',$$

since ϵ is arbitrary. The constant function 1 is the unit of $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, since 1 and δ_0 are related by (4.1). By the definition of $\|\cdot\|'$, we have $\|1\|' \le \|\delta_0\| = 1$. Since $\|1\|' = \|1^2\|' \le \|1\|' \cdot \|1\|'$ and $1 \neq 0$, we have $1 \le \|1\|'$; so that $\|1\|' = 1$ and $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is a Banach algebra with unit by Theorem 4.2. Define $\phi : \bar{\mathcal{M}}(B[0,T]) \to \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ by

$$\phi([\mu]) = \int_{B[0,T]} J(x,f)\,d\mu(f)$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$. Let $c_1, c_2 \in \mathbb{C}$ and $[\mu_1], [\mu_2] \in \bar{\mathcal{M}}(B[0,T])$. If $[\mu_1] = [\mu_2]$, then $\mu_1 \sim \mu_2$. By the definition of $\sim$, we have $\phi([\mu_1]) = \phi([\mu_2])$, which implies that ϕ is well-defined. Now we have

$$\begin{aligned}
\phi(c_1[\mu_1] + c_2[\mu_2]) &= \phi([c_1\mu_1 + c_2\mu_2]) \\
&= \int_{B[0,T]} J(x,f)\,d(c_1\mu_1 + c_2\mu_2)(f) \\
&= c_1\phi([\mu_1]) + c_2\phi([\mu_2]).
\end{aligned}$$

Since $J(x, f + g) = J(x,f)J(x,g)$, for $f, g \in B[0,T]$ and for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$, we also have

$$\begin{aligned}
\phi([\mu_1][\mu_2]) &= \phi([\mu_1 * \mu_2]) \\
&= \int_{B[0,T]} J(x,f)\,d(\mu_1 * \mu_2)(f) \\
&= \int_{B[0,T]}\int_{B[0,T]} J(x, f+g)\,d\mu_1(f)\,d\mu_2(g) \\
&= \left[\int_{B[0,T]} J(x,f)\,d\mu_1(f)\right]\left[\int_{B[0,T]} J(x,g)\,d\mu_2(g)\right] \\
&= \phi([\mu_1])\phi([\mu_2]).
\end{aligned}$$

By the definition of $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, it is obvious that ϕ is onto. If $\phi([\mu_1]) = \phi([\mu_2])$, then

$$\int_{B[0,T]} J(x,f)d\mu_1(f) = \phi([\mu_1]) = \phi([\mu_2]) = \int_{B[0,T]} J(x,f)d\mu_2(f)$$

for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$; so that $\mu_1 \sim \mu_2$. Thus we have $[\mu_1] = [\mu_2]$, which implies that ϕ is one-to-one. Moreover, we have

$$\begin{aligned} \|\phi([\mu_1])\|' &= \left\| \int_{B[0,T]} J(\cdot,f)d\mu_1(f) \right\|' = \inf\{\|\mu\| : \mu \in \mathcal{M}(B[0,T]) \text{ and } \mu \sim \mu_1\} \\ &= \inf\{\|\mu\| : \mu \in [\mu_1]\} = \|[\mu_1]\|; \end{aligned}$$

so that ϕ is an isometric Banach algebra isomorphism. □

Theorem 4.4. *$\bar{\mathcal{S}}'_{\alpha,\beta;\varphi} \subseteq \bar{\mathcal{S}}_{\alpha,\beta;\varphi}$, which is the Banach algebra of functions given by (2.1).*

Proof. If $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, then there exists $\mu' \in \mathcal{M}(B[0,T])$ such that (4.1) holds. Note that $B[0,T] \subseteq L^2_{\alpha,\beta}[0,T]$; so that if $E \in \mathcal{B}(L^2_{\alpha,\beta}[0,T])$, then $E \cap B[0,T] \in \mathcal{A}'$ by the definition of $\mathcal{A}'$. Define a measure μ on $L^2_{\alpha,\beta}[0,T]$ by

$$\mu(E) = \mu'(E \cap B[0,T]) \text{ for all } E \in \mathcal{B}(L^2_{\alpha,\beta}[0,T]).$$

Then we have, for $w_{\alpha,\beta;\varphi}$-almost everywhere $x \in C[0,T]$,

$$\begin{aligned} F(x) &= \int_{B[0,T]} J(x,f)d\mu'(f) = \int_{B[0,T]} \exp\{iI_{\alpha,\beta}(f)(x)\}d\mu'(f) \\ &= \int_{L^2_{\alpha,\beta}[0,T]} \exp\{iI_{\alpha,\beta}(f)(x)\}d\mu(f), \end{aligned} \tag{4.3}$$

by Theorem 3.8 of [3], which completes the proof. □

Remark 4.5. It is not obvious whether $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is isometrically embedded in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ or not. By Theorem 4.4, we can know that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is only a subspace of $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ as a vector space.

Corollary 4.6. *If $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, then $\|F\| \le \|F\|'$, where $\|F\|$ is the norm of F in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$; so that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is continuously embedded in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$.*

Proof. Let $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$. Take any μ' in $\mathcal{M}(B[0,T])$ such that (4.1) holds. Let μ be the measure defined in the proof of Theorem 4.4. Since F and μ are related by (2.1) from (4.3), we have $\|F\| \le \|\mu\|$; so that

$$\|F\| \le \|\mu\| = \mathrm{var}_{B[0,T]}\mu + \mathrm{var}_{L^2_{\alpha,\beta}[0,T]-B[0,T]}\mu = \mathrm{var}_{B[0,T]}\mu' + 0 = \|\mu'\|.$$

Since μ' is an arbitrary measure satisfying (4.1), we have

$$\|F\| \le \inf\{\|\mu'\|\} = \|F\|';$$

so that the proof of this corollary is completed. □

Remark 4.7. Assume that the following condition [4] holds: φ is a probability measure on $\mathbb{R}$, α is absolutely continuous, β' is continuous, positive, and bounded away from 0, and $L^2_{\alpha,\beta}[0,T] = L^2_{0,\beta}[0,T]$ as sets. Then $\|F\| = \|F\|'$, for $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, by the uniqueness of the measure which is related by (4.1); so that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ is isometrically embedded in $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$. In particular, if $\alpha(t) = 0$, $\beta(t) = t$ for $t \in [0,T]$, and $\varphi = \delta_0$, then we have $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi} = \mathcal{S}'$ and $\bar{\mathcal{S}}_{\alpha,\beta;\varphi} = \mathcal{S}$, where $\mathcal{S}'$ and $\mathcal{S}$ are the spaces of Fourier–Stieltjes transforms of measures of finite variation on $B[0,T]$ and $L^2[0,T]$, respectively. So that $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$ and $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ generalize $\mathcal{S}'$ and $\mathcal{S}$ in [2], respectively.

5. Applications to the analytic Feynman integrals

Feynman integrals are introduced by Feynman in his formulation of quantum mechanics, but they are inadequate mathematically [6]. One of approaches to define rigorously them, is to use an analytic continuation from real to imaginary time, which is now called the analytic Feynman integral [7].

In this section we evaluate analytic Feynman integrals of the functions in $\bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, which play important roles in treating the heat equation and the Schrödinger equation by integration over the Wiener space [1].

Theorem 5.1. *For $\mu \in \mathcal{M}(B[0,T])$ and $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, let F and μ be related by (4.1). Then we have, for $\lambda > 0$,*

$$J_F(\lambda) = \varphi(\mathbb{R}) \int_{B[0,T]} \exp\biggl\{-\frac{1}{2\lambda}\int_0^T [f(t)]^2 d\beta(t) + i\lambda^{-\frac{1}{2}} \int_0^T f(t)d\alpha(t)\biggr\} d\mu(f). \quad (5.1)$$

In addition, if there exists $M > 0$ satisfying

$$\int_{B[0,T]} \exp\biggl\{\mathrm{Re}(i\lambda^{-\frac{1}{2}}) \int_0^T f(t)d\alpha(t)\biggr\} d|\mu|(f) \le M, \quad (5.2)$$

for any $\lambda \in \mathbb{C}_+$, then $\int^{anw_\lambda}_{C[0,T]} F(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (5.1). Moreover, if (5.2) holds for all $\lambda \in \{z \in \mathbb{C} : \mathrm{Re} z \ge 0, z \ne 0\}$, then, for any nonzero real q, $\int^{anf_q}_{C[0,T]} F(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (5.1) with replacing λ by $-iq$.

Proof. Let $\varphi_0 = \frac{1}{\varphi(\mathbb{R})}\varphi$. Then φ_0 is a probability measure on $\mathbb{R}$; so that $w_{\alpha,\beta;\varphi_0}$ is also a probability measure on $C[0,T]$, and $w_{\alpha,\beta;\varphi} = \varphi(\mathbb{R})w_{\alpha,\beta;\varphi_0}$ by their definitions. Now the null sets with respect to $w_{\alpha,\beta;\varphi}$ are equivalent to the null sets with respect to $w_{\alpha,\beta;\varphi_0}$. Moreover, as a function on $(C[0,T], w_{\alpha,\beta;\varphi_0})$, $\int_0^T f(t)dx(t)$ is normally distributed with the mean $\int_0^T f(t)d\alpha(t)$ and the variance $\|f\|^2_{0,\beta}$ for

$f \in L^2_{\alpha,\beta}[0,T]$ (see [3]). We now have, for $\lambda > 0$,

$$\begin{aligned} J_F(\lambda) &= \int_{C[0,T]} F(\lambda^{-\frac{1}{2}}x)dw_{\alpha,\beta;\varphi}(x) \\ &= \varphi(\mathbb{R})\int_{B[0,T]}\int_{C[0,T]}\exp\bigg\{i\lambda^{-\frac{1}{2}}\int_0^T f(t)dx(t)\bigg\}dw_{\alpha,\beta;\varphi_0}(x)d\mu(f) \\ &= \varphi(\mathbb{R})\int_{B[0,T]}\exp\bigg\{-\frac{1}{2\lambda}\int_0^T[f(t)]^2d\beta(t)+i\lambda^{-\frac{1}{2}}\int_0^T f(t)d\alpha(t)\bigg\}d\mu(f), \end{aligned}$$

by Fubini's theorem, since $B[0,T] \subseteq L^2_{\alpha,\beta}[0,T]$, which proves (5.1). If (5.2) holds, then we have the remainder part of this theorem by the analytic continuation and the dominated convergence theorem. □

By letting $M = \|\mu\|$ in (5.2) of Theorem 5.1, we now have the following corollary.

Corollary 5.2. *For $\mu \in \mathcal{M}(B[0,T])$ and $F \in \bar{\mathcal{S}}'_{\alpha,\beta;\varphi}$, let F and μ be related by (4.1). If $\int_0^T f(t)d\alpha(t) = 0$ for μ almost everywhere $f \in B[0,T]$, then we have, for any $\lambda \in \mathbb{C}_+$,*

$$\int_{C[0,T]}^{anw_\lambda} F(x)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})\int_{B[0,T]}\exp\bigg\{-\frac{1}{2\lambda}\int_0^T[f(t)]^2d\beta(t)\bigg\}d\mu(f). \quad (5.3)$$

Moreover, for any nonzero real q, $\int_{C[0,T]}^{anf_q} F(x)dw_{\alpha,\beta;\varphi}(x)$ is given by the right-hand side of (5.3) with replacing λ by $-iq$.

Remark 5.3. All the results of this paper are independent of choice of the initial measure φ; that is, they does not depend on particular initial positions of the generalized Wiener paths.

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